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DOMINATION NUMBER IN THE ANNIHILATING-SUBMODULE GRAPH OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Let M be a module over a commutative ring R. The annihilating-submodule graph of M, denoted by $AG(M)$, is a simple undirected graph in which a non-zero submodule N of M is a vertex if and only if there exists a non-zero proper submodule K of M such that $NK = (0)$, where NK, the product of N and K, is denoted by $(N : M)(K : M)M$ and two distinct vertices N and K are adjacent if and only if $NK = (0)$. This graph is a submodule version of the annihilating-ideal graph and under some conditions, is isomorphic with an induced subgraph of the Zariski topology-graph $G(\tau_T)$ which was introduced in [H. Ansari-Toroghy and S. Habibi, Comm. Algebra, 42(2014), 3283-3296]. In this paper, we study the domination number of $AG(M)$ and some connections between the graph-theoretic properties of $AG(M)$ and algebraic properties of module M.

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1. Introduction

Throughout this paper R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M.

Define $(N :_R M)$ or simply $(N : M) = \{r \in R|\ rM \subseteq N\}$ for any $N \leq M$. We denote $((0) : M)$ by $Ann_R(M)$ or simply $Ann(M)$. M is said to be faithful if $Ann(M) = (0)$. Let $N, K \leq M$. Then the product of N and K, denoted by NK, is defined by $(N : M)(K : M)M$ (see [6]). Define $ann(N)$ or simply $annN = \{m \in M|\ m(N : M) = 0\}$.

The prime spectrum of M is the set of all prime submodules of M and denoted by $Spec(M)$, $Max(M)$ is the set of all maximal submodules of M, and $J(M)$, the jacobson radical of M, is the intersection of all elements of $Max(M)$, respectively.

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There are many papers on assigning graphs to rings or modules (see, for example, [5,7,13,14]). The annihilating-ideal graph $AG(R)$ was introduced and studied in [14]. $AG(R)$ is a graph whose vertices are ideals of R with nonzero annihilators and in which two vertices I and J are adjacent if and only if $IJ = (0)$. Later, it was modified and further studied by many authors (see [1,2,3,20,22]).

In [7], the present authors introduced and studied the graph $G(\tau_T)$ (resp. $AG(M)$), called the *Zariski topology-graph* (resp. *the annihilating-submodule graph*), where T is a non-empty subset of $Spec(M)$.

$AG(M)$ is an undirected simple graph with vertices $V(AG(M))= \{N \leq M|$ there exists $(0) \neq K < M$ with $NK = (0)\}$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$ (see [8,9]). Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M$ with $(N : M) \neq Ann(M)|$ there exists a submodule $K < M$ with $(K : M) \neq Ann(M)$ and $NK = (0)\}$. By [7, Theorem 3.4], one conclude that $AG(M)^*$ is a connected subgraph. Note that M is a vertex of $AG(M)$ if and only if there exists a nonzero proper submodule N of M with $(N : M) = Ann(M)$ if and only if every nonzero submodule of M is a vertex of $AG(M)$. Clearly, if M is not a vertex of $AG(M)$, then $AG(M) = AG(M)^*$. In [10, Lemma 2.8], we showed that under some conditions, $AG(M)$ is isomorphic with an induced subgraph of the Zariski topology-graph $G(\tau_T)$.

In this paper, we study the domination number of $AG(M)$ and some connections between the graph-theoretic properties of $AG(M)$ and algebraic properties of module M.

A prime submodule of M is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [18].

The notations $Z(R)$ and $Nil(R)$ will denote the set of all zero-divisors, the set of all nilpotent elements of R, respectively. Also, $Z_R(M)$ or simply $Z(M)$, the set of zero divisors on M, is the set $\{r \in R|\ rm = 0$ for some $0 \neq m \in M\}$. If $Z(M) = 0$, then we say that M is a domain. An ideal $I \leq R$ is said to be nil if I consist of nilpotent elements.

Let us introduce some graphical notions and denotations that are used in what follows: A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. The number of edges incident at x in G is called the degree of the vertex x in G and is denoted by $d_G(x)$ or simply $d(x)$. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i. The distance between two vertices x and y, denoted $d(x,y)$, is the length of the shortest path from x to y. The diameter of a connected graph G is the maximum distance between two distinct vertices of G. For any vertex x of a connected graph G, the eccentricity of x, denoted $e(x)$, is the maximum of the distances from x to the other vertices of G. The set of vertices with minimum eccentricity is called the center of the graph G, and this minimum eccentricity value is the radius of G. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \setminus U$ adjacent to at least one vertex of U and $N[U] = N(U) \cup \{U\}$.

A graph H is a subgraph of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A subgraph H of G is a spanning subgraph of G if $V(H) = V(G)$. A spanning subgraph H of G is called a perfect matching of G if every vertex of G has degree 1. A subset S of the vertex set $V(G)$ is called independent if no two vertices of S are adjacent in G.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $cl(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$.

A graph G is a split graph if $V(G)$ can be partitioned into two subsets A and B such that the subgraph induced by A in G is a clique in G, and B is an independent subset of $V(G)$.

A subset D of $V(G)$ is called a dominating set if every vertex of G is either in D or adjacent to at least one vertex in D. The domination number of G, denoted by $\gamma(G)$, is the number of vertices in a smallest dominating set of G. A total dominating set of a graph G is a dominating set S such that every vertex is adjacent to a vertex in S. The total domination number of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A dominating set of cardinality $\gamma(G)$ ($\gamma_t(G)$) is called a γ-set (γ_t-set). A dominating set D is a connected dominating set if the subgraph $< D >$ induced by D is a connected subgraph of G. The connected domination number of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of G. A dominating set D is a clique dominating set if the subgraph $< D >$ induced by D is complete in G. The clique domination number $\gamma_{cl}(G)$ of G equals the minimum cardinality of a clique dominating set of G. A dominating set D is a paired-dominating set if the subgraph $< D >$ induced by D has a perfect matching. The paired-domination number $\gamma_{pr}(G)$ of G equals the minimum cardinality of a paired-dominating set of G.

A vertex u is a neighbor of v in G, if uv is an edge of G, and $u \neq v$. The set of all neighbors of v is the open neighborhood of v or the neighbor set of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v in G.

Let S be a dominating set of a graph G, and $u \in S$. The private neighborhood of u relative to S in G is the set of vertices which are in the closed neighborhood of u, but not in the closed neighborhood of any vertex in $S \setminus \{u\}$. Thus the private neighborhood $P_N(u, S)$ of u with respect to S is given by $P_N(u, S) = N[u] \setminus (\cup_{v \in S \setminus \{u\}} N[v])$. A set $S \subseteq V(G)$ is called irredundant if every vertex v of S has at least one private neighbor. An irredundant set S is a maximal irredundant set if for every vertex $u \in V \setminus S$, the set $S \cup \{u\}$ is not irredundant. The irredundance number $ir(G)$ is the minimum cardinality of maximal irredundant sets. There are so many domination parameters in the literature and for more details one can refer [16].

A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is, U and V are each independent sets and is denoted by $B_{n,m}$, where V and U are of size n and m, respectively. A complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. We denote by C_n and P_n a cycle and a path of order n, respectively (see [15]).

In Section 2, a dominating set of $AG(M)$ is constructed using elements of the center when M is an Artinian module. Also we prove that the domination number of $AG(M)$ is equal to the number of factors in the Artinian decomposition of M and we also find several domination parameters of $AG(M)$. In Section 3, some relations between the domination numbers and the total domination numbers of annihilating-submodule graphs are studied. Also, we study the domination number of the annihilating-submodule graphs for reduced rings with finitely many minimal primes and faithful modules.

The following results are useful for further reference in this paper.

Proposition 1.1. *Suppose that e is an idempotent element of R. We have the following statements.*

(a) $R = R_1 \times R_2$, *where* $R_1 = eR$ *and* $R_2 = (1 - e)R$.
(b) $M = M_1 \times M_2$, *where* $M_1 = eM$ *and* $M_2 = (1 - e)M$.
(c) *For every submodule N of M, $N = N_1 \times N_2$ such that N_1 is an R_1-submodule M_1, N_2 is an R_2-submodule M_2, and* $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$.

(d) *For submodules N and K of M, $NK = N_1K_1 \times N_2K_2$ such that $N = N_1 \times N_2$ and $K = K_1 \times K_2$.*

(e) *Prime submodules of M are $P \times M_2$ and $M_1 \times Q$, where P and Q are prime submodules of M_1 and M_2, respectively.*

Proof. This is clear. □

We need the following results.

Lemma 1.2. (See [4, Proposition 7.6].) *Let $R_1, R_2, \ldots, R_n$ be non-zero ideals of R. Then the following statements are equivalent:*

(a) $R = R_1 \times \ldots \times R_n$;

(b) *As an abelian group R is the direct sum of $R_1, \ldots, R_n$;*

(c) *There exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ with $1 = e_1 + \ldots + e_n$, and $R_i = Re_i$, $i = 1, \ldots, n$.*

Lemma 1.3. (See [17, Theorem 21.28].) *Let I be a nil ideal in R and $u \in R$ be such that $u + I$ is an idempotent in R/I. Then there exists an idempotent e in uR such that $e - u \in I$.*

Lemma 1.4. (See [9, Lemma 2.4].) *Let N be a minimal submodule of M and let $Ann(M)$ be a nil ideal. Then we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.*

Proposition 1.5. *Let $R/Ann(M)$ be an Artinian ring and let M be a finitely generated module. Then every nonzero proper submodule N of M is a vertex in $AG(M)$.*

Theorem 1.6. (See [9, Theorem 2.5].) *Let $Ann(M)$ be a nil ideal. There exists a vertex of $AG(M)$ which is adjacent to every other vertex if and only if $M = eM \oplus (1-e)M$, where eM is a simple module and $(1-e)M$ is a prime module for some idempotent $e \in R$, or $Z(M) = Ann((N : M)M)$, where N is a nonzero proper submodule of M or M is a vertex of $AG(M)$.*

Theorem 1.7. (See [9, Theorem 3.3].) *Let M be a faithful module. Then the following statements are equivalent.*

(a) $\chi(AG(M)^*) = 2$.

(b) *$AG(M)^*$ is a bipartite graph with two nonempty parts.*

(c) *$AG(M)^*$ is a complete bipartite graph with two nonempty parts.*

(d) *Either R is a reduced ring with exactly two minimal prime ideals, or $AG(M)^*$ is a star graph with more than one vertex.*

Corollary 1.8. (See [9, Corollary 3.5].) *Let R be a reduced ring and assume that M is a faithful module. Then the following statements are equivalent.*

(a) $\chi(AG(M)^*) = 2$.
(b) $AG(M)^*$ *is a bipartite graph with two nonempty parts.*
(c) $AG(M)^*$ *is a complete bipartite graph with two nonempty parts.*
(d) *R has exactly two minimal prime ideals.*

Theorem 1.9. (See [8, Theorem 2.7].) *If $AG(M)$ is a tree, then either $AG(M)$ is a star graph or $AG(M) \cong P_4$. Moreover, $AG(M) \cong P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule.*

Proposition 1.10. (See [16, Proposition 3.9].) *Every minimal dominating set in a graph G is a maximal irredundant set of G.*

2. Domination number in the annihilating-submodule graph for Artinian modules

The main goal in this section, is to obtain the value certain domination parameters of the annihilating-submodule graph for Artinian modules.

Recall that M is a vertex of $AG(M)$ if and only if there exists a nonzero proper submodule N of M with $(N : M) = Ann(M)$ if and only if every nonzero submodule of M is a vertex of $AG(M)$. In this case, the vertex N is adjacent to every other vertex. Hence $\gamma(AG(M)) = 1$ and $\gamma_t(AG(M)) = 2$. So we assume that **throughout this paper** M **is not a vertex of** $AG(M)$. Clearly, if M is not a vertex of $AG(M)$, **then** $AG(M) = AG(M)^*$**.**

We start with the following remark which completely characterizes all modules for which $\gamma(AG(M)) = 1$.

Remark 2.1. Let $Ann(M)$ be a nil ideal. By Theorem 1.6, there exists a vertex of $AG(M)$ which is adjacent to every other vertex if and only if $M = eM \oplus (1-e)M$, where eM is a simple module and $(1-e)M$ is a prime module for some idempotent $e \in R$, or $Z(M) = Ann((N : M)M)$, where N is a nonzero proper submodule of M or M is a vertex of $AG(M)$. Now, let $Ann(M)$ be a nil ideal and M be a domain module. Then $\gamma(AG(M)) = 1$ if and only if $M = eM \oplus (1-e)M$, where eM is a simple module and $(1-e)M$ is a prime module for some idempotent $e \in R$.

Theorem 2.2. *Let M be a finitely generated Artinian local module. Assume that N is the unique maximal submodule of M. Then the radius of $AG(M)$ is* 0 *or* 1 *and the center of $AG(M)$ is $\{K \subseteq ann(N) \mid K \neq (0)$ is a submodule in $M\}$.*

Proof. If N is the only non-zero proper submodule of M, then $AG(M) \cong K_1$, $e(N) = 0$ and the radius of $AG(M)$ is 0. Assume that the number of non-zero proper submodules of M is greater than 1. Since M is finitely generated Artinian module, there exists $m \in \mathbb{N}$, $m > 1$ such that $N^m = (0)$ and $N^{m-1} \neq (0)$. For any non-zero submodule K of M, $KN^{m-1} \subseteq NN^{m-1} = (0)$ and so $d(N^{m-1}, K) = 1$. Hence $e(N^{m-1}) = 1$ and so the radius of $AG(M)$ is 1. Suppose K and L are arbitrary non-zero submodules of M and $K \subseteq ann(N)$. Then $KL \subseteq KN = (0)$ and hence $e(K) = 1$. Suppose $(0) \neq K' \nsubseteq ann(N)$. Then $K'N \neq (0)$ and so $e(K') > 1$. Hence the center of $AG(M)$ is $\{K \subseteq ann(N) | \ K \neq (0)$ is a submodule in $M\}$. □

Corollary 2.3. *Let M be a finitely generated Artinian local module and N is the unique maximal submodule of M. Then the following hold good.*

(a) $\gamma(AG(M)) = 1$.

(b) *D is a γ-set of $AG(M)$ if and only if $D \subseteq ann(N)$.*

Proof. (a) Trivial from Theorem 2.2.
(b) Let $D = \{K\}$ be a γ-set of $AG(M)$. Suppose $K \nsubseteq ann(N)$. Then $KN \neq (0)$ and so N is not dominated by K, a contradiction. Conversely, suppose $D \subseteq ann(N)$. Let K be an arbitrary vertex in $AG(M)$. Then $KL \subseteq NL = (0)$ for every $L \in D$. i.e., every vertex K is adjacent to every $L \in D$. If $|D| > 1$, then $D \setminus \{L'\}$ is also a dominating set of $AG(M)$ for some $L' \in D$ and so D is not minimal. Thus $|D| = 1$ and so D is a γ-set by (a). □

Theorem 2.4. *Let $M = \oplus_{i=1}^n M_i$, where M_i is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then the radius of $AG(M)$ is 2 and the center of $AG(M)$ is $\{K \subseteq J(M) | \ K \neq (0)$ is a submodule in $M\}$.*

Proof. Let $M = \oplus_{i=1}^n M_i$, where M_i is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Let J_i be the unique maximal submodule in M_i with nilpotency n_i. Note that $Max(M) = \{N_1, \ldots, N_n | \ N_i = M_1 \oplus \ldots \oplus M_{i-1} \oplus J_i \oplus M_{i+1} \oplus \ldots \oplus M_n, 1 \leq i \leq n\}$ is the set of all maximal submodules in M. Consider $D_i = (0) \oplus \ldots \oplus (0) \oplus J_i^{n_i - 1} \oplus (0) \oplus \ldots \oplus (0)$ for $1 \leq i \leq n$. Note that $J(M) = J_1 \oplus \ldots \oplus J_n$ is the Jacobson radical of M and any non-zero submodule in M is adjacent to D_i for some i. Let K be any non-zero submodule of M. Then $K = \oplus_{i=1}^n K_i$, where K_i is a submodule of M_i.
Case 1. If $K = N_i$ for some i, then $KD_j \neq (0)$ and $KN_j \neq (0)$ for all $j \neq i$. Note that $N(K) = \{(0) \oplus \ldots \oplus (0) \oplus L_i \oplus (0) \oplus \ldots \oplus (0) | \ J_i L_i = (0)$, L_i is a nonzero submodule in $M_i\}$. Clearly $N(K) \cap N(N_j) = (0)$, $d(K, N_j) \neq 2$ for all $j \neq i$, and so $K - D_i - D_j - N_j$ is a path in $AG(M)$. Therefore $e(K) = 3$ and so $e(N) = 3$

for all $N \in Max(M)$.

Case 2. If $K \neq D_i$ and $K_i \subseteq J_i$ for all i. Then $KD_i = (0)$ for all i. Let L be any non-zero submodule of M with $KL \neq (0)$. Then $LD_j = (0)$ for some j, $K - D_j - L$ is a path in $AG(M)$ and so $e(K) = 2$.

Case 3. If $K_i = M_i$ for some i, then $KD_i \neq (0)$, $KN_i \neq (0)$ and $KD_j = (0)$ for some $j \neq i$. Thus $K - D_j - D_i - N_i$ is a path in $AG(M)$, $d(K, N_i) = 3$ and so $e(K) = 3$. Thus $e(K) = 2$ for all $K \subseteq J(M)$. Further note that in all the cases center of $AG(M)$ is $\{K \subseteq J(M) | \ K \neq (0) \text{ is a submodule in } M\}$. □

In view of Theorems 2.2 and 2.4, we have the following corollary.

Corollary 2.5. *Let $M = \oplus_{i=1}^{n} M_i$, where M_i is a simple module for all $1 \leq i \leq n$ and $n \geq 2$. Then the radius of $AG(M)$ is 1 or 2 and the center of $AG(M)$ is $\cup_{i=1}^{n} D_i$, where $D_i = (0) \oplus \ldots \oplus (0) \oplus M_i \oplus (0) \oplus \ldots \oplus (0)$ for $1 \leq i \leq n$.*

Proposition 2.6. *Let $M = \oplus_{i=1}^{n} M_i$, where M_i is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$ ($M \neq M_1 \oplus M_2$, where M_1 and M_2 are simple modules). Then*

(a) $\gamma(AG(M)) = n$.
(b) $ir(AG(M)) = n$.
(c) $\gamma_c(AG(M)) = n$.
(d) $\gamma_t(AG(M)) = n$.
(e) $\gamma_{cl}(AG(M)) = n$.
(f) $\gamma_{pr}(AG(M)) = n$, *if n is even and* $\gamma_{pr}(AG(M)) = n + 1$, *if n is odd.*

Proof. Let J_i be the unique maximal submodule in M_i with nilpotency n_i. Let $\Omega = \{D_1, D_2, \ldots, D_n\}$, where $D_i = (0) \oplus \ldots \oplus (0) \oplus J_i^{n_i - 1} \oplus (0) \oplus \ldots \oplus (0)$ for $1 \leq i \leq n$. Note that any non-zero submodule in M is adjacent to D_i for some i. Therefore $N[\Omega] = V(AG(M))$, Ω is a dominating set of $AG(M)$ and so $\gamma(AG(M)) \leq n$. Suppose S is a dominating set of $AG(M)$ with $|S| < n$. Then there exists $N \in Max(M)$ such that $NK \neq (0)$ for all $K \in S$, a contradiction. Hence $\gamma(AG(M)) = n$. By Proposition 1.10, Ω is a maximal irredundant set with minimum cardinality and so $ir(AG(M)) = n$. Clearly $< \Omega >$ is a complete subgraph of $AG(M)$. Hence $\gamma_c(AG(M)) = \gamma_t(AG(M)) = \gamma_{cl}(AG(M)) = n$. If n is even, then $< \Omega >$ has a perfect matching and so Ω is a paired-dominating set of $AG(M)$. Thus $\gamma_{pr}(AG(M)) = n$. If n is odd, then $< \Omega \cup K >$ has a perfect matching for some $K \in V(AG(M)) \setminus \Omega$. and so $\Omega \cup K$ is a paired-dominating set of $AG(M)$. Thus $\gamma_{pr}(AG(M)) = n$ if n even and $\gamma_{pr}(AG(M)) = n + 1$ if n is odd. □

Let $M = \oplus_{i=1}^{n} M_i$, where M_i is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Then by Theorem 2.4, radius of $AG(M)$ is 2. Further, by Proposition 2.6, the domination number of $AG(M)$ is equal to n, where n is the number of distinct maximal submodules of M. However, this need not be true if the radius of $AG(M)$ is 1. For, consider the ring $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules. Then $AG(M)$ is a star graph and so has radius 1, whereas M has two distinct maximal submodules. The following corollary shows that a more precise relationship between the domination number of $AG(M)$ and the number of maximal submodules in M, when M is finite.

Corollary 2.7. *Let M be a finitely generated Artinian module, M is a faithful module, and $\gamma(AG(M)) = n$. Then either $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules or M has n maximal submodules.*

Proof. When $\gamma(AG(M)) = 1$, the proof follows from [9, Corollary 2.12]. If $\gamma(AG(M)) = n$, where $n \geq 2$, then M can not be $M = M_1 \oplus M_2$, where M_1 and M_2 are simple modules. Hence $M = \oplus_{i=1}^{m} M_i$, where M_i is a finitely generated Artinian local module for all $1 \leq i \leq m$ and $m \geq 2$. By Proposition 2.6, $\gamma(AG(M)) = m$. Hence by assumption $m = n$, i.e., $M = \oplus_{i=1}^{n} M_i$, where M_i is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. One can see now that M has n maximal submodules. □

3. The relationship between $\gamma_t(AG(M))$ and $\gamma(AG(M))$

The main goal in this section is to study the relation between $\gamma_t(AG(M))$ and $\gamma(AG(M))$.

Theorem 3.1. *Let M be a module. Then*

$$\gamma_t(AG(M)) = \gamma(AG(M)) \text{ or } \gamma_t(AG(M)) = \gamma(AG(M)) + 1.$$

Proof. Assume that $\gamma_t(AG(M)) \neq \gamma(AG(M))$ and D is a γ-set of $AG(M)$. If $\gamma(AG(M)) = 1$, then it is clear that $\gamma_t(AG(M)) = 2$. So let $\gamma(AG(M)) > 1$ and put $k = Max\{n|$ there exist $L_1, \ldots, L_n \in D$ such that $\sqcap_{i=1}^{n} L_i \neq 0\}$. Since $\gamma_t(AG(M)) \neq \gamma(AG(M))$, we have $k \geq 2$. Let $L_1, \ldots, L_k \in D$ be such that $\sqcap_{i=1}^{k} L_i \neq 0$. Then $S = \{\sqcap_{i=1}^{k} L_i, annL_1, \ldots, annL_k\} \cup D \setminus \{L_1, \ldots, L_k\}$ is a γ_t-set. Hence $\gamma_t(AG(M)) = \gamma(AG(M)) + 1$. □

Example 3.2. Let C_n and P_n be a cycle and a path with n vertices, respectively.

(a) Clearly, $\gamma(C_n) = \gamma(P_n) = [n/3]$ (see [19, Example 1]).

(b) Let $\mathbb{Z}_2 \times \mathbb{Z}_3$ as $\mathbb{Z}_{12}$-module. It is easy to see that $AG(\mathbb{Z}_2 \times \mathbb{Z}_3) = P_2$ and $\gamma_t(P_2) = 2 = \gamma(P_2) + 1$.

(c) By [12, Lemma 10.9.5], for any split graph G, $\gamma_t(G) = \gamma(G)$. Let $\mathbb{Z}_3 \times \mathbb{Z}_4$ as $\mathbb{Z}_{24}$-module. The split graph $AG(\mathbb{Z}_3 \times \mathbb{Z}_4) = P_4$ and $\gamma_t(P_4) = \gamma(P_4) = 2$.

In the following result we find the total domination number of $AG(M)$.

Theorem 3.3. *Let S be the set of all maximal elements of the set $V(AG(M))$. If $|S| > 1$, then $\gamma_t(AG(M)) = |S|$.*

Proof. Suppose that S is the set of all maximal elements of the set $V(AG(M))$. Let $K \in S$. First we show that $K = ann(annK)$ and there exists $m \in M$ such that $K = ann(Rm)$. Since $annK \neq 0$, there exists $0 \neq m \in annK$. Hence $K \subseteq ann(annK) \subseteq ann(Rm)$. Thus by the maximality of K, we have $K = ann(annK) = ann(Rm)$. For any $K \in S$, choose $m_K \in M$ such that $K = ann(Rm_K)$. We assert that $D = \{Rm_K |\ K \in S\}$ is a total dominating set of $AG(M)$. Since for every $L \in V(AG(M))$ there exists $K \in S$ such that $L \subseteq K = ann(Rm_K)$, L and Rm_K are adjacent. Also for each pair $K, K' \in S$, we have $(Rm_K)(Rm_{K'}) = 0$. Namely, if there exists $m \in (Rm_K)(Rm_{K'}) \setminus \{0\}$, then $K = K' = ann(Rm)$. Thus $\gamma_t(AG(M)) \leq |S|$. To complete the proof, we show that each element of an arbitrary γ_t-set of $AG(M)$ is adjacent to exactly one element of S. Assume to the contrary, that a vertex L' of a γ_t-set of $AG(M)$ is adjacent to K and K', for $K, K' \in S$. Thus $K = K' = annL'$, which is impossible. Therefore $\gamma_t(AG(M)) = |S|$. □

The following corollary is a connection between Sections 2 and 3.

Corollary 3.4. *Let $M = \oplus_{i=1}^{n} M_i$, where M_i is a finitely generated Artinian local module for all $1 \leq i \leq n$, $n \geq 2$ ($M \neq M_1 \oplus M_2$, where M_1 and M_2 are simple modules). Then $\gamma_t(AG(M)) = \gamma(AG(M)) = |Max(M)|$.*

Proof. Let $M = \oplus_{i=1}^{n} M_i$, where (M_i, J_i) is a finitely generated Artinian local module for all $1 \leq i \leq n$ and $n \geq 2$. Recall that $Max(M) = \{N_1, \ldots, N_n |\ N_i = M_1 \oplus \ldots \oplus M_{i-1} \oplus J_i \oplus M_{i+1} \oplus \ldots \oplus M_n, 1 \leq i \leq n\}$. By Proposition 1.5, every nonzero proper submodule of M is a vertex in $AG(M)$. So the set of maximal elements of $V(AG(M))$ and $Max(M)$ are equal and hence by Theorem 3.3, $\gamma_t(AG(M)) = |Max(M)|$. Finally, the result follows from Proposition 2.6. □

Example 3.5. Let $\mathbb{Z}_3 \times \mathbb{Z}_4$ as $\mathbb{Z}_{24}$-module. $S = \{(0) \times \mathbb{Z}_4, \mathbb{Z}_3 \times \bar{2}\mathbb{Z}_4\}$ is the set of all maximal elements of $AG(\mathbb{Z}_3 \times \mathbb{Z}_4)$ and $\gamma_t(AG(\mathbb{Z}_3 \times \mathbb{Z}_4)) = \gamma_t(P_4) = 2 = |S|$.

Theorem 3.6. *Let R be a reduced ring, M is a faithful module, and $|Min(R)| < \infty$. If $\gamma(AG(M)) > 1$, then $\gamma_t(AG(M)) = \gamma(AG(M)) = |Min(R)|$.*

Proof. Since R is reduced, M is a faithful module, and $\gamma(AG(M)) > 1$, we have $|Min(R)| > 1$. Suppose that $Min(R) = \{p_1, \ldots, p_n\}$. If $n = 2$, the result follows from Corollary 1.8. Therefore, suppose that $n \geq 3$. We define $\widehat{p_iM} = p_1 \ldots p_{i-1}p_{i+1} \ldots p_nM$, for every $i = 1, \ldots, n$. Clearly, $\widehat{p_iM} \neq 0$, for every $i = 1, \ldots, n$. Since R is reduced, we deduce that $\widehat{p_iM}p_iM = 0$. Therefore, every p_iM is a vertex of $AG(M)$. If K is a vertex of $AG(M)$, then by [11, Corollary 3.5], $(K : M) \subseteq Z(R) = \cup_{i=1}^{n} p_i$. It follows from the Prime Avoidance Theorem that $(K : M) \subseteq p_i$, for some i, $1 \leq i \leq n$. Thus p_iM is a maximal element of $V(AG(M))$, for every $i = 1, \ldots, n$. From Theorem 3.3, $\gamma_t(AG(M)) = |Min(R)|$. Now, we show that $\gamma(AG(M)) = n$. Assume to the contrary, that $B = \{J_1, \ldots, J_{n-1}\}$ is a dominating set for $AG(M)$. Since $n \geq 3$, the submodules p_iM and p_jM, for $i \neq j$ are not adjacent (from $p_ip_j = 0 \subseteq p_k$ it would follow that $p_i \subseteq p_k$ or $p_j \subseteq p_k$ which is not true). Because of that, we may assume that for some $k < n-1$, $J_i = p_iM$ for $i = 1, \ldots, k$, but none of the other of submodules from B are equal to some p_sM (if $B = \{p_1M, \ldots, p_{n-1}M\}$, then p_nM would be adjacent to some p_iM, for $i \neq n$). So every submodule in $\{p_{k+1}M, ..., p_nM\}$ is adjacent to a submodule in $\{J_{k+1}, ..., J_{n-1}\}$. It follows that for some $s \neq t$, there is an l such that $(p_sM)J_l = 0 = (p_tM)J_l$. Since $p_s \nsubseteq p_t$, it follows that $J_l \subseteq p_tM$, so $J_l^2 = 0$, which is impossible, since the ring R is reduced. So $\gamma_t(AG(M)) = \gamma(AG(M)) = |Min(R)|$. □

Theorem 3.3 leads to the following corollary.

Corollary 3.7. *Let R be a reduced ring, M is a faithful module, and $|Min(R)| < \infty$. If $\gamma(AG(M)) > 1$, then the following are equivalent.*

(a) $\gamma(AG(M)) = 2$.

(b) $AG(M) = B_{n,m}$ *such that* $n, m \geq 2$.

(c) $AG(M) = K_{n,m}$ *such that* $n, m \geq 2$.

(d) *R has exactly two minimal primes.*

Proof. Follows from Theorem 3.3 and Corollary 1.8. □

In the following theorem the domination number of bipartite annihilating-submodule graphs is given.

Theorem 3.8. *Let M be a faithful module. If $AG(M)$ is a bipartite graph, then $\gamma(AG(M)) \leq 2$.*

Proof. Let M be a faithful module. If $AG(M)$ is a bipartite graph, then from Theorem 1.7, either R is a reduced ring with exactly two minimal prime ideals, or $AG(M)$ is a star graph with more than one vertex. If R is a reduced ring with exactly two minimal prime ideals and $\gamma(AG(M)) = 1$, then we are done. If R is a reduced ring with exactly two minimal prime ideals and $\gamma(AG(M)) > 1$, then the result follows by Corollary 3.7. If $AG(M)$ is a star graph with more than one vertex, then we are done. $\square$

Theorem 3.9. *If R is a Notherian ring and M a finitely generated module, then $\gamma(AG(M)) \leq |Ass(M)| < \infty$.*

Proof. By [21], Since R is a Notherian ring and M a finitely generated module, $|Ass(M)| < \infty$. Let $Ass(M) = \{p_1, ..., p_n\}$, where $p_i = ann(m_i)$ for some $m_i \in M$ for every $i = 1, \ldots, n$. Set $A = \{Rm_i | 1 \leq i \leq n\}$. We show that A is a dominating set of $AG(M)$. Clearly, every Rm_i is a vertex of $AG(M)$, for $i = 1, \ldots, n$ $((p_iM)(m_iR) = 0)$. If K is a vertex of $AG(M)$, then [21, Corollary 9.36] implies that $(K : M) \subseteq Z(M) = \cup_{i=1}^{n} p_i$. It follows from the Prime Avoidance Theorem that $(K : M) \subseteq p_i$, for some i, $1 \leq i \leq n$. Thus $K(Rm_i) = 0$, as desired. $\square$

The remaining result of this paper provides the domination number of the annihilating-submodule graph of a finite direct product of modules.

Theorem 3.10. *For a module M, which is a product of two (nonzero) modules, one of the following holds.*

(a) *If $M \cong F \times D$, where F is a simple module and D is a prime module, then $\gamma(AG(M)) = 1$.*
(b) *If $M \cong D_1 \times D_2$, where D_1 and D_2 are prime modules which are not simple, then $\gamma(AG(M)) = 2$.*
(c) *If $M \cong M_1 \times D$, where M_1 is a module which is not prime and D is a prime module, then $\gamma(AG(M)) = \gamma(AG(M_1)) + 1$.*
(d) *If $M \cong M_1 \times M_2$, where M_1 and M_2 are two modules which are not prime, then $\gamma(AG(M)) = \gamma(AG(M_1)) + \gamma(AG(M_2))$.*

Proof. Parts (a) and (b) are trivial.

(c) Without loss of generality, one can assume that $\gamma(AG(M_1)) < \infty$. Suppose that $\gamma(AG(M_1)) = n$ and $\{K_1, \ldots, K_n\}$ is a minimal dominating set of $AG(M_1)$. It is not hard to see that $\{K_1 \times 0, \ldots, K_n \times 0, 0 \times D\}$ is the smallest dominating set of $AG(M)$.

(d) We may assume that $\gamma(AG(M_1)) = m$ and $\gamma(AG(M_2)) = n$, for some positive integers m and n. Let $\{K_1, \ldots, K_m\}$ and $\{L_1, \ldots, L_n\}$ be two minimal dominating sets in $AG(M_1)$ and $AG(M_2)$, respectively. It is easy to see that $\{K_1 \times 0, \ldots, K_m \times 0, 0 \times L_1 \ldots 0 \times L_n\}$ is the smallest dominating set in $AG(M)$. $\square$

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