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TREED DOMAINS

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Abstract. We establish that treed domains are well behaved in Zafrullah's sense and have locally polynomial depth 1. For the DW-domains R of Mimouni, such that $I^{-1} \neq R$ for each nontrivial finitely generated ideal I of R, likewise results are proven. We study some special treed domains and show in particular that the Nagata ring of an integral domain R is (locally) divided if and only if R is (locally) divided and quasi-Prüfer. We show that the small finitistic dimension of a local treed domain is 1 and calculate the small finitistic dimension of localizations of polynomial rings over a treed domain.

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1. Introduction

A *treed domain* is an integral domain R, whose incomparable prime ideals are coprime. In other words, the spectrum $\mathrm{Spec}(R)$, endowed with the natural partial ordering, is a tree. Going-down domains, (locally) divided domains, and i-domains are treed domains (see [3] and [27] for information on these classes of domains, mainly investigated by Dobbs and Papick). Note also that Olberding proved that stable domains are treed [26, Proposition 4.11]. This paper is devoted to the study of treed domains. We give new properties of these domains, especially about their grades and homological properties. We also examine how the treed hypothesis acts on some classes of domains. In order to explain what our aims are, we introduce some notation. Integral domains considered in this paper are not fields and local rings may not be Noetherian. We denote by $\mathcal{I}_f(R)$ the set of all finitely generated ideals $I \neq 0$ of a ring R and $\mathrm{Max}(R)$ is the set of all its maximal ideals. If $\mathrm{gr}(I, M)$ is the classical grade of an ideal I of a ring R on an R-module M, we set $\mathrm{gr}(I) := \mathrm{gr}(I, R)$ and define the depth of a ring R by $\mathrm{depth}(R) := \sup[\mathrm{gr}(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R)]$. Dobbs proved recently that $\mathrm{depth}(R_{\mathfrak{p}}) = 1$ for each $0 \neq \mathfrak{p} \in \mathrm{Spec}(R)$, where R is a treed domain [4]. Dobbs's result may be deduced from a paper by Huckaba and Papick [17, Lemma 2.1 and Proposition 2.5]. Adapting Hochster's approach to a theory of grade [14], Northcott defined the *polynomial (or true) grade* of an ideal $I \neq R$ on an R-module M as $\mathrm{Gr}(I, M) := \lim_{n\to\infty} \mathrm{gr}(I[X_1, \dots, X_n], M[X_1, \dots, X_n])$ [25, Chapter 5]. We set $\mathrm{Gr}(I) := \mathrm{Gr}(I, R)$. Hochster's equivalent definition is $\mathrm{Gr}(I) = \sup \mathrm{gr}(IS)$, where S runs over the class of faithfully flat R-algebras S. We define the *polynomial depth* of a ring R as $\text{p-depth}(R) := \sup[\mathrm{Gr}(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(R)]$. Hence, in case R is an integral domain, $\text{p-depth}(R) = 1$ if and only if $\mathrm{Gr}(\mathfrak{m}) = 1$ for each $\mathfrak{m} \in \mathrm{Max}(R)$. Sakaguchi observed that $\text{p-depth}(V) = 1$, for a valuation domain V [33, Corollary 5.6]. In this introduction, we only give a sketch of the paper and refer the reader to the next sections for more information on definitions. The aim of Section 2 is to proving that a treed domain has polynomial depth 1 (see Theorem 2.4). We also show that a treed domain is well behaved. In Section 3, we consider the condition (†), which implies that an integral domain has polynomial depth 1. This condition is closely linked to Nagata rings and is verified by treed domains. Theorem 3.4 asserts that this condition on an integral domain R is equivalent to $I^{-1} \neq R$ for each $I \in \mathcal{I}_f(R) \setminus \{R\}$. In other words, R is a DW-domain of Mimouni [23]. In Section 4 we examine some special treed domains. The main result is Theorem 4.2, which states that the Nagata ring $R(X)$ of an integral domain R is (locally) divided if and only if R is quasi-Prüfer and (locally) divided. In Section 5, we show that the small finitistic dimension of a local treed domain is 1 (Theorem 5.2). We thus recover a result of S. Glaz in case R is a Gaussian domain: a *Gaussian ring* R is such that a simplified content formula holds (see [12]). If R is a coherent Gaussian ring, then its small finitistic dimension is ≤ 1 [12, Theorem 3.2]. It is enough to recall that R is a Gaussian domain if and only if R is Prüfer [12, Introduction]. We also compute the small finitistic dimension of local rings of polynomial rings over a treed domain.

2. Properties of treed domains

We use some sets of prime ideals, linked to polynomial grade 1. We denote by $\mathrm{Ass}(M)$ the set of all *weak Bourbaki associated prime ideals* of an R-module M and by $\mathrm{Att}(M)$ the set of all *Northcott attached prime ideals* of M (see [25, p.178], [19] and [29, Section 5]). For an integral domain R, we denote by K its quotient field and set $\mathrm{k}(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ for $\mathfrak{p} \in \mathrm{Spec}(R)$. We introduce $\mathrm{Ass}(K/R)$ and $\mathrm{Att}(K/R)$. A prime ideal $\mathfrak{p}$ of R belongs to $\mathrm{Ass}(K/R)$ if and only if $\mathfrak{p}$ is a minimal prime ideal of $(a) : (b)$, for some $a, b \in R$ such that $a \neq 0$; so that

$\mathrm{Ass}(K/R) = \cup[\mathrm{Ass}(R/(a)) \mid 0 \neq a \in R]$. Also, $\mathfrak{p}$ belongs to $\mathrm{Att}(K/R)$ if and only if for each $I \in \mathcal{I}_f(R)$ with $I \subseteq \mathfrak{p}$, there are some $a, b \in R$ with $a \neq 0$, such that $I \subseteq (a) : (b) \subseteq \mathfrak{p}$. We set $\mathrm{Att}^*(K/R) := \cup[\mathrm{Att}(R/(a)) \mid 0 \neq a \in R]$. Hence, $\mathfrak{p} \in \mathrm{Att}^*(K/R)$ if and only if there is some $0 \neq a \in R$, such that for each $I \in \mathcal{I}_f(R)$ with $I \subseteq \mathfrak{p}$, there is some $b \in R$, such that $I \subseteq (a) : (b) \subseteq \mathfrak{p}$. Then we have $\mathrm{Ass}(K/R) \subseteq \mathrm{Att}^*(K/R) \subseteq \mathrm{Att}(K/R)$ (see [29, Section 5] and [30, Definition 1.4], where $\mathrm{Ass}(K/R)$ and $\mathrm{Att}^*(K/R)$ appear under a different name). Moreover, each nonzero prime ideal of R is a set union of some elements of $\mathrm{Ass}(K/R)$, because a minimal prime ideal of a nonzero principal ideal of R belongs to $\mathrm{Ass}(K/R)$.

We gather below some results stated in our paper [30]. Many authors have used t-ideals for characterizing some domains properties. The t-operation associated to an integral domain R is as follows. For a nonzero ideal I of R, set $I_t := \cup[(J^{-1})^{-1} \mid J \in \mathcal{I}_f(R), J \subseteq I]$. Then I is dubbed a *t-ideal* if $I = I_t$. A prime ideal $\mathfrak{p} \neq 0$ of a domain R is called *well behaved* by Zafrullah if $\mathfrak{p}$ and $\mathfrak{p}R_{\mathfrak{p}}$ are t-ideals [35] and R is well behaved if each nonzero prime ideal is well behaved. As usual $\mathrm{D}(I)$ is the set of all prime ideals $\mathfrak{p}$ of a ring R, such that $I \not\subset \mathfrak{p}$ and $\mathrm{Spec}(R)$ is a topological space, whose open subsets are the $\mathrm{D}(I)$'s, where I is an ideal of R.

Proposition 2.1. *Let R be an integral domain and $0 \neq \mathfrak{p} \in \mathrm{Spec}(R)$. Then* p-depth$(R_{\mathfrak{p}}) = 1$ *is equivalent to $\mathfrak{p} \in \mathrm{Att}^*(K/R)$. The elements of $\mathrm{Att}^*(K/R)$ are prime t-ideals. If I is an ideal of R, then $\mathrm{Ass}(K/R) \subseteq \mathrm{D}(I) \Leftrightarrow \mathrm{Ass}(K/R) \subseteq \mathrm{D}(I_t)$, and* Ass *can be replaced with* Att^*.

Proof. Use [30, Proposition 1.23 and Theorem 1.5] for the first part. The second part is an easy consequence of the first. □

Proposition 2.2. *The set of all well behaved prime ideals of an integral domain R is* $\mathrm{Att}(K/R)$.

Proof. It is enough to translate [35, Proposition 1.1]. □

Kang stated the next result for a local treed domain [20, Theorem 3.19].

Theorem 2.3. *A treed domain R is well behaved.*

Proof. We first observe that an ideal, which is a directed set union of t-ideals, is itself a t-ideal. To complete the proof, it is enough to combine the following facts. A nonzero prime ideal of a treed domain is a directed set union of minimal prime ideals of principal ideals. Moreover, these minimal prime ideals are t-ideals, because the elements of $\mathrm{Att}^*(K/R)$ are t-ideals by Proposition 2.1. □

We now generalize the above mentioned result of Dobbs [4].

Theorem 2.4. *Each prime ideal* $\mathfrak{p} \neq 0$ *of a treed domain* R *verifies* $\text{p-depth}(R_{\mathfrak{p}}) = 1$*; so that* $\text{p-depth}(R) = 1$ *and* $\text{depth}(R) = 1$*. Moreover,* $\text{Att}^*(K/R) = \text{Att}(K/R) = \text{Spec}(R) \setminus \{0\}$ *is compact.*

Proof. Let a be a nonzero nonunit element of R, then $\text{Gr}(Ra) = 1$, because the polynomial grade of an ideal is less than the cardinal of a system of generators of the ideal [25, 5.5, Theorem 13]. Let $\mathfrak{p} \neq 0$ be a prime ideal of R, then $\mathfrak{p}R_{\mathfrak{p}}$ is the set union of linearly ordered prime ideals $\mathfrak{p}_i$ such that $\mathfrak{p}_i = \sqrt{(x_i)}$ for some $x_i \in R_{\mathfrak{p}}$, because $\mathfrak{p}$ is the set union of some minimal prime ideals of principal ideals and $R_{\mathfrak{p}}$ is treed. Now $\text{Gr}(I) = \text{Gr}(\sqrt{I})$ for any ideal I of a ring R by [25, 5.5, Theorem 12]. Arguing as in the proof of [25, 5.5, Theorem 11], we get that $\text{Gr}(I) = \sup \text{Gr}(J_i)$, if I is the set union of a directed set of ideals $\{J_i\}$. These last facts combine to yield that $\text{Gr}(\mathfrak{p}R_{\mathfrak{p}}) = 1$. Then observing that (regular) R-sequences are preserved by flat extensions, we get that $\text{Gr}(I) \leq \text{Gr}(IS)$ if $R \to S$ is a flat ring morphism and I an ideal of R. It follows that $\text{p-depth}(R) = 1$, because $R \to R_{\mathfrak{p}}$ is flat. To complete the proof, use Proposition 2.1. Therefore, if R is a treed domain, $\text{Att}^*(K/R) = \text{Spec}(R) \setminus \{0\}$ holds. □

The preceding theorems are recovered in the more general setting of Section 3.

Remark 2.5. By the above proof, $\text{Gr}(I) \leq \text{Gr}(IS)$ if $R \to S$ is a flat ring morphism and I an ideal of R. Moreover, equality holds if $R \to S$ is faithfully flat by [25, Remark, p.63]. It follows easily that the polynomial grade is preserved by a faithfully flat morphism $R \to S$, such that $\text{Max}(S) = \{\mathfrak{m}S \mid \mathfrak{m} \in \text{Max}(R)\}$. But this is false for arbitrary faithfully flat morphisms like $R \to R[X]$ (see Remark 5.5).

The following ideal theoretic characterization of treed rings is reminiscent of the theory of Prüfer domains.

Proposition 2.6. *Let* R *be a ring. The following statements are equivalent:*

(1) R *is treed;*
(2) $I \cap (J + K) = (I \cap J) + (I \cap K)$ *for each ideal* I *and each pair of radical ideals* J, K *of* R*;*
(3) $I \cap (P + Q) = (I \cap P) + (I \cap Q)$ *for each ideal* I *and each pair of prime ideals* P, Q *of* R.

Proof. Assume that R is treed. In order to show that (2) holds, we can assume that R is local. In that case, J and K are comparable prime ideals. Then the verification of the equation is routine. Conversely, suppose that (3) holds. Let P and Q be non comparable prime ideals of R and pick some $a \in P \setminus Q$ and some $b \in Q \setminus P$. Set $I = (a+b)$, we have $I \cap (P+Q) = (I \cap P) + (I \cap Q)$ and $a+b$ belongs to $(I \cap P) + (I \cap Q)$; so that $a+b = u(a+b) + v(a+b)$, where $u \in P$ and $v \in Q$, because $a+b \notin P$ and $a+b \notin Q$. Hence $1-u \in Q$ implies that $1 \in P+Q$ and (1) is proved. □

A ring is called a *Baer ring* (respectively, a *weak Baer ring*) if the annihilator of each of its ideals (respectively, principal ideals) is generated by an idempotent. We say that a ring is a *strong (weak) Baer ring* if each of its reduced factor rings is a (weak) Baer ring. We proved that a ring R is strong Baer if and only if $(I : J) + (J : I) = R$ for all pairs of radical ideals I, J of R [29, Proposition 4.25]. Such a ring is treed. Moreover, when R is quasi-Noetherian (each ideal of R has finitely many minimal prime ideals), then R is treed if and only if R is strong (weak) Baer [29, Théorème 4.28].

3. Integral domains with polynomial depth 1

We intend to give criteria for an integral domain to have polynomial depth 1. We need some notation. Let R be a ring and $Z \subseteq \mathrm{Spec}(R)$. The *generalization* $Z^{\downarrow}$ of Z is $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{q} \text{ for some } \mathfrak{q} \in Z\}$. We set $\mathcal{U}(Z) = \cup[\mathfrak{p} \mid \mathfrak{p} \in Z]$ and $Z[X] = \{\mathfrak{p}[X] \mid \mathfrak{p} \in Z\} \subseteq \mathrm{Spec}(R[X])$, where X is an indeterminate over R.

We will consider the *Nagata ring* $R(X)$ of an integral domain R; that is the localization of $R[X]$, with respect to the set complement N of $\mathcal{U}(\mathrm{Max}(R)[X])$. It is well known that $f(X) \in N \Leftrightarrow \mathrm{c}(f(X)) = R$, where $\mathrm{c}(f(X))$ is the content ideal of $f(X)$. We also consider the localization $R\{X\}$ of $R[X]$, with respect to the set complement U of $\mathcal{U}(\mathrm{Ass}(K/R)[X])$. It was introduced by Kang under an equivalent definition. The next result may be found in [30, Definition 2.1, Theorem 2.2].

Proposition 3.1. *Let R be an integral domain, then $f(X) \in U \Leftrightarrow \mathrm{c}(f(X))^{-1} = R$ and $\mathcal{U}(\mathrm{Ass}(K/R)[X]) = \mathcal{U}(\mathrm{Att}^*(K/R)[X])$. Then, $R \to R(X) \to R\{X\}$ is factored by injective flat morphisms, the first one being faithfully flat. Moreover, $R(X) = R\{X\}$ if and only if $N = U$.*

The technical condition $N = U$ is the key of our next results.

Definition 3.2. We say that an integral domain R has condition (†) if $N = U$; or equivalently,

$$(\dagger) : I^{-1} = R \Leftrightarrow I = R \text{ for each } I \in \mathcal{I}_f(R).$$

Mimouni has recently introduced DW-domains R, in which the w-operation is trivial [23]. Then [23, Proposition 2.2] shows that the condition (†) holds on R if and only if R is a DW-domain. We refer the reader to Mimouni's paper for more information on the w-operation and DW-domains.

We need the following lemma, where $\mathrm{T}(I) := \cup [R :_K I^n \mid n \in \mathbb{N}]$ is the *(Nagata) ideal transform* of an ideal I of an integral domain R. We recall that an ideal I of an integral domain R is called t-finite if $I_t = J_t$ for some $J \in \mathcal{I}_f(R)$.

Lemma 3.3. *Let R be an integral domain and I an ideal of R.*

(1) $\mathrm{Ass}(K/R) = \mathrm{Att}^*(K/R) = \mathrm{Att}(K/R)$ *if* $\mathrm{Ass}(K/R)$ *is compact.*
(2) $\mathrm{Ass}(K/R) \subseteq \mathrm{D}(I)$ *(respectively;* $\mathrm{Att}^*(K/R) \subseteq \mathrm{D}(I)) \Rightarrow I^{-1} = R$.
(3) $I^{-1} = R \Rightarrow \mathrm{Att}(K/R) \subseteq \mathrm{D}(I)$ *if I is t-finite.*
(4) *If* $\mathrm{Att}^*(K/R)$ *is compact (e.g. either R is treed or* $\mathrm{Ass}(K/R)$ *is compact), then* $\mathrm{Att}^*(K/R) \subseteq \mathrm{D}(I)$ *implies* $I_t = R$.
(5) *If* $\mathrm{Att}^*(K/R)$ *is compact, the prime t-ideals of R are in* $\mathrm{Att}^*(K/R)^{\downarrow}$.
(6) $\mathrm{T}(I) = R \Leftrightarrow \mathrm{Ass}(K/R) \subseteq \mathrm{D}(I) \Leftrightarrow \mathrm{Att}^*(K/R) \subseteq \mathrm{D}(I)$, *when I is t-finite.*

Proof. (1) is a consequence of [29, Proposition 5.5]. We show (2). It is enough to suppose that $\mathrm{Ass}(K/R) \subseteq \mathrm{D}(I)$. In that case $a/b \in I^{-1} \Leftrightarrow I \subseteq (b) : (a)$ implies $(b) : (a) = R$; so that $I^{-1} = R$. Assume that the hypothesis of (3) holds. If I is t-finite, there is some $J \subseteq I$ in $\mathcal{I}_f(R)$ such that $I^{-1} = J^{-1}$ by [8, p.324]. Suppose that $J \subseteq P$ for some $P \in \mathrm{Att}(K/R)$. Then by [29, p.712], there is some $k \in K$, whose class in K/R is nonzero, and such that $Jk \subseteq R$; whence $k \in J^{-1} = R$, a contradiction. It follows that $\mathrm{Att}(K/R) \subseteq \mathrm{D}(J) \subseteq \mathrm{D}(I)$. Under the hypotheses of (4), there is some finitely generated ideal $J \subseteq I$, such that $\mathrm{Att}^*(K/R) \subseteq \mathrm{D}(J)$; so that $J^{-1} = R$ by (2). Then $I_t = R$, because $(J^{-1})^{-1} \subseteq I_t$. Then (5) is a consequence of (4). We show (6). To begin with, assume that $I \in \mathcal{I}_f(R)$ and denote by $\bar{k}$ an arbitrary element of K/R. Then $\mathrm{T}(I) \neq R$ is equivalent to the existence of some $\bar{k} \neq 0$, such that $I^n\bar{k} = 0$, which in turn is equivalent to $\mathrm{Ass}(K/R) \cap \mathrm{V}(I) \neq \emptyset$ by [29, Définition 5.1(3)]. We can replace Ass with Att^*, because $\mathrm{Ass}(K/R) \subseteq \mathrm{Att}^*(K/R) \subseteq \mathrm{Att}(K/R)$ and $\mathrm{Ass}(K/R) \cap \mathrm{V}(I) \neq \emptyset \Leftrightarrow \mathrm{Att}(K/R) \cap \mathrm{V}(I) \neq \emptyset$ by [29, Proposition 5.2]. Now suppose that I is t-finite; so that $I_t = J_t$ for some $J \in \mathcal{I}_f(R)$. Then we have $\mathrm{T}(I) = \mathrm{T}(I_t)$ for each ideal I by [5, Proposition 3.4] and [5, Proof of Corollary 3.5]. To conclude, it is enough to use the end of Proposition 2.1. □

We generalize some results of Huckaba and Papick on treed domains [28, Proposition 3.3] and [17, Proposition 2.2]. Note that not all treed domains R have $\mathrm{Ass}(K/R)$ compact. It is enough to consider a valuation ring $(V, \mathfrak{m})$, such that

$\mathfrak{m} = \cup[\mathfrak{p} \mid \mathfrak{p} \subset \mathfrak{m}]$ (see [28, p.223]). But $\mathrm{Att}^*(K/R)$ is compact if R is a treed domain. The above mentioned authors introduced for an integral domain R the condition (†) of Definition 3.2, which involves that $\mathrm{depth}(R) = 1$ [17, Lemma 2.1]. We show that a germane result is true when depth is replaced with p-depth. A subset Z of $\mathrm{Spec}(R)$ is called *dilated* (respectively, *fathomable*) if for any ideal (respectively, finitely generated ideal) I of R, the following condition holds: $I \subseteq \mathcal{U}(Z) \Rightarrow I \subseteq P$ for some $P \in Z$ [29]. To verify that Z is dilated, it is enough to check that the defining condition holds for all prime ideals I.

Theorem 3.4. *The following statements are equivalent for an integral domain R:*

1. *R verifies condition* (†) *(i.e. R is a DW-domain);*
2. $\mathrm{Ass}(K/R)$ *(respectively,* $\mathrm{Att}^*(K/R)$*) is fathomable;*
3. $\mathrm{T}(I) \neq R$ *for each* $I \in \mathcal{I}_f(R) \setminus \{R\}$*;*
4. $I^{-1} \neq R$ *for each* $I \in \mathcal{I}_f(R) \setminus \{R\}$*.*

If one of the above statements holds, R has polynomial depth 1.

Proof. In view of Proposition 3.1 and Lemma 3.3, it is enough to observe that each statement amounts to saying that $\mathrm{Att}^*(K/R) \cap \mathrm{V}(I) \neq \emptyset$ for each $I \in \mathcal{I}_f(R) \setminus \{R\}$ and that each maximal ideal is the set union of some elements of $\mathrm{Ass}(K/R)$ (respectively, $\mathrm{Att}^*(K/R)$) (see the end of the first paragraph of Section 2). Assume that the condition (†) holds and suppose that $\text{p-depth}(R) \geq 2$. There is some maximal ideal $\mathfrak{m}$ of R such that $\mathrm{Gr}(\mathfrak{m}) = \mathrm{Gr}(\mathfrak{m}[X]) \geq 2$. We deduce from [34, Proposition 3.4] that $\mathrm{gr}(\mathfrak{m}[X]) \geq 2$. Therefore, there is an $R[X]$-sequence $a, b \in \mathfrak{m}[X]$. Then $a + bX^n$ is contained in $\mathfrak{m}[X]$, for each integer $n > 0$; so that $a + bX^n \in \mathfrak{p}[X]$ for some $\mathfrak{p} \in \mathrm{Att}^*(K/R)$. Pick an integer n such that the coefficients of a and b are the coefficients of $a + bX^n$. In that case $J := (a, b) \subseteq \mathfrak{p}[X]$. We claim that $\mathfrak{p}[X] \in \mathrm{Att}^*(K(X)/R[X])$. Indeed, there is some nonzero $r \in R$ such that $\mathfrak{p} \in \mathrm{Att}(R/Rr)$. The claim follows by [19, Theorem 2.5], because $\{\mathfrak{p}[X] \mid \mathfrak{p} \in \mathrm{Att}(R/Rr)\} = \mathrm{Att}(R[X]/rR[X])$. We are lead to a contradiction by Lemma 3.3(3), because $J^{-1} = R[X]$ by [21, Exercise 1,p.102]. Therefore, R has p-depth 1. □

Remark 3.5. We give some examples of integral domain R, with $\text{p-depth}(R) = 1$. Clearly, a treed domain verifies (†) and is therefore a DW-domain. But a DW-domains is not necessarily treed. Consider the DW-domain R exhibited in [23, Example 2.10]. Then R has a prime ideal P, whose height is 2, and R_P is a Krull domain. Since a treed Krull domain must have dimension one (see for instance the study of treed PIT domains at the beginning of Section 4), R cannot be treed.

(1) The condition (3) of Theorem 3.4 holds if $\mathrm{T}(\mathfrak{m}) \neq R$ for each $\mathfrak{m} \in \mathrm{Max}(R)$, and in particular, if $\mathfrak{m} : \mathfrak{m} \neq R$ (respectively, $\mathfrak{m}^{-1} \neq R$) for each $\mathfrak{m} \in \mathrm{Max}(R)$. Note also that the above conditions are verified if they are verified locally on $\mathrm{Max}(R)$. In that case, p-depth$(R_{\mathfrak{m}}) = 1$ for each $\mathfrak{m} \in \mathrm{Max}(R)$.

(2) Assume that R is absolutely fathomable; that is, each subset of $\mathrm{Spec}(R)$ is fathomable. Then the statements of Theorem 3.4 are locally verified. Therefore, p-depth$(R_P) = 1$ for each nonzero $P \in \mathrm{Spec}(R)$. For instance, let C be a ring of real continuous functions on a topological space E. Then each integral factor ring of C has p-depth 1. This follows from [29, Exemple 3.20].

(3) Assume that R is an integral domain such that each compact open subset of $\mathrm{Spec}(R)$ is affine. Then $I\mathrm{T}(I) = \mathrm{T}(I)$ holds for each $I \in \mathcal{I}_f(R)$ [5, Theorem 4.4]; so that $\mathrm{T}(I) \neq R$ for $I \neq R$. This condition is verified if $\sqrt{I}$ is locally the radical of a nonzero principal ideal for each $I \in \mathcal{I}_f(R)$ [5, Corollary 4.6]. In particular, the compact open subsets of the spectrum of a Prüfer domain are affine.

(4) Consider an integral domain R such that $\mathrm{T}(bc) = \mathrm{T}(b) + \mathrm{T}(c)$ for each $b, c \in R$. In other words R is a *T_3-domain*. Then a T_3-domain has depth 1 [7, Corollary 4.5.19]. We do not know whether any T_3-domain has polynomial depth 1, but there is an answer for classes of integral domains described in (5) and (6).

(5) It is known that $R = \cap[R_P \mid P \in \mathrm{Ass}(K/R)]$ [17, Theorem 2.0]. If this representation is of finite character, then for each $I \in \mathcal{I}_f(R)$, there are some $b, c \in I$ such that $\mathrm{T}(I) = \mathrm{T}(b, c) = R_b \cap R_c$ [18, Theorem 2.6]. This holds for a Noetherian domain or a Krull domain. Consider a T_3-domain, whose above representation is of finite character. Then $I\mathrm{T}(I) = \mathrm{T}(I)$ for each $I \in \mathcal{I}_f(R)$ by [7, Theorem 4.5.14]. Therefore, (3) can be applied and p-depth$(R) = 1$.

(6) Let $(R, \mathfrak{m})$ be a local domain, such that $\mathrm{T}(\mathfrak{m}) = K$. Then R is a T_1-domain and p-depth$(R) = 1$. An example is given by [7, Example 4.5.10].

(7) Sakaguchi proved that p-depth$(R) \leq \mathrm{Dim}_v(R)$ for a local ring R, where Dim_v denotes the valuative dimension [32]. Therefore, a local integral domain whose valuative dimension is 1 has p-depth 1. We can add that a local domain R with valuative dimension 2, which is not Cohen-Macaulay in Sakaguchi's sense (see [33]), has p-depth 1.

Theorem 3.6. *Let R be an integral domain. The following statements are equivalent:*

(1) $\mathrm{Att}^*(K/R)$ *is compact and each element of* $\mathrm{Max}(R)$ *is a t-ideal;*
(2) $\mathrm{Max}(R) \subseteq \mathrm{Att}^*(K/R)$*;*

(3) $\mathrm{Att}^*(K/R)[X]$ *is fathomable (dilated) and each element of* $\mathrm{Max}(R)$ *is a* t*-ideal;*

(4) $R_{\mathfrak{m}}$ *has p-depth* 1 *for each* $\mathfrak{m} \in \mathrm{Max}(R)$*;*

(5) R *has condition (†) (i.e. R is a DW-domain) and* $\mathrm{Att}^*(K/R)$ *is compact;*

(6) $\mathrm{T}(I) \neq R$ *for each* $I \in \mathcal{I}_f(R) \setminus \{R\}$ *and* $\mathrm{Att}^*(K/R)$ *is compact;*

(7) $R_{\mathfrak{m}}$ *verifies any of the above statements for each* $\mathfrak{m} \in \mathrm{Max}(R)$.

If any of the above statements holds, the elements of $\mathrm{Max}(R)$ *are well behaved.*

Proof. (1) ⇒ (2) by Lemma 3.3(5). Now (2) ⇒ (1) by Proposition 2.1 and because for $S \subseteq \mathrm{Spec}(R)$, the condition $\mathrm{Max}(R) \subseteq S$ implies that S is compact. Indeed, an open subset of $\mathrm{Spec}(R)$, containing $\mathrm{Max}(R)$, is $\mathrm{Spec}(R)$. Then (1) ⇔ (3) by [30, Lemma 2.18], in case we consider the dilated case. The fathomable case is a consequence of the continuity of the map $P \mapsto P[X]$, because [29, Proposition 2.2] asserts that a subset Z is dilated if and only Z is fathomable and compact. Then (2) ⇔ (4) by Proposition 2.1. Now (3) ⇔ (5) ⇔ (6) by Theorem 3.4. □

Remark 3.7. A treed domain verifies the statement (4) of Theorem 3.6 by Theorem 2.4. Actually, $\text{p-depth}(R_{\mathfrak{p}}) = 1$ holds for each prime ideal $\mathfrak{p} \neq 0$ of an integral domain R if and only if $\mathrm{Spec}(R) \setminus \{0\} = \mathrm{Att}^*(K/R)$. Note that Papick proved for a treed domain R that $\mathrm{Ass}(K/R)$ is compact if and only if $\mathrm{Max}(R) \subseteq \mathrm{Ass}(K/R)$. In that case we know that $\mathrm{Ass}(K/R) = \mathrm{Att}^*(K/R)$. Note also that an integral domain with Noetherian spectral space verifies condition (1) if and only if each maximal ideal is a t-ideal. We can also consider Glaz-Vasconcelos's H-domains. An integral domain R is called an *H-domain* if each ideal J of R such that $J^{-1} = R$ contains an ideal $I \in \mathcal{I}_f(R)$ such that $I^{-1} = R$ [11, Section 3]. Then an integral domain R is an H-domain if and only if $\mathrm{Ass}(K/R)$ is compact and $P^{-1} \neq R$ for each $P \in \mathrm{Ass}(K/R)$ [11, 3.2b]. It follows from Lemma 3.3(1) that condition (1) holds for an H-domain if and only if each maximal ideal of R is a t-ideal, or equivalently, is a divisorial ideal. This last point is a consequence of [15, Proposition 2.4], because a divisorial ideal is a t-ideal.

4. Special treed domains

The treed condition, combined with another property of integral domains, is very strong. Quasi-Prüfer domains are involved in the following. An integral domain R is called *quasi-Prüfer* if, among many equivalent conditions, its integral closure R' is a Prüfer domain [7, Section 6.5]. Going-down domains, locally divided domains, and i-domains are treed domains, mainly studied by Dobbs and Papick (see [3] and

[27]). We give only here some definitions. An integral domain R is called *going-down* if each of its overrings S defines a going-down extension $R \subseteq S$, whereas R is called an *i-domain* if each of its overrings S defines an i-extension $R \subseteq S$ (a ring morphism is called an i-morphism if its spectral map is injective). Equivalently, R is an i-domain if and only if R is quasi-Prüfer and $R \subseteq R'$ is an i-extension. An integral domain R is *divided* if each prime ideal is comparable to each ideal, or equivalently, $\mathfrak{p}R_\mathfrak{p} = \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$. Incidentally, we observe that the going-down property of a domain can be checked by considering some special prime ideals. Kaplansky defines the going-down property of a ring morphism at a prime ideal [21, Exercise 36, p.44]. It is easy to show that an integral domain R is going-down if and only if R is treed and for each overring S, the extension $R \subseteq S$ has the going-down property at each minimal prime ideal P of a principal ideal (respectively, at each $P \in \mathrm{Ass}(K/R)$, $P \in \mathrm{Att}^*(K/R)$). In the same way, an integral domain R is divided if and only if each minimal prime ideal $\mathfrak{p}$ of a nonzero principal ideal (respectively, each $\mathfrak{p} \in \mathrm{Ass}(K/R)$ or $\mathrm{Att}^*(K/R)$) is divided. We first give some examples showing that the treed condition is drastic.

An integral domain is called a *PIT domain* if it satisfies the conclusion of the Principal Ideal Theorem, namely each minimal prime ideal of a nonzero principal ideal has height 1. For instance, Noetherian domains and weakly Krull domains R (such that $R = \cap\,[R_\mathfrak{p} \mid \mathrm{ht}(\mathfrak{p}) = 1]$ and the intersection has finite character) are PIT domains [31, Proposition 3.1]. If R is a treed PIT domain, each prime ideal is a set union of linearly ordered minimal prime ideals of principal ideals. It follows that R is a one-dimensional domain.

An integral domain R is a *Prüfer v-multiplication domain* (PVMD) if and only if $R_\mathfrak{p}$ is a valuation domain at each maximal t-ideal $\mathfrak{p}$ [24, Corollary 4.3]. Therefore, a treed domain R is a PVMD if and only if R is a Prüfer domain. Examples of PVMD's are Krull domains, GCD domains, integrally closed coherent domains, etc. We generalize below this result.

The domain R, with quotient field K, is called a *UMT-domain* if every *upper to* 0 (a nonzero prime ideal of $R[X]$ which is contracted from K[X]) is a maximal t-ideal. The class of UMT-domains is closely linked to the class of PVMDs, since a domain R is a PVMD if and only if R is an integrally closed UMT-domain [16, Proposition 3.2]. Moreover, an integral domain R is quasi-Prüfer if and only if each of its overrings is a UMT-domain [6, Corollary 3.11]. A t-linked overring of a UMT-domain is again a UMT domain [16, p.1962]. Now in view of [6, p.1022], the overrings of a treed domain R are t-linked, because each nonzero maximal ideal of

R is a t-ideal. It follows that a treed domain R is a UMT-domain if and only if R is quasi-Prüfer. In that case, each overring of R is a UMT-domain. Hence, an i-domain is a UMT-domain.

Now let X be an indeterminate over a treed domain R, with quotient field K. Since R verifies (†) by Remark 3.5, $R(X) = R[X]_U$, where $R[X] \setminus U = \cup\,[\mathfrak{p}[X] \mid \mathfrak{p} \in \mathrm{Spec}(R)]$ by Proposition 3.1 and Theorem 2.4. Hence the prime ideals of $R(X)$ are of the form $\mathfrak{q}R(X)$ where $\mathfrak{q} \subseteq \mathfrak{p}[X]$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$ by [30, Lemma 2.8]. Assume in addition that R is quasi-Prüfer. Then a prime ideal $\mathfrak{q}$ of $R[X]$ such that $\mathfrak{q} \subseteq \mathfrak{p}[X]$ is of the form $\mathfrak{q} = (\mathfrak{q} \cap R)[X]$ [7, p. 212]. In that case, the spectral map of $R \to R(X)$ is a homeomorphism, with inverse map $\mathfrak{p} \mapsto \mathfrak{p}R(X)$. Moreover, each local ring of $R(X)$ is of the form $R_\mathfrak{p}(X) = R[X]_{\mathfrak{p}[X]} = R(X)_{\mathfrak{p}R(X)}$ for some prime ideal $\mathfrak{p}$ of R. Working a little bit, we could get the following result. If R is an integral domain, then $R(X)$ is treed if and only if R is treed and quasi-Prüfer [2, Theorem 2.10]. In that case $R(X)$ is quasi-Prüfer. In [2], we may see that $R(X)$ is going-down if and only if R is going-down and quasi-Prüfer. Moreover, R is an i-domain if and only if $R(X)$ is an i-domain. We now intend to characterize integral domains R, such that $R(X)$ is (locally) divided or such that $R(X)$ is a PVD, since the study of these classes is missing in [2]. Before that, we need a result.

Proposition 4.1. *Let $f : R \to S$ be an injective ring morphism between integral domains, whose spectral map is a homeomorphism.*

(1) $S_\mathfrak{q} = S_\mathfrak{p}$ *for each* $\mathfrak{q} \in \mathrm{Spec}(S)$ *and* $\mathfrak{p} := f^{-1}(\mathfrak{q})$.
(2) *If R is (locally) divided, then S is (locally) divided.*

Proof. As ${}^af$ is a homeomorphism, we get that $\mathfrak{q} = \sqrt{\mathfrak{p}S}$. Indeed, ${}^af(\mathrm{V}(\mathfrak{q})) = \mathrm{V}(\mathfrak{p})$, because ${}^af$ is closed, and ${}^af^{-1}(\mathrm{V}(\mathfrak{p})) = \mathrm{V}(\mathfrak{p}S)$ combine to yield the desired equation. Consider the multiplicative subset $f(R \setminus \mathfrak{p})$, with saturated associated multiplicative subset T, then $S \setminus T = \cup[Q \mid Q \in \mathrm{Spec}(S)\,\text{such that}\,f^{-1}(Q) \subseteq \mathfrak{p}]$. But $f^{-1}(Q) \subseteq \mathfrak{p}$ implies that $Q \subseteq \mathfrak{q}$, because $Q = \sqrt{f^{-1}(Q)S}$. From $\mathfrak{q} \subseteq S \setminus T$, we deduce $T = S \setminus \mathfrak{q}$ and $S_\mathfrak{q} = S_\mathfrak{p}$. Assume that R is divided. Let $\mathfrak{q} \in \mathrm{Spec}(S)$ be lying over $\mathfrak{p} \in \mathrm{Spec}(R)$. It follows that $\mathfrak{p}R_\mathfrak{p} = \mathfrak{p}$. Let $x \in \mathfrak{q}S_\mathfrak{q}$. From $S_\mathfrak{q} = S_\mathfrak{p}$ and $\mathfrak{q} = \sqrt{\mathfrak{p}S}$, we deduce that there is some integer n such that $x^n \in \mathfrak{p}R_\mathfrak{p}S = \mathfrak{p}S$; so that $x \in \mathfrak{q}$. Therefore, $\mathfrak{q}S_\mathfrak{q} = \mathfrak{q}$ for each $\mathfrak{q}$ implies that S is divided. The local statement follows easily, because $R_\mathfrak{p} \to S_\mathfrak{p} = S_\mathfrak{q}$ is a spectral homeomorphism. □

Theorem 4.2. *Let R be an integral domain. Then $R(X)$ is (locally) divided if and only if R is quasi-Prüfer and (locally) divided.*

Proof. Assume that R is quasi-Prüfer and (locally) divided. Then R is treed and $R(X)$ is (locally) divided by Proposition 4.1, because $R \to R(X)$ is a spectral homeomorphism. Conversely, assume that $R(X)$ is divided. Then $R(X)$ is treed and hence R is quasi-Prüfer [2, Theorem 2.10]. Moreover, R is divided because faithful flatness descends the divided property, since $IR(X) \cap R = I$ for each ideal I of R and $R \to R(X)$ is spectrally surjective. Now if $R(X)$ is locally divided, $R(X)$ is treed and then R is quasi-Prüfer and treed. Since $R_{\mathfrak{p}}(X) = R(X)_{\mathfrak{p}R(X)}$ is divided for each $\mathfrak{p} \in \mathrm{Spec}(R)$, so is $R_{\mathfrak{p}}$ by the above result. □

Consider now the PVD context. A local integral domain $(R, \mathfrak{m})$ is called a *pseudo-valuation domain* if there is a valuation overring V such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$, or equivalently, there is a valuation overring V, with maximal ideal $\mathfrak{m}$ [13, Theorem 2.7]. Assume that $(R, \mathfrak{m})$ is a PVD, associated to a valuation domain V. Then $V(X)$ is a valuation overring of $R(X)$ and $\mathfrak{m}(X) = \mathfrak{m}R(X)$ is a common ideal of these two domains. Thus $R(X)$ is a PVD. The converse is easily gotten by using the criterion of [13, Theorem 1.4] and the faithful flatness of $R \to R(X)$.

5. Homological properties

We next give some homological considerations. The following result is a direct application of [19, Theorem 5.4], because $\mathrm{Gr}(\mathfrak{p}) = 1$ for each $\mathfrak{p} \in \mathrm{Ass}(R/I)$ in an integral domain R with polynomial depth 1 and ideal I.

Theorem 5.1. *Let R be an integral domain with polynomial depth* 1 *and I an ideal. Then I is projective (equivalently, invertible) if and only if I has a resolution of finite length by finitely generated projective R-modules.*

Alfonsi defined the *small finitistic dimension* $\mathrm{fpd}(M)$ of an R-module M [1, Définition 2.1]. For $M = R$, we recover the usual definition $\sup(\mathrm{proj.dim} M) = \mathrm{fpd}(R)$, where the R-modules M are such that $\mathrm{proj.dim} M < \infty$ and M has a finite projective resolution. Then we have $\mathrm{Gr}(\mathfrak{m}, M) = \mathrm{fpd}(M)$ for a local ring $(R, \mathfrak{m})$ and an R-module M [1, Corollaire 2.7]. We get immediately the following result.

Theorem 5.2. *Let R be a local domain, with polynomial depth* 1*, then* $\mathrm{fpd}(R) = 1$.

Corollary 5.3. *Let R be an integral domain and $X_1, \ldots, X_n$ indeterminates over R. For $\mathfrak{q} \in \mathrm{Spec}(R[X_1, \ldots, X_n])$ and $\mathfrak{p} = \mathfrak{q} \cap R$, such that* $\text{p-depth}(R_{\mathfrak{p}}) = 1$*, then* $\mathrm{Gr}(R[X_1, \ldots, X_n]_{\mathfrak{q}}) = \mathrm{fpd}(R[X_1, \ldots, X_n]_{\mathfrak{q}}) = 1 + \dim(\mathrm{k}(\mathfrak{p})[X_1, \ldots, X_n]_{\mathfrak{q}})$.

Proof. It is enough to use [1, Corollaire 2.12]. □

Corollary 5.4. *Let* $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ *be a local faithfully flat morphism such that* $\mathfrak{m} = \mathfrak{n}S$ *(e.g.* $R \to R(X)$ *or a henselization morphism). If* $\mathrm{fpd}(R) = 1$*, so does* S*.*

Proof. Use Remark 2.5. □

Remark 5.5. Sakaguchi proved that $\mathrm{Gr}(M[X]_{\mathfrak{q}}) = \mathrm{Gr}(M_{\mathfrak{p}}) + 1$ for an R-module M, $\mathfrak{q} \in \mathrm{Spec}(R[X])$ and $\mathfrak{p} = \mathfrak{q} \cap R$ [33, Theorem 3.5]. Using either this result or the above corollary, we get that $\mathrm{Gr}(R[X]_{\mathfrak{q}}) = \mathrm{fpd}(R[X]_{\mathfrak{q}}) = 2$ if $\mathfrak{q} \in \mathrm{Max}(R[X])$ and p-depth$(R_{\mathfrak{q}\cap R}) = 1$. Thus Corollary 5.4 is not valid for arbitrary faithfully flat morphisms.

Remark 5.6. McDowell defined a *pseudo-Noetherian ring* R as a coherent ring, such that for each nonzero finitely presented R-module M and each $I \in \mathcal{I}_f(R)$, then $I \subseteq \mathrm{Z}(M)$ implies that there is some $m \neq 0$ in M such that $Im = 0$ [22]. Now a treed coherent domain R is locally pseudo-Noetherian. To see this, we can assume that $(R, \mathfrak{m})$ is local. In view of [29, p.712], $\mathrm{Z}(M) = \cup[\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Att}(M)]$ and the previous reference allows us to conclude, because $\mathrm{Att}(M)$ is linearly ordered. Note that the Ext-grade of a local pseudo-Noetherian ring is equal to its small finitistic dimension [22, Theorem 2.5]. If in addition R is treed, we recover Alphonsi's result by using [29, Proposition 6.32]. As a consequence of Theorem 5.2, $\mathrm{fpd}(R) = \mathrm{Fpd}(R)$, the finitistic dimension of R, does not hold when R is a treed coherent domain. Deny, then a valuation ring has finitistic dimension 1, and hence is Noetherian [9, Theorem 2.5.14]. Note also that if $(R, \mathfrak{m})$ is a treed regular local domain, then $1 = \mathrm{Gr}(\mathfrak{m})$ equals the weak global dimension of R [10, Lemma 3]. It follows that R is a valuation domain [9, Corollary 4.2.6].

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References

[1] B. Alfonsi, *Grade non-noetherien*, Comm. Algebra, 9 (1981), 811–840.

[2] D. F. Anderson, D. E. Dobbs and M. Fontana, *On treed Nagata rings*, J. Pure Appl. Algebra, 61 (1989), 107–122.

[3] D. E. Dobbs, *Divided rings and going-down*, Pacific J. Math. 67 (1976), 353–363.

[4] D. E. Dobbs, *Treed domains have grade 1*, Int. J. Commut. Rings, 2 (2003), 43–46.

[5] M. Fontana, *Kaplansky ideal transforms: A survey*, in: Advances in commutative ring theory (editors D. E. Dobbs, M. Fontana and S. Kabbaj), in: Lect. Notes Pure Appl. Math., vol. 205, *Marcel Dekker* , (New York-Basel, 1999), pp.271–306.

[6] M. Fontana, S. Gabelli, and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra, 26 (1998), 1017–1039.

[7] M. Fontana, J. A. Huckaba and I. J. Papick, Prüfer domains, *Marcel Dekker*, New York-Basel, 1997.

[8] S. Gabelli and M. Roitman, *Maximal divisorial ideals and t-maximal ideals*, JP. J. Algebra Number Theory Appl., 4 (2004), 323–336.

[9] S. Glaz, Commutative coherent rings, *Springer-Verlag*, New York-Heidelberg-Berlin, 1989.

[10] S. Glaz, *On the coherence and weak dimension of the rings $R\langle X\rangle$ and $R(X)$*, Proc. Amer. Math. Soc. 106 (1989), 579-587.

[11] S. Glaz and W. V. Vasconcelos, *Flat ideals II*, Manuscripta Math. 22 (1977), 325–341.

[12] S. Glaz, *The weak dimension of Gaussian rings*, Proc. Amer. Math. Soc. 133 (2005), 2507–2513.

[13] J. R. Hedstrom and E. Houston, *Pseudo-valuation domains*, Pacific J. Math. 75 (1978), 137–147.

[14] M. Hochster, *Grade-sensitive modules and perfect modules*, Proc. London Math. Soc. 29 (1974), 55–76.

[15] E. Houston and M. Zafrullah, *Integral domains in which each t-ideal is divisorial*, Michigan Math. J. 35 (1988), 291–300.

[16] E. Houston and M. Zafrullah, *On t-invertibility II*, Comm. Algebra, 17 (1989), 1955–1969.

[17] J. A. Huckaba and I. J. Papick, *A localization of $R[x]$*, Canad. J. Math. 33 (1981), 103–115.

[18] D. Katz and L. J. Ratliff, Jr, *Two notes on ideal transforms*, Math. Proc. Camb. Phil. Soc. 102 (1987), 389–397.

[19] J. Iroz and D. E. Rush, *Associated prime ideals in non-noetherian rings*, Canad. J. Math. 36 (1984), 344–360.

[20] B. G. Kang, *Prüfer v-multiplication domains and the ring $R[X]_{N_v}$*, J. Algebra, 123 (1989), 151–170.

[21] I. Kaplansky, *Commutative rings*, Allyn and Bacon, Boston, 1970.

[22] K. P. McDowell, *Pseudo-noetherian rings*, Canad. Math. Bull., 19 (1976), 77–84.

[23] A. Mimouni, *Integral domains in which each ideal is a w-ideal*, Comm. Algebra, 33 (2005), 1345–1355.

[24] J. Mott and M. Zafrullah, *On Prüfer v-multiplication domains*, Manuscripta Math., 35 (1981), 1–26.

[25] D. G. Northcott, Finite free resolutions, *Cambridge University Press*, vol 71, Cambridge-London-New York-Melbourne, 1976.

[26] B. Olberding, *On the structure of stable domains*, Comm. Algebra, 30 (2002), 877–895.

[27] I. J. Papick, *Topologically defined classes of going-down domains*, Bull. Amer. Math. Soc, 81 (1975), 718–721.

[28] I. J. Papick, *Super-primitive elements*, Pacific J. Math, 105 (1983), 217–226.

[29] G. Picavet, *Parties sondables d'un spectre et profondeur*, Boll. Unione Mat. Ital. Sez B Artic. Ric. Mat., 7(8) (1994), 677–730.

[30] G. Picavet, *About GCD domains*, in: Advances in commutative ring theory, in: Lect. Notes Pure Appl. Math., vol. 205 (editors D. E. Dobbs, M. Fontana and S. Kabbaj) (Marcel Dekker, New York-Basel, 1999), pp.501–519.

[31] G. Picavet and M. Picavet-L'Hermitte, *When is length a length function?*, J. Algebra , 293 (2005), 561–594.

[32] M. Sakaguchi, *A note on the polynomial grade and the valuative dimension*, Hiroshima Math. J. 8 (1978), 327–333.

[33] M. Sakaguchi, *Generalized Cohen-Macaulay rings*, Hiroshima Math. J. 10 (1980), 615–634.

[34] H. Uda, *LCM-stableness in ring extensions*, Hiroshima Math. J. 13 (1983), 357–377.

[35] M. Zafrullah, *Well behaved prime ideals*, J. Pure Appl. Algebra, 65 (1990), 199–207.

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