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# $C$-CANONICAL MODULES

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Abstract. Let $C$ be a semidualizing module over a commutative Noetherian local ring $R$. In this paper we introduce a new class of modules, namely $C$-canonical modules which are a generalization of canonical modules. It is shown that if the canonical module exists then the $C$-canonical module exists and the converse holds under special conditions. Also, a new characterization of Gorenstein local rings is given via $C$-canonical modules.

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## 1. Introduction

Throughout this introduction $(R, \mathfrak{m})$ is a commutative Noetherian local ring of dimension $n$, $\mathrm{E}_R(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$ and $\mathrm{H}^n_{\mathfrak{m}}(R)$ is the $n$-th local cohomology module of $M$ with respect to $\mathfrak{m}$.

Grothendieck [11] defined a canonical module over a complete local ring and called it a module of dualizing differentials; see [11, page 94]. Herzog and Kunz defined a canonical module for $R$ as a finitely generated $R$-module $K$ for which $K \otimes_R \hat{R} \cong \mathrm{Hom}_R(\mathrm{H}^n_{\mathfrak{m}}(R), \mathrm{E}_R(R/\mathfrak{m}))$ [12, Definition 5.6]. In [13] M. Hochster and C. Huneke, defined a canonical module as a finitely generated $R$-module $K$, for which $\mathrm{Hom}_R(K, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(R)$. By a dualizing module over a Cohen-Macaulay local ring, we mean a finitely generated maximal Cohen-Macaulay $R$-module with finite injective dimension of type 1 (see Section 2 for the definition of type). A canonical module of a Cohen-Macaulay local ring (if it exists) actually is the dualizing module. Canonical modules play an important role in studying Cohen-Macaulay local rings.

It is known that a canonical module (if it exists) is unique up to isomorphism [3, Theorem 12.1.6]. Canonical modules in general are studied extensively in the literature. Aoyoma [2] proved excellent results concerning behavior of canonical modules under flat base change, the endomorphism ring of canonical modules, and the trivial extension of the ring by canonical modules. In [1], the author proved that if the canonical module has finite projective dimension, then it is isomorphic to $R$. Also, it is known that if $R$ is a homomorphic image of a Gorenstein local ring, then it has the canonical module, and the converse holds when $R$ is Cohen-Macaulay. In fact, when $R$ is a Cohen-Macaulay local ring, Foxby [8], and Reiten [16] proved (independently) that if the canonical module exists, then the trivial extension of $R$ by the canonical module is a Gorenstein local ring, and thus $R$ is a homomorphic image of a Gorenstein local ring.

A semidualizing $R$-module is a finitely generated $R$-module $C$ such that the homothety map $R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism, and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. These modules, were introduced by Foxby [8], Vasconcelos [19], and Golod [10] independently. The ring itself and the dualizing module (if it exists) are examples of semidualizing modules. Semidualizing modules have been studied by many researchers; see, for example, [5], [6], [9], [15], [18], [20]. Also, we refer the reader to [21] for detailed results concerning semidualizing modules.

The main goal of this paper is to generalize the concept of canonical modules for semidualizing modules. To do this, we define a $C$-canonical module (or a canonical module for $C$) as a finitely generated $R$-module $K$ such that $\mathrm{Hom}_R(K, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(C)$, where $C$ is a semidualizing module.

In Section 3, we prove that a canonical module for $C$ (if it exists) is unique up to isomorphism. Also, if $R$ has a canonical module, then every semidualizing module $C$ has a $C$-canonical module. We shall show that if a semidualizing module $C$ has a canonical module and belongs to $\mathcal{A}_C(R)$ (the Auslander class of $C$), then $R$ has a canonical module.

In Section 4, we discuss the case where $R$ is a Cohen-Macaulay ring. Let us denote the canonical module of a semidualizing module $C$ by $\omega_C$. As an application, we prove some new results concerning the existence of the canonical module over Cohen-Macaulay rings via $C$-canonical modules. For instance, Theorem 4.1 says that if the canonical module for a semidualizing $R$-module $C$ exists, then $\omega_R$ exists. More precisely:

**Theorem:** Let $R$ be a Cohen-Macaulay ring. The following are equivalent:

(i) $\omega_R$ exists;
(ii) $\omega_C$ exists for every semidualizing module $C$;
(iii) $\omega_C$ exists for some semidualizing module $C$.

It is known that a Cohen-Macaulay local ring is Gorenstein if and only if $R \cong \omega_R$. By the following result (which is Corollary 4.3), one can replace $R$ by an arbitrary semidualizing module.

**Theorem:** Let $R$ be a Cohen-Macaulay ring. The following are equivalent:

(i) $R$ is Gorenstein;

(ii) $\omega_C \cong C$ for every semidualizing module $C$;

(iii) $\omega_C \cong C$ for some semidualizing module $C$.

Sharp [17] showed that over a Cohen-Macaulay local ring with the canonical module $\omega_R$, any maximal Cohen-Macaulay $R$-module with finite injective dimension, is equal to a finite direct sum of copies of $\omega_R$ (see [17, Theorem 2.1 (v)]). By the following result (Theorem 4.9), we obtain a similar representation for some subclasses of maximal Cohen-Macaulay $R$-modules, via $C$-canonical modules.

**Theorem:** Let $R$ be a Cohen-Macaulay ring, $C$ be a semidualizing module, and suppose $\omega_C$ exists. Let $M$ be a maximal Cohen-Macaulay $R$-module with $\mathcal{I}_C$-id$(M) < \infty$. Then $M \cong \bigoplus_{i=1}^{t} \omega_C$ for some positive number $t$.

## 2. Preliminaries

Throughout this paper $R$ is a commutative Noetherian local ring with nonzero identity and dim $R = n$. We denote the maximal ideal of $R$ by $\mathfrak{m}$, and the residue field $R/\mathfrak{m}$ by $k$. The minimal number of generators of a finitely generated $R$-module $M$ is denoted by $\mu(M)$, which is equal to $\mathrm{vdim}_k(M \otimes_R k)$. The type of a finitely generated $R$-module $M$ is denoted by $r_R(M)$ and is defined by $r_R(M) = \mathrm{vdim}_k \mathrm{Ext}_R^t(k, M)$ where $t = \mathrm{depth}_R M$. In particular when $M$ is an Artinian $R$-module, one has $r(M) = \mathrm{vdim}_k \mathrm{Soc}\,(M)$, where $\mathrm{Soc}\,(M) = (0 :_M \mathfrak{m})$. The $\mathfrak{m}$-adic completion of $R$ is denoted by $\hat{R}$ . We use $\mathrm{E}_R(k)$ to denote the injective hull of the residue field $k$. For each $i \in \mathbb{N} \bigcup \{0\}$, the $i$-th local cohomology module of $M$ with respect to the ideal $\mathfrak{a}$ is defined by

$$\mathrm{H}_{\mathfrak{a}}^i(M) = \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}_R^i(R/\mathfrak{a}^n, M).$$

We refer the reader to [3] for more details about local cohomology modules.

**Definition 2.1.** A finitely generated $R$-module $C$ is called a *semidualizing* $R$-module if the homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism, and $\mathrm{Ext}_R^i(C, C) = 0$ for all $i > 0$.

For example, the ring itself is always a semidualizing $R$-module. Also, a dualizing module of a Cohen-Macaulay local ring is a semidualizing $R$-module.

**Definition 2.2.** Let $C$ be a semidualizing module and $M$ be an $R$-module. Then $M$ is called $C$-*injective* if $M \cong \mathrm{Hom}_R(C, I)$ for some injective $R$-module $I$. The class of $C$-injective $R$-modules is denoted by $\mathcal{I}_C(R)$. Also, $M$ is called $C$-*projective* if $M \cong C \otimes_R P$ for some projective $R$-module $P$. The class of $C$-projective $R$-modules is denoted by $\mathcal{P}_C(R)$.
Consider the complex:

$$X = 0 \to M \to B^0 \to B^1 \to \cdots \to B^n \to \dots$$

where each $B^i$ is a $C$-injective $R$-module. This complex is called an augmented $\mathcal{I}_C-$injective resolution for $M$ whenever the following complex is exact:

$$C \otimes_R X = 0 \to C \otimes_R M \to C \otimes_R B^0 \to C \otimes_R B^1 \to \cdots \to C \otimes_R B^n \to \dots$$

Also, $\mathcal{I}_C$-*injective dimension* of $M$ (or $\mathcal{I}_C$-id$(M)$) is defined as:

$$\mathcal{I}_C\text{-id}(M) := inf\{sup\{n \geq 0 \mid X_n \neq 0\} \ X \ is \ an \ \mathcal{I}_C\text{-}injective \ resolution \ of \ M\}.$$

The terms $\mathcal{P}_C$-*projective resolution* and $\mathcal{P}_C$-*projective dimension* of $M$ ($\mathcal{P}_C$-pd(M)) are defined dually. These concepts are completely discussed in [18].

**Definition 2.3.** The *Auslander class with respect to* $C$, denoted by $\mathcal{A}_C(R)$, is a class of $R$-modules $M$ such that

(i) *the natural map* $M \to \mathrm{Hom}_R(C, C \otimes_R M)$ *is an isomorphism and*
(ii) $\mathrm{Tor}_i^R(C, M) = 0 = \mathrm{Ext}_R^i(C, C \otimes_R M)$ *for all* $i \geq 1$.

**Definition 2.4.** The *Bass class with respect to* $C$, denoted by $\mathcal{B}_C(R)$, is a class of $R$-modules $M$ such that

(i) *the evaluation map* $C \otimes_R \mathrm{Hom}_R(C, M) \to M$ *is an isomorphism and*
(ii) $\mathrm{Ext}_R^i(C, M) = 0 = \mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, M))$ *for all* $i \geq 1$.

Next, we recall some known results concerning semidualizing $R$-modules which will be needed throughout this paper.

**Proposition 2.5.** *Let* $C$ *be a semidualizing module.*

(i) $C$ *is a faithful* $R$*-module and therefore* $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$ *and* $\dim C = \dim R$. *Also,* $\mathrm{Ass}_R(R) = \mathrm{Ass}_R(C)$.
(ii) *A sequence* $\underline{x}$ *of elements of* $R$ *is an* $R$*-sequence if and only if it is a* $C$*-sequence and in this situation* $C/\underline{x}C$ *is a semidualizing* $R/\underline{x}R$*-module. Moreover,* $\mathrm{depth}_R(R) = \mathrm{depth}_R(C)$.
(iii) *One has* $\mathrm{Hom}_R(C, M) \neq 0$ *for any nonzero* $R$*-module* $M$.

(iv) *If* $\varphi : R \to S$ *is a flat ring homomorphism then* $C \otimes_R S$ *is a semidualizing* $S$*-module. The converse holds when* $\varphi$ *is a faithfully flat ring homomorphism.*

**Proof.** For $(i)$ see [21, Proposition 2.1.16], for $(ii)$ see [21, Theorem 2.2.6], for $(iii)$ see [21, Corollary 2.1.17] and for $(iv)$ see [21, Proposition 2.2.1]. □

**Theorem 2.6.** ([18, Theorem 2.11]) *Let* $C$ *be a semidualizing* $R$*-module and let* $M$ *be an* $R$*-module.*

(i) $\mathcal{I}_C\text{-id}_R(M) = \text{id}_R(C \otimes_R M)$ *and* $\text{id}_R(M) = \mathcal{I}_C\text{-id}_R(\text{Hom}_R(C, M))$.

(ii) $\mathcal{P}_C\text{-pd}_R(M) = \text{pd}_R(\text{Hom}_R(C, M))$ *and* $\text{pd}_R(M) = \mathcal{P}_C\text{-pd}_R(C \otimes_R M)$.

**Theorem 2.7.** *Let* $C$ *be a semidualizing module.*

(i) *If any two* $R$*-modules in a short exact sequence are in* $\mathcal{A}_C(R)$*, respectively* $\mathcal{B}_C(R)$*, then so is the third.*

(ii) $\mathcal{A}_C(R)$ *(resp.* $\mathcal{B}_C(R)$*) contains every* $R$*-module of finite projective dimension (resp. injective dimension).*

(iii) $\mathcal{A}_C(R)$ *(resp.* $\mathcal{B}_C(R)$*) contains every* $R$*-module of finite* $\mathcal{I}_C$*-injective dimension (resp.* $\mathcal{P}_C$*-projective dimension).*

**Proof.** $(i)$ See [21, Proposition 3.1.7].
$(ii)$ See [21, Proposition 3.1.9 and Proposition 3.1.10].
$(iii)$ See [18, Corollary 2.9]. □

**Theorem 2.8.** ([18, Theorem 2.8]) *Let* $C$ *be a semidualizing* $R$*-module and* $M$ *be an* $R$*-module.*

(i) $M \in \mathcal{B}_C(R)$ *if and only if* $\text{Hom}_R(C, M) \in \mathcal{A}_C(R)$.

(ii) $M \in \mathcal{A}_C(R)$ *if and only if* $C \otimes_R M \in \mathcal{B}_C(R)$.

## 3. Main results

In this section $C$ is a semidualizing $R$-module. We generalize some concepts which are stated in chapter 12 of [3].

**Definition 3.1.** A $C$*-canonical module* (or a canonical module for $C$) is a finitely generated $R$-module $K$ such that $\text{Hom}_R(K, \text{E}_R(R/\mathfrak{m})) \cong \text{H}^n_{\mathfrak{m}}(C)$.

**Example 3.2.** If $D$ is the dualizing module for $R$, then $R$ is a $D$-canonical module.

**Remark 3.3.** If $R$ is a homomorphic image of an $n'$-dimensional Gorenstein local ring $R'$, then by the Local Duality Theorem ([3, Theorem 11.2.6]), $K = \text{Ext}^{n'-n}_{R'}(C, R')$ is a $C$-canonical module.

**Lemma 3.4.** *Let $R$ be a complete local ring. Then $C$ has a canonical module and any two $C$-canonical modules are isomorphic.*

**Proof.** By Remark 3.3, and Cohen's Structure Theorem [14, Theorem 29.4], $C$ has a canonical module. If $K$ and $K'$ are $C$-canonical modules, then

$$\mathrm{Hom}_R(K, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{Hom}_R(K', \mathrm{E}_R(R/\mathfrak{m})).$$

Now $K \cong K'$, by Matlis Duality Theorem [3, Theorem 10.2.12]. □

**Theorem 3.5.** *Let $K$ be a finitely generated $R$-module. Then $K$ is a $C$-canonical module if and only if $K \otimes_R \hat{R}$ is a $C \otimes_R \hat{R}$-canonical module.*

**Proof.** Note that $\mathrm{E}_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}}) \cong \mathrm{E}_R(R/\mathfrak{m}) \otimes_R \hat{R}$ and by [3, Theorem 4.3.2], $\mathrm{H}^n_{\mathfrak{m}}(C) \otimes_R \hat{R} \cong \mathrm{H}^n_{\hat{\mathfrak{m}}}(C \otimes_R \hat{R})$. Let $K$ be a $C$-canonical module. Then by [14, Theorem 7.11], and [3, Theorem 4.3.2], there are $\hat{R}$-isomorphisms

$$\begin{aligned} \mathrm{H}^n_{\hat{\mathfrak{m}}}(C \otimes_R \hat{R}) &\cong \mathrm{H}^n_{\mathfrak{m}}(C) \otimes_R \hat{R} \\ &\cong \mathrm{Hom}_R(K, \mathrm{E}_R(R/\mathfrak{m})) \otimes_R \hat{R} \\ &\cong \mathrm{Hom}_{\hat{R}}(K \otimes_R \hat{R}, \mathrm{E}_R(R/\mathfrak{m}) \otimes_R \hat{R}) \\ &\cong \mathrm{Hom}_{\hat{R}}(K \otimes_R \hat{R}, \mathrm{E}_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}})). \end{aligned}$$

Hence $K \otimes_R \hat{R}$ is a $C \otimes_R \hat{R}$-module.
Conversely, suppose that $K \otimes_R \hat{R}$ is a $C \otimes_R \hat{R}$-canonical $\hat{R}$-module. Therefore,

$$\mathrm{Hom}_{\hat{R}}(K \otimes_R \hat{R}, \mathrm{E}_{\hat{R}}(\hat{R}/\hat{\mathfrak{m}})) \cong \mathrm{H}^n_{\hat{\mathfrak{m}}}(C \otimes_R \hat{R}).$$

Using [14, Theorem 7.11], and [3, Theorem 4.3.2], we get

$$\mathrm{Hom}_R(K, \mathrm{E}_R(R/\mathfrak{m})) \otimes_R \hat{R} \cong \mathrm{H}^n_{\mathfrak{m}}(C) \otimes_R \hat{R}.$$

But both $R$-modules $\mathrm{Hom}_R(K, \mathrm{E}_R(R/\mathfrak{m}))$ and $\mathrm{H}^n_{\mathfrak{m}}(C)$ are Artinian, so that $\mathrm{Hom}_R(K, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(C)$ (See [3, Exercise 8.2.4]). Hence $K$ is a $C$-canonical module. □

**Theorem 3.6.** *Suppose that $K$ and $K'$ are two $C$-canonical modules. Then $K \cong K'$.*

**Proof.** By Theorem 3.5, $K \otimes_R \hat{R}$ and $K' \otimes_R \hat{R}$ are $C \otimes_R \hat{R}$-canonical modules, so that by Lemma 3.4, $K \otimes_R \hat{R} \cong K' \otimes_R \hat{R}$. Now the result follows from [12, Lemma 5.8]. □

**Remark 3.7.** Suppose that there exists a $C$-canonical module. By Theorem 3.6, this $C$-canonical module is unique up to isomorphism. We shall denote this module by $\omega_C$.

**Theorem 3.8.** (a) *If* $\omega_R$ *exists, then* $\operatorname{Hom}_R(C, \omega_R) \cong \omega_C$. *Moreover,* $\operatorname{Supp} \omega_R = \operatorname{Supp} \omega_C$.

(b) *Suppose that* $\omega_C$ *exists.*

($i$) $\operatorname{Ass}_R \omega_C = \{\mathfrak{p} \in Spec(R) : \dim R/\mathfrak{p} = n\}$.

($ii$) *Let* $\dim R = n > 0$, *and let* $a_1, \dots a_n$ *be a system of parameters for* $R$. *Then* $a_1$ *is a non-zerodivisor on* $\omega_C$ *and if* $n \geq 2$, *then* $a_1, a_2$ *is an* $\omega_C$*-sequence.*

**Proof.** (a) By tensoring both sides of $\operatorname{Hom}_R(\omega_R, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(R)$ with $C$, we get

$$C \otimes_R \operatorname{Hom}_R(\omega_R, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(R) \otimes_R C.$$

But by [3, Lemma 10.2.16],

$$C \otimes_R \operatorname{Hom}_R(\omega_R, \mathrm{E}_R(R/\mathfrak{m})) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, \omega_R), \mathrm{E}_R(R/\mathfrak{m}))$$

and $\mathrm{H}^n_{\mathfrak{m}}(R) \otimes_R C \cong \mathrm{H}^n_{\mathfrak{m}}(C)$ by [3, Exercise 6.1.10]. Next we show that $\operatorname{Supp} \omega_R = \operatorname{Supp} \omega_C$. Let $\mathfrak{p} \in \operatorname{Supp} \omega_R$. By Proposition 2.5 (iv), $C_\mathfrak{p}$ is a semidualizing $R_\mathfrak{p}$-module and by Proposition 2.5 (iii), $\operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, (\omega_R)_\mathfrak{p}) \neq 0$. Hence $(\omega_C)_\mathfrak{p} \neq 0$ and thus $\mathfrak{p} \in \operatorname{Supp} \omega_C$. The converse inclusion holds because

$$\operatorname{Supp} \omega_C = \operatorname{Supp} (\operatorname{Hom}_R(C, \omega_R)) \subseteq \operatorname{Supp} \omega_R.$$

(b) The proof is similar to [3, Theorem 12.1.9]. □

The following result, shows that under special conditions, the existence of $\omega_C$ guarantees the existence of $\omega_R$.

**Theorem 3.9.** *Suppose that* $\omega_C$ *exists. Then the following are equivalent:*

($i$) $\omega_C \in \mathcal{A}_C(R)$;

($ii$) $\mathrm{H}^n_{\mathfrak{m}}(R) \in \mathcal{A}_C(R)$;

($iii$) $\omega_R$ *exists and belongs to* $\mathcal{B}_C(R)$.

*If the above equivalent conditions hold, then* $\omega_R \cong C \otimes_R \omega_C$.

**Proof.** ($i$) $\Rightarrow$ ($ii$) One has

$$\operatorname{Hom}_R(\omega_C, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(C) \cong \mathrm{H}^n_{\mathfrak{m}}(R) \otimes_R C.$$

Thus by hypothesis and [21, Proposition 3.3.1], $C \otimes_R \mathrm{H}^n_{\mathfrak{m}}(R) \in \mathcal{B}_C(R)$ and by Theorem 2.8, $\mathrm{H}^n_{\mathfrak{m}}(R) \in \mathcal{A}_C(R)$, as desired.

*(ii)* $\Rightarrow$ *(iii)* Since $\mathrm{H}^n_{\mathfrak{m}}(R) \in \mathcal{A}_C(R)$, it follows that

$$\mathrm{H}^n_{\mathfrak{m}}(R) \cong \mathrm{Hom}_R(C, C \otimes_R \mathrm{H}^n_{\mathfrak{m}}(R)) \cong \mathrm{Hom}_R(C, \mathrm{H}^n_{\mathfrak{m}}(C)).$$

Therefore, using Hom-Tensor adjointness and by $\mathrm{Hom}_R(\omega_C, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(C)$, we can deduce that $\mathrm{Hom}_R(C \otimes_R \omega_C, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(R)$, which shows that $\omega_R$ exists and is isomorphic to $C \otimes_R \omega_C$. Also, $\omega_R \in \mathcal{B}_C(R)$ holds by $\mathrm{H}^n_{\mathfrak{m}}(R) \in \mathcal{A}_C(R)$ and [21, Proposition 3.3.1].
*(iii)* $\Rightarrow$ *(i)* This is clear by Theorems 3.8 and 2.8. $\square$

Let $u_R(0)$ be the intersection of the primary components $\mathfrak{q}$ of the zero ideal of $R$ for which $\dim R/\mathfrak{q} = n$.

**Proposition 3.10.** *Let $S(C)$ denote the set of all submodules of $C$. Set*

$$\Sigma := \{N \in S(C) \ : \dim N < n\}$$

$$= \{N \in S(C),\ N_{\mathfrak{p}} = 0 \quad \textit{for all } \mathfrak{p} \in \mathrm{Spec(R)} :\ \dim\ R/\mathfrak{p} = n\}.$$

*By* [3, Lemma 7.3.1]*, $\Sigma$ has a largest element $N'$. Then the following hold:*

(i) $u_R(0) = \mathrm{Ann}_R(C/N')$*;*

(ii) *If $\omega_C$ exists, then it is annihilated by $u_R(0)$;*

(iii) *Let $K$ be a finitely generated $R$-module which is annihilated by $u_R(0)$. Set $G := C/N'$ and $\overline{R} := R/u_R(0)$ and $\overline{\mathfrak{m}} := \mathfrak{m}/u_R(0)$. Then $K$ is a $C$-canonical module if and only if*

$$\mathrm{Hom}_{\overline{R}}(K, \mathrm{E}_{\overline{R}}(\overline{R}/\overline{\mathfrak{m}})) \cong \mathrm{H}^n_{\overline{\mathfrak{m}}}(G).$$

**Proof.** (i) If $\dim R/\mathfrak{p} = n$ then

$$\mathrm{Ann}_R(C/N')R_{\mathfrak{p}} = \mathrm{Ann}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}/N'_{\mathfrak{p}}) = \mathrm{Ann}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}) = 0$$

and the last equality holds because $C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-module. Hence we have $\dim(\mathrm{Ann}_R(C/N')) < n$ and by [3, Exercise 12.1.11], $\mathrm{Ann}_R(C/N') \subseteq u_R(0)$. For the converse inclusion, let $r \in u_R(0)$. We show that $\dim(rC) < n$. Suppose that $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\dim R/\mathfrak{p} = n$. Note that $rR_{\mathfrak{p}} = 0$ by [3, Exercise 12.1.11], and therefore $(rC)_{\mathfrak{p}} = rR_{\mathfrak{p}}C_{\mathfrak{p}} = 0$. This shows that $\dim(rC) < n$. Since $N'$ is the largest element of $\Sigma$, hence $rC \subseteq N'$. Therefore, $r(C/N') = 0$, and we have $u_R(0) \subseteq \mathrm{Ann}_R(C/N')$.

(ii) By [3, Lemma 7.3.1], we have $\mathrm{H}^n_{\mathfrak{m}}(C) \cong \mathrm{H}^n_{\mathfrak{m}}(G)$. But by part (i), $u_R(0)$ annihilates $G$, therefore it annihilates $\mathrm{H}^n_{\mathfrak{m}}(C)$. As $\mathrm{Hom}_R(\omega_C, \mathrm{E}_R(R/\mathfrak{m})) \cong \mathrm{H}^n_{\mathfrak{m}}(C)$, we conclude that $u_R(0)$ annihilates $\mathrm{Hom}_R(\omega_C, \mathrm{E}_R(R/\mathfrak{m}))$. By [3, Remark 10.2.2],

$\mathrm{Hom}_R(\omega_C, \mathrm{E}_R(R/\mathfrak{m}))$ and $\omega_C$ have the same annihilator, therefore $u_R(0)$ annihilates $\omega_C$.

(iii) Let $K$ be a finitely generated $R$-module which is annihilated by $u_R(0)$. Using ([3, Theorem 4.2.1]), and $\mathrm{H}^n_{\mathfrak{m}}(C) \cong \mathrm{H}^n_{\mathfrak{m}}(G)$, we get $\mathrm{H}^n_{\mathfrak{m}}(C) \cong \mathrm{H}^n_{\overline{\mathfrak{m}}}(G)$. Set $E := \mathrm{E}_R(R/\mathfrak{m})$. In view of [3, Lemma 10.1.16], we have

$$\begin{aligned}\mathrm{Hom}_R(K, E) &\cong \mathrm{Hom}_R(K \otimes_R \overline{R}, E)\\ &\cong \mathrm{Hom}_R(K, \mathrm{Hom}_R(\overline{R}, E))\\ &\cong \mathrm{Hom}_R(K, (0 :_E u_R(0)))\\ &\cong \mathrm{Hom}_{\overline{R}}((K, (0 :_E u_R(0)))\\ &\cong \mathrm{Hom}_{\overline{R}}(K, \mathrm{E}_{\overline{R}}(\overline{R}/\overline{m})).\end{aligned}$$

Thus there is an $R$-isomorphism $\mathrm{Hom}_R(K, E) \cong \mathrm{H}^n_{\mathfrak{m}}(C)$ if and only if there is an $\overline{R}$-isomorphism $\mathrm{Hom}_{\overline{R}}(K, \mathrm{E}_{\overline{R}}(\overline{R}/\overline{m})) \cong \mathrm{H}^n_{\overline{\mathfrak{m}}}(G)$. □

**Theorem 3.11.** *Suppose that $\omega_C$ exists. Then $(0 :_R \omega_C) = (0 :_R \mathrm{H}^n_{\mathfrak{m}}(C)) = u_R(0)$.*

**Proof.** By [3, Remark 10.2.2], we have $(0 :_R \omega_C) = (0 :_R \mathrm{H}^n_{\mathfrak{m}}(C))$. Note that $u_R(0) \subseteq (0 :_R \omega_C)$ by Proposition 3.10 (ii). First suppose that $R$ is a homomorphic image of a Gorenstein local ring $(R', \mathfrak{m}')$ and therefore by [3, Remark 12.1.14], we may assume that $\dim R' = n$. As $\omega_R$ exists, we have $\omega_C \cong \mathrm{Hom}_R(C, \omega_R)$. It is enough to show that $(0 :_R \omega_C)R_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R/\mathfrak{p} = n$. But

$$(0 :_R \omega_C)R_{\mathfrak{p}} = (0 :_{R_{\mathfrak{p}}} (\omega_C)_{\mathfrak{p}}) = (0 :_{R_{\mathfrak{p}}} \mathrm{Hom}(C, \omega_R)_{\mathfrak{p}}) = (0 :_{R_{\mathfrak{p}}} \mathrm{Hom}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}, (\omega_R)_{\mathfrak{p}})).$$

By [3, Theorem 12.1.18 (ii)], one has $(\omega_R)_{\mathfrak{p}} \cong \omega_{R_{\mathfrak{p}}}$. Note that $R_{\mathfrak{p}}$ is a 0-dimensional local ring and $\omega_{R_{\mathfrak{p}}}$ is the canonical module for $R_{\mathfrak{p}}$, so that $\omega_{R_{\mathfrak{p}}} \cong \mathrm{E}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$. Hence

$$(0 :_R \mathrm{Hom}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}, (\omega_R)_{\mathfrak{p}})) = (0 :_{R_{\mathfrak{p}}} \mathrm{Hom}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}, \mathrm{E}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})) = (0 :_{R_{\mathfrak{p}}} C_{\mathfrak{p}}) = 0.$$

We have proved the theorem when $R$ is a homomorphic image of a Gorenstein local ring. Now let $R$ be an arbitrary ring. An argument as in [3, Theorem 12.1.15], shows that $(0 :_R \omega_C) = u_R(0)$. □

**Theorem 3.12.** *Suppose that $\omega_C$ exists.*

(i) *One has*

$$\mathrm{Supp}\, \omega_C = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{ht}\, \mathfrak{p} + \dim R/\mathfrak{p} = n\};$$

*moreover, $\omega_C$ and $\mathrm{Hom}_R(\omega_C, \omega_C)$ satisfy the condition $S_2$.*

(ii) *If $R$ is a homomorphic image of a Gorenstein local ring, then* $(\omega_C)_\mathfrak{p} \cong \omega_{C_\mathfrak{p}}$ *for each* $\mathfrak{p} \in \operatorname{Supp} \omega_C$.

**Proof.** First suppose that $R$ is a homomorphic image of a Gorenstein local ring $R'$ with $\dim R' = n$. Hence $\omega_R$ and $\omega_C$ exist, one has $\omega_C \cong \operatorname{Hom}_R(C, \omega_R)$, and $\operatorname{Supp} \omega_R = \operatorname{Supp} \omega_C$ by Theorem 3.8 (a). Now by [3, Theorem 12.1.18], the result follows. Next we prove that $\omega_C$ and $\operatorname{Hom}_R(\omega_C, \omega_C)$ are $S_2$ $R$-modules. But let us first prove $(ii)$. Let $\mathfrak{p} \in \operatorname{Supp} \omega_C$. By [3, Theorem 12.1.18], $\omega_{R\mathfrak{p}}$ exists and we have $\omega_{R\mathfrak{p}} \cong (\omega_R)_\mathfrak{p}$. Hence

$$(\omega_C)_\mathfrak{p} \cong (\operatorname{Hom}_R(C, \omega_R))_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, (\omega_R)_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, \omega_{R_\mathfrak{p}}) \cong \omega_{C_\mathfrak{p}}.$$

It remains to show that $\omega_C$ and $\operatorname{Hom}_R(\omega_C, \omega_C)$ satisfy the condition $S_2$. Note that by Theorem 3.8 (b), $\operatorname{detph}_{R_\mathfrak{p}} (\omega_C)_\mathfrak{p} \geq \min\{2, \dim R_\mathfrak{p}\}$ as $(\omega_C)_\mathfrak{p}$ is a $C_\mathfrak{p}$-canonical module. Also, by [4, Exercise 1.4.19],

$$\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{Hom}_R((\omega_C)_\mathfrak{p}, (\omega_C)_\mathfrak{p}) \geq \min\{2, \operatorname{depth}_{R_\mathfrak{p}}(\omega_C)_\mathfrak{p}\} \geq \min\{2, \dim R_\mathfrak{p}\}.$$

Thus we have proved the theorem when $R$ is a homomorphic of a Gorenstein local ring. Now let $R$ be an arbitrary ring. Note that $\omega_{C\otimes_R \hat{R}} \cong \omega_C \otimes_R \hat{R}$ and $\hat{R}$ is a homomorphic image of a regular local ring, so that $\omega_C \otimes_R \hat{R}$ and $\operatorname{Hom}_{\hat{R}}(\omega_{C\otimes_R \hat{R}}, \omega_{C\otimes_R \hat{R}})$ are $S_2$ $\hat{R}$-modules. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. There exists a prime ideal of $\hat{R}$ lying over $\mathfrak{p}$. Suppose that $\beta$ is the minimal ideal among these primes. We claim that $\beta$ is a minimal prime ideal of $\mathfrak{p}\hat{R}$. To see this, suppose that $\mathfrak{p}\hat{R} \subseteq \mathfrak{q} \subseteq \beta$ for some $\mathfrak{q} \in \operatorname{Spec}(\hat{R})$. Then

$$\mathfrak{p} \subseteq (\mathfrak{p}\hat{R})^c \subseteq \mathfrak{q}^c \subseteq \beta^c = \mathfrak{p}.$$

Now minimality of $\beta$ implies that $\beta = \mathfrak{q}$ as required. Thus $\dim \hat{R}_\beta / \mathfrak{p}\hat{R}_\beta = 0$, so that by [14, Theorem 15.1], we have $\operatorname{ht}_{\hat{R}} \beta = \operatorname{ht}_R \mathfrak{p}$.

Consider the flat local ring homomorphism $R_\mathfrak{p} \longrightarrow \hat{R}_\beta$. By [14, Theorem 23.3],

$$\operatorname{depth}_{\hat{R}_\beta}(\operatorname{Hom}((\omega_C)_\mathfrak{p}, (\omega_C)_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \hat{R}_\beta) = \operatorname{depth}_{R_\mathfrak{p}} \operatorname{Hom}_{R_\mathfrak{p}}((\omega_C)_\mathfrak{p}, (\omega_C)_\mathfrak{p}).$$

Thus $\operatorname{depth}_{R_\mathfrak{p}} \operatorname{Hom}_{R_\mathfrak{p}}((\omega_C)_\mathfrak{p}, (\omega_C)_\mathfrak{p}) \geq \min\{2, \operatorname{ht}_{\hat{R}} \beta\} = \min\{2, \operatorname{ht}_R \mathfrak{p}\}$. With a similar argument for $(\omega_C)_\mathfrak{p}$ we see that $\omega_C$ is $S_2$. For the rest, let $\mathfrak{p} \in \operatorname{Spec}(R)$, so that we can choose $\beta$ as above. Using the $\hat{R}_\beta$-isomorphism

$$(\omega_C \otimes_R \hat{R}) \otimes_{\hat{R}} \hat{R}_\beta \cong (\omega_C \otimes_R R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \hat{R}_\beta$$

we have $\mathfrak{p} \in \operatorname{Supp} \omega_C$ if and only if $\beta \in \operatorname{Supp}_{\hat{R}} \omega_C \otimes_R \hat{R}$. Now suppose that $\mathfrak{p} \in \operatorname{Supp} \omega_C$. Then $\beta \in \operatorname{Supp}_{\hat{R}} \omega_C \otimes_R \hat{R}$. Thus $\operatorname{ht}_{\hat{R}} \beta + \dim_{\hat{R}} \hat{R}/\beta = n$. But $\dim \hat{R}/\beta = \dim R/\mathfrak{p}$ and $\operatorname{ht}_{\hat{R}} \beta = \operatorname{ht}_R \mathfrak{p}$. Hence $\operatorname{ht}_R \mathfrak{p} + \dim R/\mathfrak{p} = n$ if and only if $\operatorname{ht}_{\hat{R}} \beta + \dim_{\hat{R}} \hat{R}/\beta = n$. Thus $\beta \in \operatorname{Supp} \omega_C \otimes_R \hat{R}$ and therefore $\mathfrak{p} \in \operatorname{Supp} \omega_C$. $\square$

We end this section by asking two questions.

**Question 3.13.** *Suppose that $\omega_C$ exists for some semidualizing module. Can we conclude that $\omega_R$ exists?*

**Question 3.14.** *Suppose that $\omega_C \cong \omega_{C'}$ for two semidualizing modules $C$ and $C'$. Does this imply that $C \cong C'$?*

These questions will be answered when $R$ is a Cohen-Macaulay ring (see Theorem 4.1, and Corollary 4.5 (ii)).

## 4. The Cohen-Macaulay case

In this section, we present some results concerning $C$-canonical $R$-modules, when $R$ is a Cohen-Macaulay local ring and $C$ is a semidualizing module. By [3, Corollary 12.1.21], we know that if $\omega_R$ exists then it has finite injective dimension and $\omega_R$ is the dualizing module.

**Theorem 4.1.** *Let $R$ be a Cohen-Macaulay ring. The following are equivalent:*

(i) *$\omega_R$ exists;*
(ii) *$\omega_C$ exists for every semidualizing module $C$;*
(iii) *$\omega_C$ exists for some semidualizing module $C$.*

**Proof.** $(i) \Rightarrow (ii)$ This is clear by Theorem 3.8 (a).
$(ii) \Rightarrow (iii)$ It is obvious.
$(iii) \Rightarrow (i)$ We claim that $C \otimes_R \omega_C \cong \omega_R$. In view of Theorem 3.5, it is enough for us to show that $(C \otimes_R \omega_C) \otimes_R \hat{R} \cong \omega_{\hat{R}}$. But $(C \otimes_R \omega_C) \otimes_R \hat{R} \cong (C \otimes_R \hat{R}) \otimes_{\hat{R}} (\omega_C \otimes_R \hat{R})$ and by Theorem 3.5, $\omega_C \otimes_R \hat{R}$ is a canonical module for $\hat{R}$-semidualizing module $C \otimes_R \hat{R}$. Thus we assume that $R$ is complete, and therefore $\omega_R$ exists. Hence by Theorem 3.8 (a), $\omega_C \cong \mathrm{Hom}_R(C, \omega_R)$. Note that $R$ is Cohen-Macaulay and therefore $\mathrm{id}_R(\omega_R) < \infty$. It follows from Theorem 2.7 that $\omega_R \in \mathcal{B}_C(R)$, and therefore $C \otimes_R \omega_C \cong C \otimes_R \mathrm{Hom}_R(C, \omega_R) \cong \omega_R$. □

**Corollary 4.2.** *Let $R$ be a Cohen-Macaulay ring and suppose $\omega_C$ exists. Then $\mathcal{I}_C\text{-id}\,\omega_C < \infty$.*

**Proof.** By Theorem 4.1, and Theorem 3.8 (a), $\omega_R$ exists and $\omega_C$ is isomorphic to $\mathrm{Hom}_R(C, \omega_R)$. Now Theorem 2.6 implies that $\mathcal{I}_C\text{-id}\,\omega_C = \mathrm{id}_R\,\omega_R = n < \infty$. □

**Corollary 4.3.** *Let $R$ be a Cohen-Macaulay ring. The following are equivalent:*

(i) *$R$ is Gorenstein;*
(ii) *$\omega_C \cong C$ for every semidualizing module $C$;*

(iii) *$\omega_C \cong C$ for some semidualizing module $C$.*

**Proof.** $(i) \Rightarrow (ii)$ By [21, Corollary 4.1.11], the only semidualizing module over a Gorenstein local ring is $R$ itself and so $(ii)$ holds.
$(ii) \Rightarrow (iii)$ Is clear.
$(iii) \Rightarrow (i)$ By Corollary 4.2, we have $\mathcal{I}_C\text{-id}\, C < \infty$ and therefore $C \in \mathcal{A}_C(R)$ by Theorem 2.7. Hence $C$ is projective by [21, Corollary 4.3.2]. As $R$ is a local ring and $C$ is an indecomposable $R$-module, $C$ is a free $R$-module of rank 1 which shows that $C \cong R$. Therefore, $\text{id}_R\, R < \infty$. □

**Remark 4.4.** Let $C$ be a semidualizing $R$-module. If $R$ is a Cohen-Macaulay ring and $\omega_C$ exists, then by Theorem 4.1, $\omega_R$ exsits and $\omega_C \cong \text{Hom}_R(C, \omega_R)$. In this situation, by [21, Corollary 4.1.3], $\omega_C$ is a semidualizing $R$-module.

If $R$ is a Cohen-Macaulay local ring with the dualizing module $D$, then $D \cong \omega_R$ and $R \cong \omega_D$. Next we generalize this fact for any two semidualizing $R$-modules.

**Corollary 4.5.** *Let $R$ be a Cohen-Macaulay ring and $C$ and $C'$ be two semidualizing modules.*

(i) *One has $\omega_C \cong C'$ if and only if $\omega_{C'} \cong C$.*
(ii) *If $\omega_C \cong \omega_{C'}$, then $C \cong C'$.*

**Proof.** (i) Suppose that $\omega_C \cong C'$. Then $\text{Hom}_R(C, \omega_R) \cong C'$. Since $C$ is a maximal Cohen-Macaulay $R$-module, hence by [4, Theorem 3.3.10], we have

$$\begin{aligned} C &\cong \text{Hom}_R(\text{Hom}_R(C, \omega_R), \omega_R) \\ &\cong \text{Hom}_R(C', \omega_R) \cong \omega_{C'}. \end{aligned}$$

(ii) The proof is similar to $(i)$. □

Using Theorem 4.1, and [4, Theorem 3.3.10 and Proposition 3.3.11], one has the following result.

**Proposition 4.6.** *Let $R$ be a Cohen-Macaulay ring and suppose that $\omega_C$ exists. Then $\mu(\omega_C) = r(C)$ and $r(\omega_C) = \mu(C)$.*

**Lemma 4.7.** *Let $R$ be a Cohen-Macaulay ring and suppose that $\omega_C$ exists. Then any $R$-sequence $\underline{x}$ is an $\omega_C$-sequence and we have $\omega_C/\underline{x}\omega_C \cong \omega_{C/\underline{x}C}$.*

**Proof.** By Remark 4.4, $\omega_C$ is a semidualizing module. Hence any $R$-sequence is an $\omega_C$-sequence by Proposition 2.5 $(ii)$. For the rest, note that $\omega_C \cong \text{Hom}_R(C, \omega_R)$. Therefore,

$$\omega_C \otimes_R R/\underline{x} \cong \omega_C/\underline{x}\omega_C \cong R/\underline{x}R \otimes_R \text{Hom}_R(C, \omega_R).$$

By [4, Proposition 3.3.3], the right-hand side is isomorphic to $\mathrm{Hom}_{R/\underline{x}R}(C/\underline{x}C, \omega_R/\underline{x}\omega_R)$ and this is isomorphic to $\omega_{C/\underline{x}C}$. □

**Theorem 4.8.** *Let $R$ be a Cohen-Macaulay ring and suppose that $\omega_C$ exists. Let $N$ be a maximal Cohen-Macaulay $R$-module in $\mathcal{A}_C(R)$. Then:*

(i) $\mathrm{Hom}_R(N, \omega_C)$ *is a maximal Cohen-Macaulay $R$-module in $\mathcal{A}_C(R)$.*

(ii) $N \cong \mathrm{Hom}_R(\mathrm{Hom}_R(N, \omega_C), \omega_C)$ *(that is, $N$ is $\omega_C$-reflexive).*

(iii) *For any $R$-sequence $\underline{x}$ one has*

$$\mathrm{Hom}_R(N, \omega_C) \otimes_R R/\underline{x}R \cong \mathrm{Hom}_{R/\underline{x}R}(N/\underline{x}N, \omega_C/\underline{x}\omega_C).$$

**Proof.** (i) By Theorem 4.1, $\omega_R$ exists. Also, $\mathrm{Ext}^j_R(N, \omega_R) = 0$ for all $j > 0$ and $\mathrm{Hom}_R(N, \omega_R)$ is a maximal Cohen-Macaulay $R$-module by [4, Theorem 3.3.10]. Thus by [21, Proposition 3.3.16], we have $\mathrm{Hom}_R(N, \omega_R) \in \mathcal{B}_C(R)$. Since $\mathrm{Ext}^j_R(C, \mathrm{Hom}_R(N, \omega_R)) = 0$ for all $j > 0$, so that by [4, Proposition 3.3.3], $\mathrm{Hom}_R(C, \mathrm{Hom}_R(N, \omega_R))$ is a maximal Cohen-Macaulay $R$-module and by Theorem 2.8, it belongs to $\mathcal{A}_C(R)$. But this module is isomorphic to $\mathrm{Hom}_R(N, \mathrm{Hom}_R(C, \omega_R))$ $\cong \mathrm{Hom}_R(N, \omega_C)$. This completes the proof of (i).

(ii) Using $\omega_C \cong \mathrm{Hom}_R(C, \omega_R)$ and Hom-Tensor adjointness, one has

$$\begin{aligned}\mathrm{Hom}_R(\mathrm{Hom}_R(N, \omega_C), \omega_C) &\cong \mathrm{Hom}_R(C, \mathrm{Hom}_R(\mathrm{Hom}_R(N, \omega_C), \omega_R)) \\ &\cong \mathrm{Hom}_R(C, \mathrm{Hom}_R(\mathrm{Hom}_R(C \otimes_R N, \omega_R), \omega_R)).\end{aligned}$$

By [7, Lemma 2.11], $N \otimes_R C$ is a maximal Cohen-Macaulay $R$-module, so that by [4, Theorem 3.3.10 (d)],

$$C \otimes_R N \cong \mathrm{Hom}_R(\mathrm{Hom}_R(C \otimes_R N, \omega_R), \omega_R).$$

Hence

$$\mathrm{Hom}_R(\mathrm{Hom}_R(N, \omega_C), \omega_C) \cong \mathrm{Hom}_R(C, C \otimes_R N).$$

As $N \in \mathcal{A}_C(R)$, the right-hand side is isomorphic to $N$, as desired.

(iii) By [4, Proposition 3.3.3], and part (i), the following isomorphism holds:

$$\mathrm{Hom}_R(C, \mathrm{Hom}_R(N, \omega_R)) \otimes_R R/\underline{x}R \cong \\ \mathrm{Hom}_{R/\underline{x}R}(C/\underline{x}C, \mathrm{Hom}_R(N, \omega_R)/\underline{x}\mathrm{Hom}_R(N, \omega_R)).$$

The left-hand side is isomorphic to $\mathrm{Hom}_R(N, \omega_C) \otimes_R R/\underline{x}R$. Again by [4, Proposition 3.3.3], we conclude that the right-hand side is isomorphic to

$$\mathrm{Hom}_{R/\underline{x}R}(C/\underline{x}C, \mathrm{Hom}_{R/\underline{x}R}(N/\underline{x}N, \omega_R/\underline{x}\omega_R))$$

and by using Hom-Tensor adjointness, the later is isomorphic to

$$\mathrm{Hom}_{R/\underline{x}R}(N/\underline{x}N, \mathrm{Hom}_{R/\underline{x}R}(C/\underline{x}C, \omega_R/\underline{x}\omega_R)) \cong \mathrm{Hom}_{R/\underline{x}R}(N/\underline{x}N, \omega_C/\underline{x}\omega_C)$$

where the last isomorphism is obtained from Lemma 4.7. □

**Theorem 4.9.** *Let $R$ be a Cohen-Macaulay ring and suppose that $\omega_C$ exists and $M$ is a maximal Cohen-Macaulay $R$-module with $\mathcal{I}_C$-id$(M) < \infty$. Then $M \cong \bigoplus_{i=1}^t \omega_C$ for some positive number $t$.*

**Proof.** By Theorem 2.6, we have $\mathrm{id}_R(C \otimes_R M) < \infty$ and $M \in \mathcal{A}_C(R)$ by Theorem 2.7. Also, $C \otimes_R M$ is a maximal Cohen-Macaulay $R$-module by [7, Lemma 2.11]. Hence by [17, Theorem 2.1(v)], we get $C \otimes_R M \cong \bigoplus_{i=1}^t \omega_R$. Applying $\mathrm{Hom}_R(C, -)$ on both sides, we get

$$\mathrm{Hom}_R(C, C \otimes_R M) \cong \bigoplus_{i=1}^t \mathrm{Hom}_R(C, \omega_R) \cong \bigoplus_{i=1}^t \omega_C.$$

But the left-hand side is isomorphic to $M$ because $M \in \mathcal{A}_C(R)$. □

**Remark 4.10.** According to [4, Definition 3.3.1], the canonical module over a Cohen-Macaulay local ring, is a maximal Cohen-Macaulay R-module $K$ with finite injective dimension of type 1. By the next theorem, one may define a $C$-canonical module as a maximal Cohen-Macaulay $R$-module with $r_R(K) = \mu(C)$ and $\mathcal{I}_C$-id$(K) < \infty$.

**Theorem 4.11.** *Let $R$ be a Cohen-Macaulay ring and $K$ be a finitely generated $R$-module. Then $K$ is a $C$-canonical module if and only if $K$ is a maximal Cohen-Macaulay module with $r_R(K) = \mu(C)$ and $\mathcal{I}_C$-id$(K) < \infty$.*

**Proof.** ($\Rightarrow$) This is clear by Corollary 4.2, Remark 4.4, and Proposition 4.6.
($\Leftarrow$) Note that $K \otimes_R \hat{R}$ is a maximal Cohen-Macaulay $\hat{R}$-module and

$$\mu(C) = r_R(K) = r_{\hat{R}}(K \otimes_R \hat{R}) = \mu(C \otimes_R \hat{R}).$$

By using Theorem 2.6, the following equalities hold:

$$\begin{aligned}\mathcal{I}_{C\otimes_R\hat{R}}\text{-id}(K \otimes_R \hat{R}) &= \mathrm{id}_{\hat{R}}((K \otimes_R \hat{R}) \otimes_{\hat{R}} (C \otimes_R \hat{R})) \\ &= \mathrm{id}_{\hat{R}}((K \otimes_R C) \otimes_R \hat{R}) \\ &= \mathrm{id}_R(K \otimes_R C) \\ &= \mathcal{I}_C\text{-id}(K) < \infty.\end{aligned}$$

Hence $I_{C\otimes_R\hat{R}}$-id$(K \otimes_R \hat{R}) < \infty$. As $\omega_{C\otimes_R\hat{R}}$ exists, it is enough for us to show that $K \otimes_R \hat{R} \cong \omega_{C\otimes_R\hat{R}}$ by Theorem 3.5. Thus we may assume that $R$ is complete and therefore $\omega_C$ exists. By Theorem 4.9, $K \cong \bigoplus_{i=1}^{t} \omega_C$ for some positive number $t$, so that $r_R(K) = t\, r(\omega_C) = t\, \mu(C)$. But $r_R(K) = \mu(C) = r(\omega_C)$ which implies that $t = 1$ and therefore $K \cong \omega_C$. □

The following result is a generalization of [4, Proposition 3.3.13].

**Theorem 4.12.** *Let $R$ be a Cohen-Macaulay ring and $C$ be a semidualizing $R$-module. For any $R$-module $Q$, the following conditions are equivalent.*

(*i*) $Q \cong \omega_C$.

(*ii*) *$Q \in \mathcal{A}_C(R)$ is a maximal Cohen-Macaulay faithful $R$-module and $r_R(Q) = \mu(C)$.*

**Proof.** $(i) \Rightarrow (ii)$. By remark 4.4, $\omega_C$ is a semidualizing $R$-module. Therefore, it is a maximal Cohen-Macaulay faithful $R$-module. Also, by Proposition 4.6, $r(\omega_C) = \mu(C)$.

$(ii) \Rightarrow (i)$. We claim that $C \otimes_R Q \cong \omega_R$. One can use [21, Proposition 3.4.7], and the exact sequence $0 \to R \xrightarrow{x} R \to R/xR \to 0$ to see that $Q/xQ \in \mathcal{A}_{C/xC}(R/xR)$ for any non-zero divisor $x$, and so that for any $R$-sequence $\underline{x}$, $Q/\underline{x}Q \in \mathcal{A}_{C/\underline{x}C}(R/\underline{x}R)$. Note that $C \otimes_R Q$ is a maximal Cohen-Macaulay $R$-module by [7, Lemma 2.11]. First we prove that $r_R(C \otimes_R Q) = 1$. Let $\underline{x}$ be a maximal $R$-sequence. Note that $r_R(Q) = v\dim_k \mathrm{Ext}_R^n(k, Q)$ and the later is equal to $v\dim_k \mathrm{Hom}_{R/\underline{x}}(k, Q/\underline{x}Q)$. Set $\overline{R} := R/\underline{x}, \overline{Q} := Q/\underline{x}Q$, $\overline{C} := C/\underline{x}C$ and $\overline{C \otimes_R Q} := (C \otimes_R Q)/\underline{x}(C \otimes_R Q)$. By using Hom-Tensor adjointness, [4, Lemma 3.1.16], and $\overline{Q} \in A_{\overline{C}}(\overline{R})$, we have the following equalities:

$$
\begin{aligned}
r_R(Q) &= v\dim_k \mathrm{Hom}_{\overline{R}}(k, \mathrm{Hom}_{\overline{R}}(\overline{C}, \overline{C} \otimes_{\overline{R}} \overline{Q})) \\
&= v\dim_k \mathrm{Hom}_{\overline{R}}(k, \mathrm{Hom}_{\overline{R}}(\overline{C}, \overline{R} \otimes_R (C \otimes_R Q)) \\
&= v\dim_k \mathrm{Hom}_{\overline{R}}(k \otimes_{\overline{R}} \overline{C}, \overline{C \otimes_R Q}) \\
&= v\dim_k \mathrm{Hom}_{\overline{R}}(\overline{C}, \mathrm{Hom}_{\overline{R}}(k, \overline{C \otimes_R Q})) \\
&= \mu_{\overline{R}}(\overline{C}) \cdot r_{\overline{R}}(\overline{C \otimes_R Q}) \\
&= \mu_R(C) \cdot r_R(C \otimes_R R).
\end{aligned}
$$

By hypothesis $r_R(Q) = \mu(C)$, thus $r_R(C \otimes_R Q) = 1$. On the other hand $Q \cong \mathrm{Hom}_R(C, C \otimes_R Q)$. As Q is a faithful R-module we get $C \otimes_R Q$ is faithful. Hence by [4, Proposition 3.3.13], $\omega_R$ exists and is isomorphic to $C \otimes_R Q$. Therefore,

$$Q \cong \mathrm{Hom}_R(C, C \otimes_R Q) \cong \mathrm{Hom}_R(C, \omega_R) \cong \omega_C.$$

□

By using Theorem 4.12 and Corollary 4.5, we have the following result.

**Corollary 4.13.** *Let $R$ be a Cohen-Macaulay ring. If $r_R(R) = \mu(C)$, then $C$ is the canonical module of $R$.*

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