EXTENSIONS OF GENERALIZED $\alpha$-RIGID RINGS

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Abstract. For a ring endomorphism $\alpha$, we introduce weak $\alpha$-rigid and weak $\alpha$-skew Armendariz rings which are a generalization of $\alpha$-rigid rings, and investigated their properties. Moreover, we prove that a ring $R$ is weak $\alpha$-rigid if and only if for any $n$, the $n$ by $n$ upper triangular matrix ring $T_n(R)$ is weak $\alpha$-rigid. If $R$ is semicommutative and weak $\alpha$-rigid, it is proven that the ring $R[x]$ and the ring $(R[x]/(x^n))$ where $(x^n)$ is the ideal generated by $x^n$ and $n$ is a positive integer, are weak $\alpha$-skew Armendariz.

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1. Introduction

Throughout this paper $R$ denotes an associative ring with identity. $\alpha : R \rightarrow R$ is an endomorphism of a ring $R$, we denote $R[x; \alpha]$ the Ore extension whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. A ring $R$ is called weak Armendariz if whenever polynomials $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $pq = 0$, then $a_i b_j$ is a nilpotent element of $R$ for each $i, j$. Recall that a ring $R$ is reduced if $R$ has no nonzero nilpotent elements. Observe that reduced rings are abelian (i.e., all idempotents are central).

Let $\alpha$ be an endomorphism of a ring $R$. According to Hong et al. [5], $R$ is called a $\alpha$-skew Armendariz ring if whenever $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{m} b_j x^j$ in $R[x; \alpha]$, $pq = 0$ implies $a_i \alpha^i(b_j) = 0$. As a generalization of the $\alpha$-skew Armendariz rings, in this paper, we introduce the notion of weak $\alpha$-skew Armendariz rings. We call a ring $R$ a weak $\alpha$-skew Armendariz ring if whenever polynomials $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x; \alpha]$ satisfy $pq = 0$, then $a_i \alpha^i(b_j)$ is a nilpotent element.

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of \( R \) for each \( i, j \). It can be easily checked that if \( R \) is a weak Armendariz ring, then it is a weak \( I_R \)-skew Armendariz ring, where \( I_R \) is an identity endomorphism of \( R \), and all \( \alpha \)-skew Armendariz rings are weak \( \alpha \)-skew Armendariz. So the weak \( \alpha \)-skew Armendariz rings are a generalization of weak Armendariz rings and \( \alpha \)-skew Armendariz rings.

According to Krempa [10], an endomorphism \( \alpha \) of a ring \( R \) is called to be rigid if \( a\alpha(a) = 0 \) implies \( a = 0 \) for \( a \in R \). We call a ring \( R \) \( \alpha \)-rigid if there exists a rigid endomorphism \( \alpha \) of \( R \). Note that any rigid endomorphism of a ring is a monomorphism, and \( \alpha \)-rigid rings are reduced rings by Hong et al. [6]. Properties of \( \alpha \)-rigid rings have been studied in Krempa [10], Hong [6], and Hirano [4]. Motivated by results in Krempa [10], Hong et al. [6] and Z. K. Liu [12], we introduce the weak \( \alpha \)-rigid rings which are a generalization of \( \alpha \)-rigid rings. Let \( \alpha \) be an endomorphism of a ring \( R \). \( R \) is said to be weak \( \alpha \)-rigid if \( a\alpha(a) \in \text{nil}(R) \Leftrightarrow a \in \text{nil}(R) \), where \( \text{nil}(R) \) is the set of nilpotent elements of \( R \). We will show that \( R \) is \( \alpha \)-rigid if and only if \( R \) is weak \( \alpha \)-rigid and reduced. So the weak \( \alpha \)-rigid ring \( R \) is a generalization of \( \alpha \)-rigid ring to the more general case where \( R \) is not assumed to be reduced.

For a ring \( R \), we denote by \( \text{nil}(R) \) the set of all nilpotent elements of \( R \) and by \( T_n(R) \) the \( n \) by \( n \) upper triangular matrix ring over \( R \).

2. Weak \( \alpha \)-rigid rings

Our focus in this section is to introduce the concept of a weak \( \alpha \)-rigid ring and study its properties. We say a ring \( R \) with an endomorphism \( \alpha \) to be weak \( \alpha \)-rigid if \( a\alpha(a) \in \text{nil}(R) \Leftrightarrow a \in \text{nil}(R) \) for \( a \in R \). It is easy to see that the notion of a weak \( \alpha \)-rigid ring generalizes that of an \( \alpha \)-rigid ring. Clearly every subring of a weak \( \alpha \)-rigid ring is a weak \( \alpha \)-rigid ring. The following example shows that there exists a weak \( \alpha \)-rigid ring \( R \) such that \( R \) is not \( \alpha \)-rigid.

**Example 2.1.** Let \( \alpha \) be an endomorphism of \( R \) and \( R \) be an \( \alpha \)-rigid ring. Let

\[
R_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}
\]

be a subring of \( T_3(R) \). The endomorphism \( \alpha \) of \( R \) is extended to the endomorphism \( \overline{\alpha} : R_3 \rightarrow R_3 \) defined by \( \overline{\alpha}((a_{ij})) = (\alpha(a_{ij})) \). We show that, (1) \( R_3 \) is a weak \( \overline{\alpha} \)-rigid ring, (2) \( R_3 \) is not \( \overline{\alpha} \)-rigid.
Suppose that
\[
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\end{pmatrix}
\alpha
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\] \in \text{nil} (R).

Then there is some positive integer \( n \) such that
\[
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\end{pmatrix}
\alpha
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\]
\[n = \begin{pmatrix}
a \alpha(a) & * & * \\
0 & a \alpha(a) & * \\
0 & 0 & a \alpha(a) \\
\end{pmatrix}
\] \[n = 0.

Thus \( a \alpha(a) \in \text{nil}(R) \). Since \( R \) is reduced, we have \( a \alpha(a) = 0 \), and so \( a = 0 \) since \( R \) is \( \alpha \)-rigid.

Hence \[
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\end{pmatrix}
\in \text{nil}(R_3).

Conversely, assume that \[
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\] \in \text{nil}(R_3). Then there is some positive integer \( n \) such that
\[
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\end{pmatrix}^n
\begin{pmatrix}
a^n & * & * \\
0 & a^n & * \\
0 & 0 & a^n \\
\end{pmatrix}
\] \[= 0.

Thus we get \( a = 0 \) because \( R \) is reduced. So
\[
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\end{pmatrix}
\begin{pmatrix}
a & b & c \\
0 & a & d \\
0 & 0 & a \\
\end{pmatrix}
\] \[= \begin{pmatrix}
0 & * & * \\
0 & 0 & * \\
0 & 0 & 0 \\
\end{pmatrix}
\] \[\in \text{nil}(R_3).

Therefore, \( R_3 \) is weak \( \overline{\alpha} \)-rigid.

(2) Since \( R_3 \) is not reduced, \( R_3 \) is not \( \overline{\alpha} \)-rigid.

**Proposition 2.2.** Let \( \alpha \) be an endomorphism of a ring \( R \). Then \( R \) is \( \alpha \)-rigid if and only if \( R \) is weak \( \alpha \)-rigid and reduced.

**Proof.** Assume that \( R \) is \( \alpha \)-rigid, then \( R \) is reduced. Now we show that \( R \) is weak \( \alpha \)-rigid. Suppose \( a \in \text{nil}(R) \), then \( a = 0 \) since \( R \) is reduced, and so \( a \alpha(a) = 0 \in \text{nil}(R) \). If \( a \alpha(a) \in \text{nil}(R) \) for \( a \in R \), then \( a \alpha(a) = 0 \), and so \( a = 0 \in \text{nil}(R) \) since \( R \) is \( \alpha \)-rigid. Therefore \( R \) is weak \( \alpha \)-rigid and reduced. Conversely, suppose that \( R \) is weak \( \alpha \)-rigid and reduced. Let \( a \alpha(a) = 0 \) for \( a \in R \), then \( a \in \text{nil}(R) \) since \( R \) is weak \( \alpha \)-rigid. Thus \( a = 0 \) because \( R \) is reduced. Hence \( R \) is \( \alpha \)-rigid. \( \square \)

**Proposition 2.3.** Let \( R \) be a weak \( \alpha \)-rigid ring and \( \text{nil}(R) \) be an ideal of \( R \). Then we have the following:
(1) If \( ab \in \text{nil}(R) \), then \( aa^m(b) \in \text{nil}(R) \), \( \alpha^n(a)b \in \text{nil}(R) \) for positive integers \( m \) and \( n \).

(2) If \( \alpha^k(ab) \in \text{nil}(R) \) for some positive integer \( k \), then \( ab \in \text{nil}(R) \), and \( ba \in \text{nil}(R) \).

(3) If \( \alpha^t(b) \in \text{nil}(R) \) for some positive integer \( t \), then \( ab \in \text{nil}(R) \), and \( ba \in \text{nil}(R) \).

**Proof.** (1) If \( ab \in \text{nil}(R) \), then \( \alpha(ab) = \alpha(a)\alpha(b) \in \text{nil}(R) \). Since \( \text{nil}(R) \) is an ideal of \( R \), we get \( ba(a)\alpha(b) = \alpha(a)(ba(a)) \in \text{nil}(R) \). So \( ba(a) \in \text{nil}(R) \), then \( ab \in \text{nil}(R) \). Continuing this procedure, we get \( \alpha^m(ab) \in \text{nil}(R) \) for any positive integer \( m \). Similarly, by \( ab \in \text{nil}(R) \), we have \( ba \in \text{nil}(R) \). Then \( \alpha^t(b)a \in \text{nil}(R) \) and so \( aa^t(b) \in \text{nil}(R) \) for any positive integer \( n \).

(2) If \( \alpha^k(ab) \in \text{nil}(R) \) for some positive integer \( k \), then \( \alpha^k(a)\alpha^k(b) = \alpha^k(ab) = \alpha(\alpha^{k-1}(ab)) \in \text{nil}(R) \) by Proposition 2.3 (1). Since \( \text{nil}(R) \) is an ideal of \( R \), \( (\alpha^{k-1}(ab))\alpha(\alpha^{k-1}(ab)) \in \text{nil}(R) \), and so \( \alpha^{k-1}(ab) \in \text{nil}(R) \) by definition. Continuing this procedure, we obtain \( ab \in \text{nil}(R) \).

(3) Employing the same method in the proof of (2), we get the result. \( \square \)

**Proposition 2.4.** Let \( R \) be weak \( \alpha \)-rigid and \( \text{nil}(R) \) be an ideal of \( R \). Then \( \alpha(e) = e \) for any central idempotent \( e \in R \).

**Proof.** Let \( e \) be a central idempotent of \( R \), then \( e(1-e) = 0 \) implies \( \alpha(e)(1-e) = 0 \) by Proposition 2.3. Thus there is some positive integer \( k \) such that \( 0 = \alpha(e)(1-e))^k = \alpha(e)(1-e) \). Hence \( \alpha(e) = \alpha(e)e \). Similarly, \( (1-e)e = 0 \) implies \( \alpha(1-e)e = 0 \), thus \( e = \alpha(e)e \) and hence \( \alpha(e) = e \). \( \square \)

Let \( \alpha \) be an endomorphism of a ring \( R \), an ideal \( I \) of \( R \) is said to be weak \( \alpha \)-rigid if \( \alpha(a) \in \text{nil}(R) \Leftrightarrow a \in \text{nil}(R) \) for \( a \in I \).

**Proposition 2.5.** Let \( R \) be an abelian ring with \( \alpha(e) = e \) for any \( e^2 = e \in R \). Then the following statements are equivalent:

(1) \( R \) is a weak \( \alpha \)-rigid ring.

(2) \( eR \) and \( (1-e)R \) are weak \( \alpha \)-rigid ideals.

**Proof.** (1) \( \Rightarrow \) (2) It is trivial.

(2) \( \Rightarrow \) (1) Let \( a \in R \) be such that \( a \in \text{nil}(R) \). Then \( ea \in \text{nil}(R) \) and \( (1-e) \in \text{nil}(R) \). Since \( eR \) and \( (1-e)R \) are weak \( \alpha \)-rigid, there exist some positive integers \( m \) and \( n \) such that \( (eaa(ea))^m = e(aa(a))^m = 0 \), and \( ((1-e)a\alpha((1-e)a))^n = (1-e)(a\alpha(a))^n = 0 \). Let \( k = \max\{m, n\} \), then \( e(aa(a))^k = 0 \) and \( (1-e)(a\alpha(a))^k = 0 \). Thus \( (a\alpha(a))^k = 0 \), hence \( a\alpha(a) \in \text{nil}(R) \).
Conversely, assume that $\alpha a(a) \in \text{nil}(R)$. Then $e\alpha ea(a) \in \text{nil}(R)$ and $(1 - e)\alpha a((1 - e)a) \in \text{nil}(R)$. Thus $ea \in \text{nil}(R)$ and $(1 - e)a \in \text{nil}(R)$ since $eR$ and $(1 - e)R$ are weak $\alpha$-rigid ideals. So $a \in \text{nil}(R)$. Therefore $R$ is weak $\alpha$-rigid. □

3. Extensions of weak $\alpha$-rigid rings

Let $\alpha$ be an endomorphism of a ring $R$. Let $T_n(R)$ denote the $n$ by $n$ upper triangular matrix ring over $R$. Then the endomorphism $\alpha$ of $R$ is extended to the endomorphism $\overline{\alpha} : T_n(R) \to T_n(R)$ defined by

$$
\overline{\alpha} \left( \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \right) = \begin{pmatrix} \alpha(a_{11}) & \alpha(a_{12}) & \cdots & \alpha(a_{1n}) \\ 0 & \alpha(a_{22}) & \cdots & \alpha(a_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha(a_{nn}) \end{pmatrix}
$$

for any $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$. Then we have the following:

**Theorem 3.1.** Let $\alpha$ be an endomorphism of a ring $R$. Then the following statements are equivalent:

1. $R$ is weak $\alpha$-rigid.
2. $T_n(R)$ is weak $\overline{\alpha}$-rigid for any positive integer $n$.

**Proof.** (1) $\implies$ (2) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \in T_n(R)$ be such that $A \in \text{nil}(T_n(R))$. Then there exists some positive integer $n$ such that

$$
\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}^n = \begin{pmatrix} a_{11}^n & * & \cdots & * \\ 0 & a_{22}^n & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^n \end{pmatrix} = 0.
$$

Thus $a_{ii} \in \text{nil}(R) (i = 1, 2, \ldots, n)$. Since $R$ is weak $\alpha$-rigid, we get that $\alpha a_i a(a_i) \in \text{nil}(R)$. So there is some positive integer $t_i$ such that $(\alpha a_i a(a_i))^{t_i} = 0$, (i =
1, 2, · · · , n). Let $t = \text{Max}\{t_i\}$, $(i = 1, 2, · · · , n)$, then

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}^t \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}
= \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}^t = 0.
$$

Hence

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}^t \in \text{nil}(T_n(R)).
$$

Now, Let $A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix} \in T_n(R)$ be such that $A \pi(A) \in \text{nil}(T_n(R))$.

There is some positive integer $n$ such that

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}^t \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}
= \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}^t = 0. \text{ Thus } a_{ii} \alpha(a_{ii}) \in \text{nil}(R),
$$

$(i = 1, 2, \cdots n)$. Hence $a_{ii} \in \text{nil}(R)$ since $R$ is weak $\alpha$-rigid, and so there is some positive integer $t_i$ such that $(a_{ii})^{t_i} = 0$. Let $t = \text{Max}\{t_i\}$, $(i = 1, 2, \cdots , n)$, then

$$
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{pmatrix}^t = \begin{pmatrix}
a_{11}^t & \cdots & * \\
0 & a_{22}^t & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}^t
\end{pmatrix}^t = 0.
$$

Hence $A \in \text{nil}(T_n(R))$. Therefore $T_n(R)$ is weak $\alpha$-rigid.
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(2) $\implies$ (1) Let $a \in \text{nil}(R)$, then $a^n = 0$ for some positive integer $n$. Let

$$A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in T_n(R),$$

then $A \in \text{nil}(T_n(R))$. Since $T_n(R)$ is weak $\alpha$-rigid,

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \alpha \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} a\alpha(a) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$\in \text{nil}(T_n(R))$. Thus $a\alpha(a) \in \text{nil}(R)$. Now let $a\alpha(a) \in \text{nil}(R)$, then

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \alpha \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \text{nil}(T_n(R)).$$

Since $T_n(R)$ is weak $\alpha$-rigid, we have

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \text{nil}(T_n(R)),$$

so $a \in \text{nil}(R)$. Hence $R$ is weak $\alpha$-rigid. \qed

Given a ring $R$ and a bimodule $R_M_R$, the trivial extension of $R$ by $M$ is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring of all matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Let $\alpha$ be an endomorphism of a ring $R$, then $\alpha$ is extended to the endomorphism $\overline{\alpha} : T(R, R) \rightarrow T(R, R)$ defined by $\overline{\alpha}\left(\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}\right) = \begin{pmatrix} \alpha(r) & \alpha(m) \\ 0 & \alpha(r) \end{pmatrix}$ for any $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \in T(R, R)$.

**Corollary 3.2.** Let $\alpha$ be an endomorphism of a ring $R$. Then the trivial extension $T(R, R)$ of $R$ by $R$ is weak $\overline{\alpha}$-rigid if and only if $R$ is weak $\alpha$-rigid.
Proof. Since $T(R, R)$ is isomorphic to the subring $\left\{ \left( \begin{array}{cc} r & m \\ 0 & r \end{array} \right) | r, m \in R \right\}$ of a ring $T_2(R)$ in Theorem 3.1. And each subring of a weak $\alpha$-rigid ring is also weak $\alpha$-rigid. So it is easy to see that $T(R, R)$ is a weak $\alpha$ rigid ring if and only if $R$ is weak $\alpha$-rigid. $\square$

We say a ring $R$ is a weak $\alpha$-skew Armendariz ring if whenever polynomials $p = \sum_{i=0}^{m} a_i x^i$, and $q = \sum_{j=0}^{n} b_j x^j$ in $R[x; \alpha]$ satisfy $pq = 0$, then $a_i \alpha^i(b_j)$ is a nilpotent element of $R$ for each $i, j$.

Theorem 3.3. Let $R$ be a weak $\alpha$-rigid ring with $\text{nil}(R)$ an ideal of $R$. Then $R$ is a weak $\alpha$-skew Armendariz ring.

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x; \alpha]$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \alpha]$ be such that $f(x)g(x) = 0$. Then $\sum_{k=0}^{m+n} \left( \sum_{i+j=k} a_i \alpha^i(b_j) \right) x^k = 0$. Then we have the following equations:

$$\sum_{i+j=k} a_i \alpha^i(b_j) = 0, \; k = 0, 1, \cdots, m+n.$$ 

We will show that $a_i \alpha^i(b_j) \in \text{nil}(R)$ by induction on $i+j$.

If $i+j = 0$, then $a_0 b_0 = 0 \in \text{nil}(R)$ and so $b_0 a_0 \in \text{nil}(R)$.

Now suppose that $k$ is a positive integer such that $a_i \alpha^i(b_j) \in \text{nil}(R)$ when $i+j < k$. We will show that $a_i \alpha^i(b_j) \in \text{nil}(R)$ when $i+j = k$.

Consider equation:

$$a_0 b_k + a_1 \alpha(b_{k-1}) + a_2 \alpha^2(b_{k-2}) + \cdots + a_k \alpha^k(b_0) = 0. \quad (1)$$

Multiplying (1) by $b_0$ from left, we have $b_0 a_k \alpha^k(b_0) = -(b_0 a_0 b_k + b_0 a_1 \alpha(b_{k-1}) + b_0 a_2 \alpha^2(b_{k-2}) + \cdots + b_0 a_{k-1} \alpha^{k-1}(b_1))$. By the induction hypothesis, $a_i \alpha^i(b_0) \in \text{nil}(R)$ for $0 \leq i < k$. Thus $a_i b_0 \in \text{nil}(R)$ by Proposition 2.3, and so $b_0 a_i \in \text{nil}(R)$ for $0 \leq i < k$. Hence $b_0 a_k \alpha^k(b_0) \in \text{nil}(R)$ since $\text{nil}(R)$ is an ideal of $R$. Thus $b_0 a_k \alpha^k(b_0) \alpha^k(a_k) = b_0 a_k \alpha^k(b_0 a_k) \in \text{nil}(R)$, this means that $b_0 a_k \in \text{nil}(R)$ and so $a_k b_0 \in \text{nil}(R)$. Thus $a_k \alpha^k(b_0) \in \text{nil}(R)$ by Proposition 2.3. Multiplying (1) by $b_1$ from the left, similarly, we have $a_{k-1} \alpha^{k-1}(b_1) \in \text{nil}(R)$.

Continuing this procedure yields that $a_i \alpha^i(b_j) \in \text{nil}(R)$ for $i+j = k$. Therefore by induction, we have $a_i \alpha^i(b_j) \in \text{nil}(R)$ for each $i, j$, and so $R$ is a weak $\alpha$-skew Armendariz ring. $\square$

Now we can give an example of a weak $\alpha$-rigid ring which is not weak $\alpha$-skew Armendariz.
Example 3.4. Let \( R \) be a ring and \( M_2(R) \) the 2 by 2 matrix ring over \( R \). Let 
\[ S = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} | A, B, C \in M_2(R) \right\}. \]
With usual matrix operations, \( S \) is a ring.

The endomorphism \( \alpha \) of \( S \rightarrow S \) is defined by 
\[ \alpha \left( \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right) = \begin{pmatrix} A & -B \\ 0 & C \end{pmatrix} \quad \text{for any} \quad \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in S. \]
It is easy to see that \( S \) is weak \( \alpha \)-rigid. Now we show that \( S \) is not weak \( \alpha \)-skew Armendariz. Let 
\[ f(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} x, \]
\[ g(x) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} x \in S[x; \alpha]. \]

Then we have \( f(x)g(x) = 0 \), but 
\[ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \]
is not a nilpotent element of \( S \). Thus \( S \) is not weak \( \alpha \)-skew Armendariz.

Corollary 3.5. Let \( \alpha \) be an endomorphism of a ring \( R \). Then \( R \) is a weak \( \alpha \)-skew Armendariz ring if and only if for any \( n \), \( T_n(R) \) is a weak \( \overline{\alpha} \)-skew Armendariz ring.

Proof. Suppose that \( T_n(R) \) is a weak \( \overline{\alpha} \)-skew Armendariz ring, then it is easy to see that \( R \) is a weak \( \alpha \)-skew Armendariz ring. Conversely, Let \( f(x) = A_0 + A_1 x + \cdots + A_p x^p \), and \( g(x) = B_0 + B_1 x + \cdots + B_q x^q \) be elements of \( T_n(R)[x; \overline{\alpha}] \). Assume that \( f(x)g(x) = 0 \). Let
\[ A_i = \begin{pmatrix} a_{i1}^1 & a_{i2}^1 & \cdots & a_{i1}^n \\ 0 & a_{i2}^2 & \cdots & a_{i2}^n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{in}^i \end{pmatrix}, \quad B_j = \begin{pmatrix} b_{11}^j & b_{12}^j & \cdots & b_{1n}^j \\ 0 & b_{22}^j & \cdots & b_{2n}^j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{jn}^j \end{pmatrix}. \]
Then from $f(x)g(x) = 0$, it follows that
\[
\left( \sum_{i=1}^{p} a_{ss}^i x^i \right) \left( \sum_{j=1}^{q} b_{ss}^j x^j \right) = 0, \quad s = 1, 2, \ldots, n.
\]
Since $R$ is a weak $\alpha$-skew Armendariz ring, there exists some positive integer $m_{ij}$ such that $(a_{ss}^i \alpha_i(b_{ss}^j))^{m_{ij}} = 0$ for any $s$ and any $i, j$. Let $m_{ij} = \max\{m_{ij1}, m_{ij2}, \cdots, m_{ijn}\}$.

Thus $(A_i \alpha_i(B_j))^{m_{ij}} \in \text{nil}(T_n(R))$. This shows that $T_n(R)$ is a weak $\pi$-skew Armendariz ring.

\textbf{Corollary 3.6.} Let $\alpha$ be an endomorphism of a ring $R$. Then $R$ is a weak $\alpha$-skew Armendariz ring if and only if the trivial extension $T(R, R)$ is a weak $\pi$-skew Armendariz ring.

\textbf{Proof.} It follows from Corollary 3.5. \hfill \square

Recall that if $\alpha$ is an endomorphism of a ring $R$, then the map $\alpha$ can be extended to an endomorphism of the polynomial ring $R[x]$ defined by $\sum_{i=0}^{m} a_i x^i \mapsto \sum_{i=0}^{m} \alpha(a_i) x^i$.

We shall also denote the extended map $R[x] \rightarrow R[x]$ by $\alpha$ and the image of $f \in R[x]$ by $\alpha(f)$. By [12, Theorem 3.8], if $R$ is semicommutative, then $R[x]$ is weak Armendariz. We have the following Theorem which is a generalization of [12, Theorem 3.8]

\textbf{Lemma 3.7.} [12] Let $R$ be a semicommutative ring. Then $\text{nil}(R)$ is an ideal of $R$.

\textbf{Lemma 3.8.} [12] Let $R$ be a semicommutative ring. If $a_0, a_1, \cdots, a_n \in \text{nil}(R)$, then $a_0 + a_1 x + \cdots + a_n x \in R[x]$ is a nilpotent element.

\textbf{Theorem 3.9.} Let $R$ be a weak $\alpha$-rigid and semicommutative ring. Then $R[x]$ is a weak $\alpha$-skew Armendariz ring.

\textbf{Proof.} Let $f = f_0 + f_1 y + \cdots + f_p y^p \in (R[x])[y; \alpha]$, and $g = g_0 + g_1 y + \cdots + g_q y^q \in (R[x])[y; \alpha]$ be such that $fg = 0$. Suppose that $f_i = \sum_{s=0}^{m_i} a_{ss}^i x^s$. Let
$m = \text{Max}\{m_i, (i = 0, 1, \cdots, p)\}$ Then each $f_i$ can be written in the form of $f_i = \sum_{s=0}^{m} a_s^i x^s$. Note that $xy = yx$ since $\alpha(1) = 1$, and $xa = ax$ for any $a \in R$. Thus

$$f = \sum_{i=0}^{p} \left( \sum_{s=0}^{m} a_s^i x^s \right) y^i = \sum_{s=0}^{m} \left( \sum_{i=0}^{p} a_s^i y^i \right) x^s. \tag{1}$$

Similarly, each $g_j$ can be written in the form of $g_j = \sum_{t=0}^{n} b_t^j x^t$, and thus

$$g = \sum_{j=0}^{q} \left( \sum_{t=0}^{n} b_t^j x^t \right) y^j = \sum_{t=0}^{n} \left( \sum_{j=0}^{q} b_t^j y^j \right) x^t. \tag{2}$$

From $fg = 0$, we have the following equations:

$$\sum_{s+t=k} \left( \sum_{i=0}^{p} a_s^i y^i \right) \left( \sum_{j=0}^{q} b_t^j y^j \right) = 0. \quad k = 0, 1, \cdots, m+n. \tag{2}$$

We will show by induction on $s + t$ that $a_s^i \alpha^i(b_t^j) \in \text{nil}(R)$ for any $0 \leq i \leq p$, and $0 \leq j \leq q$ and any $s, t$ with $s + t = 0, 1, \cdots m+n$.

If $s + t = 0$, then $s = t = 0$. Thus $(\sum_{i=0}^{p} a_s^i y^i)(\sum_{j=0}^{q} b_t^j y^j) = 0$. Since $R$ is semicommutative, $\text{nil}(R)$ is an ideal of $R$ by Lemma 3.7. Then $R$ is weak $\alpha$-skew Armendariz by Theorem 3.3. Thus $a_s^i \alpha^i(b_t^j) \in \text{nil}(R)$ for any $0 \leq i \leq p$, and any $0 \leq j \leq q$.

Now suppose that $k \leq m + n$ is such that $a_s^i \alpha^i(b_t^j) \in \text{nil}(R)$ for any $0 \leq i \leq p$, and any $0 \leq j \leq q$, and any $s, t$ with $s + t < k$, we will show that $a_s^i \alpha^i(b_t^j) \in \text{nil}(R)$ for any $0 \leq i \leq p$, and any $0 \leq j \leq q$, and any $s, t$ with $s + t = k$. From (2), we have

$$0 = \sum_{s+t=k} \left( \sum_{i=0}^{p} a_s^i y^i \right) \left( \sum_{j=0}^{q} b_t^j y^j \right) = \sum_{s+t=k} \sum_{i=0}^{p+q} \sum_{j=0}^{q} a_s^i \alpha^i(b_t^j) y^j = \sum_{t=0}^{p+q} \left( \sum_{i=0}^{p+q} \sum_{j=0}^{q} a_s^i \alpha^i(b_t^j) \right) y^j.$$
Thus

$$\sum_{s+t=k} a^0_s b^0_t = 0,$$

$$\sum_{s+t=k} a^0_s b^1_t + \sum_{s+t=k} a_s^1 \alpha(b^0_t) = 0,$$

$$\sum_{s+t=k} a_s^0 b^1_t + \sum_{s+t=k} a_s^2 \alpha(b^0_t) + \cdots + \sum_{s+t=k} a_s^i \alpha(b^0_t) = 0,$$

$$\sum_{s+t=k} a_s^p \alpha^p(b^0_t) = 0.$$

If \( s < k \). Then by the induction hypothesis, \( a^0_s b^0_t \in \text{nil}(R) \) and so \( b^0_t a^0_s \in \text{nil}(R) \) for \( s < k \). Hence \( b^0_t a^0_s b^0_t + b^0_t a^0_s b^0_{t-1} + \cdots + b^0_t a^0_{s-1} b^0_1 \in \text{nil}(R) \) since \( R \) is semicommutative. Therefore if we multiply \( \sum_{s+t=k} a^0_s b^0_t = 0 \) on the left side by \( b^0_t \), then it follows that \( b^0_t a^0_s b^0_t \in \text{nil}(R) \), and so \( b^0_t a^0_s \in \text{nil}(R) \) and \( a^0_s b^0_t \in \text{nil}(R) \). If we multiply \( \sum_{s+t=k} a^0_s b^0_t = 0 \) on the left side by \( b^1_t \), then \( b^1_t a^0_s b^0_t = -(b^0_t a^0_s b^0_t + b^0_t a^0_s b^0_{t-1} + \cdots + b^0_t a^0_{s-1} b^0_1) = 0 \) since \( R \) is semicommutative. Thus \( a^0_{s-1} b^1_t \in \text{nil}(R) \). Similarly, we can show that \( a^0_{s-2} b^2_t \in \text{nil}(R), \ldots, a^0_0 b^0_k \in \text{nil}(R) \). So we show that \( a^s_i \alpha^i(b^0_t) \in \text{nil}(R) \) for any \( s, t \) with \( s + t = k \) and any \( i, j \) with \( i + j = 0 \).

Suppose that \( t \leq p + q \) is such that \( a^s_i \alpha^i(b^0_t) \in \text{nil}(R) \) for any \( s, t \) with \( s + t = k \) and any \( i, j \) with \( i + j < l \), we will show that \( a^s_i \alpha^i(b^0_t) \in \text{nil}(R) \) for any \( s, t \) with \( s + t = k \) and any \( i, j \) with \( i + j = l \).

If \( s < k \), then by the induction hypothesis, \( a^s_i \alpha^i(b^0_t) \in \text{nil}(R) \). Thus \( a^s_i b^0_t \in \text{nil}(R) \) by Proposition 2.3, and so \( b^0_t a^s_i \in \text{nil}(R) \). If \( i < l \), then by the induction hypothesis on \( l \), \( a^0_k \alpha^l(b^0_t) \in \text{nil}(R) \) for any \( i < l \), which implies \( a^0_k b^0_t \in \text{nil}(R) \) and so \( b^0_t a^0_k \in \text{nil}(R) \) for any \( i < l \). Multiplying \( \sum_{s+t=k} a^0_s b^0_t = 0 \) on the left side by \( b^0_t \), we have \( b^0_t a^0_k \alpha^l(b^0_t) \in \text{nil}(R) \) since \( \text{nil}(R) \) is an ideal of \( R \) by Lemma 3.7. Thus \( b^0_t a^0_k \alpha^l(b^0_t) \alpha^l(a^0_k) = b^0_t a^0_k \alpha^l(b^0_t a^0_k) \in \text{nil}(R) \). Thus \( b^0_t a^0_k \in \text{nil}(R) \) which implies \( a^0_k b^0_t \in \text{nil}(R) \) and so \( a^0_k b^0_t \in \text{nil}(R) \) by Proposition 2.3. Similarly, we can show that \( a^s_i \alpha^i(b^0_t) \in \text{nil}(R) \) for any \( s, t \) with \( s + t = k \) and any \( i, j \) with \( i + j = l \).

Therefore, by induction, we have \( a^s_i \alpha^i(b^0_t) \in \text{nil}(R) \) for any \( 0 \leq i \leq p \), and \( 0 \leq j \leq q \) and any \( s, t \) with \( s + t = 0, 1, \ldots, m + n \). Now,
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\[
f_i \alpha^i(g_j) = \left( \sum_{s=0}^{m} a_i^s x^s \right) \alpha^i \left( \sum_{t=0}^{n} b_i^t x^t \right) = \left( \sum_{s=0}^{m} a_i^s x^s \right) \left( \sum_{t=0}^{n} \alpha^i(b_i^t) x^t \right) = \sum_{k=0}^{m+n} \left( \sum_{s+t=k} a_i^s \alpha^i(b_i^t) \right) x^k.
\]

Since \( R \) is semicommutative, by Lemma 3.7, \( \sum_{s+t=k} a_i^s \alpha^i(b_i^t) \in \text{nil}(R) \). Thus by Lemma 3.8, \( f_i \alpha^i(g_j) \in \text{nil}(R[x]) \). Therefore \( R[x] \) is weak \( \alpha \)-skew Armendariz. \( \square \)

**Corollary 3.10.** Let \( R \) be a weak \( \alpha \)-rigid semicommutative ring. Then \( R[x]/(x^n) \) is a weak \( \alpha \)-skew Armendariz ring where \( (x^n) \) is the ideal of \( R[x] \) generated by \( x^n \).

**Proof.** Denote \( \overline{x} \) in \( R[x]/(x^n) \) by \( u \), so \( R[x]/(x^n) = R[u] = R + Ru + \cdots + Ru^{n-1} \), where \( u \) is commutative with elements of \( R \) and \( u^n = 0 \). Let \( f, g \in R[u][y] \) be such that \( fg = 0 \). Then we can suppose that \( f = \sum_{i=0}^{n-1} f_i y^i \) and \( g = \sum_{j=0}^{n-1} g_j y^j \), and \( f_i = \sum_{s=0}^{n-1} a_i^s u^s \), and \( g_j = \sum_{t=0}^{n-1} b_i^t u^t \). Then a proof similar to Theorem 3.9, we get the result. \( \square \)

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