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ON S-PRIMARY SUBMODULES

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Abstract. Let R be a commutative ring with identity, S a multiplicatively closed subset of R, and M be an R-module. In this paper, we study and investigate some properties of S-primary submodules of M. Among the other results, it is shown that this class of modules contains the family of primary (resp. S-prime) submodules properly.

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1. Introduction

Throughout this article, all rings are commutative with identity elements and all modules are unital modules. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will denote respectively the natural numbers, the ring of integers and the field of quotients of $\mathbb{Z}$.

Consider a non-empty subset S of R. We call S a multiplicatively closed subset of R if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $ss' \in S$ for all $s, s' \in S$ [12]. Note that $S = R - p$ is a multiplicatively closed subset of R for every prime ideal p of R. Let N and K be two submodules of an R-module M and J an ideal of R. Then the residual N by K and J is defined as follows:

$$(N :_R K) = \{ r \in R \mid rK \subseteq N\},$$

$$(N :_M J) = \{ m \in M \mid Jm \subseteq N\}.$$

Particularly, we use $Ann_R(M)$ instead of $(0 :_R M)$ and $(N :_M s)$ instead of $(N :_M Rs)$, where Rs is the principal ideal generated by an element $s \in R$. The sets of prime ideals and maximal ideals of R are denoted by $Spec(R)$ and $Max(R)$, respectively.

A submodule P of M is called *prime* if $P \neq M$ and whenever $r \in R$ and $e \in M$ satisfy $re \in P$, then $r \in (P :_R M)$ or $e \in P$. The set of all prime submodules of M is denoted by $Spec(M)$ [3, 7].

In [11], the authors introduced the concept of S-prime submodules and investigated some properties of this class of modules. Let S be a multiplicatively closed subset of R and P be a submodule of M with $(P :_R M) \cap S = \emptyset$. Then P is said to be an *S-prime submodule* if there exists $s \in S$ such that whenever $rm \in P$, where $r \in R$ and $m \in M$, then $sr \in (P :_R M)$ or $sm \in P$. Particularly, an ideal I of R is said to be an *S-prime ideal* if I is an S-prime submodule of the R-module R.

The notion of S-primary submodule was introduced in [5]. Let S be a multiplicatively closed subset of R and P be a submodule of M with $(P :_R M) \cap S = \emptyset$. Then P is said to be an *S-primary submodule* if there exists $s \in S$ such that whenever $rm \in P$, where $r \in R$ and $m \in M$, then $sr \in \sqrt{(P :_R M)}$ or $sm \in P$.

In this paper, we will study the family of S-primary submodules extensively and investigate some of their properties. In fact, this family of modules is a generalization of primary (resp. S-prime) submodules.

Among the other results, we provide some notions that each one is equivalent to S-primary (Theorem 2.2). Examples 2.4 and 2.5 show that these new modules contain the family of primary and S-prime submodules properly. Further it is proved that if P is an S-primary submodule of M, then $S^{-1}P$ is also an S-primary submodule of $S^{-1}M$ (Proposition 2.7). Example 2.8 shows that the converse is not true in general. Also we show that S-primary submodules has a good behavior with Cartesian products (Theorems 2.20 and 2.21). Moreover, we provide some useful characterization concerning S-primary submodules (Theorems 2.17, 2.24 and 2.25).

2. Main results

Definition 2.1. Let S be a multiplicatively closed subset of R and P be a submodule of M with $(P :_R M) \cap S = \emptyset$. Then P is said to be an *S-primary submodule* of M if there exists $s \in S$ such that whenever $rm \in P$, where $m \in M$ and $r \in R$, then $sr \in \sqrt{(P :_R M)}$ or $sm \in P$ [5, Definition 2.27]. In particular, we say that an ideal I of R is an S-primary ideal if I is an S-primary submodule of R-module R.

Theorem 2.2. *Let S be a multiplicatively closed subset of R. For a submodule P of an R-module M with $(P :_R M) \cap S = \emptyset$, the following are equivalent:*

(a) *P is an S-primary submodule of M;*
(b) *There exists $s \in S$ such that for every $r \in R$, the endomorphism $r : s(M/P) \to s(M/P)$ given by $s\overline{m} = sm + P \mapsto rs\overline{m} = rsm + P$ is injective or $(rs)^t(M/P) = (\overline{0})$ for some $t \in \mathbb{N}$;*
(c) *There exists $s \in S$ such that whenever $rN \subseteq P$, where N is a submodule of M and $r \in R$, then $sr \in \sqrt{(P :_R M)}$ or $sN \subseteq P$;*
(d) *There exists $s \in S$ such that whenever $JN \subseteq P$, where N is a submodule of M and J is an ideal of R, then $sJ \subseteq \sqrt{(P :_R M)}$ or $sN \subseteq P$.*

Proof. (a)⇒(b) By hypothesis, there exists $s \in S$ such that for every $r \in R$ and $m \in M$ if $rm \in P$, then we have $sm \in P$ or $sr \in \sqrt{(P :_R M)}$. Now for each $r \in R$, we define the endomorphism $r : s(M/P) \to s(M/P)$ by $sm+P \mapsto rsm+P$. We show that this endomorphism is injective or $rs \in \sqrt{(P :_R M)}$. Assume $rs \notin \sqrt{(P :_R M)}$. Then we show the other part holds. To see let $\overline{rsm} = rsm+P = P = \overline{0}$. So we have $(rs)m \in P$. So by hypothesis, $sm \in P$ or $s(rs) = rs^2 \in \sqrt{(P :_R M)}$. We conclude $sr \in \sqrt{(P :_R M)}$, which is a contradiction. Hence $sm \in P$, as required.

(b)⇒(a) It is clear.

(a)⇒(c) It is clear.

(c)⇒(d) Let $JN \subseteq P$, where J is an ideal of R and N is a submodule of M. We will show that there exists $s \in S$ such that $sN \subseteq P$ or $sJ \subseteq \sqrt{(P :_R M)}$. Clearly, we have $rN \subseteq P$ for every $r \in J$. So by part (c), there exists $s \in S$ such that $sN \subseteq P$ or $sr \in \sqrt{(P :_R M)}$ for every $r \in J$, as desired.

(d)⇒(a) Take $r \in R$ and $m \in M$ with $rm \in P$. Now, put $J = Rr$ and $N = Rm$. Then we can conclude that $JN = Rrm \subseteq P$. By assumption, there is an $s \in S$ so that $sJ = Rrs \subseteq \sqrt{(P :_R M)}$ or $sN = Rsm \subseteq P$ and so either $sr \in \sqrt{(P :_R M)}$ or $sm \in P$, as required. □

Lemma 2.3. *Let M be an R-module and S a multiplicatively closed subset of R. Then we have the following.*

(a) *If P is a primary submodule of M such that $(P :_R M) \cap S = \emptyset$, then P is an S-primary submodule of M.*
(b) *If P is an S-primary submodule of M and $S \subseteq u(R)$, where $u(R)$ denotes the set of units in R, then P is a primary submodule of M.*

Proof. This is clear. □

By setting $S = \{1\}$, we conclude that every primary submodule is an S-primary submodule by Lemma 2.3. The following example shows that the converse is not true in general.

Example 2.4. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus (\bigoplus_{i=1}^{n} \mathbb{Z}_{p_i})$, where p_i are distinct positive prime integers. Take the submodule $P = (0)$ and the multiplicatively closed subset

$$S = \{1, p_1^{m_1} p_2^{m_2} ... p_n^{m_n} \mid \forall i \in \{1, 2, ..., n\},\ m_i \in \mathbb{N} \cup \{0\}\ \}.$$

First note that $(P :_{\mathbb{Z}} M) = (0)$ and $p_1 p_2 ... p_n (0, \overline{1}, \overline{1}, ..., \overline{1}) = (0, \overline{0}, \overline{0}, ..., \overline{0}) \in P$. Since $p_1 p_2 ... p_n \notin \sqrt{(P :_{\mathbb{Z}} M)}$ and $(0, \overline{1}, \overline{1}, ..., \overline{1}) \notin P$, P is not a primary submodule of M. Put $s = p_1p_2...p_n$ and let

$$k(\frac{a}{b}, \overline{x_1}, \overline{x_2}, ..., \overline{x_n}) = (\frac{ka}{b}, \overline{kx_1}, \overline{kx_2}, ..., \overline{kx_n}) \in P,$$

where $k \in \mathbb{Z}$ and $(\frac{a}{b}, \overline{x_1}, \overline{x_2}, ..., \overline{x_n}) \in M$. Then $ka = 0$. This yields that $k = 0$ or $a = 0$. If $k = 0$, there is nothing to prove. Thus assume that $a = 0$. Then $s(\frac{a}{b}, \overline{x_1}, \overline{x_2}, ..., \overline{x_n}) \in P$. Therefore, P is an S-primary submodule of M.

We recall that a submodule P of an R-module M is *S-prime* if there exists $s \in S$ such that whenever $rm \in P$, where $r \in R$ and $m \in M$, then $sr \in (P :_R M)$ or $sm \in P$ [11]. Clearly, every S-prime submodule is S-primary. The following example shows that the converse is not true in general.

Example 2.5. Consider $M = \mathbb{Z}_4$ as a $\mathbb{Z}$-module. Set $S = \mathbb{Z}\backslash 2\mathbb{Z}$ and $P = (\overline{0})$. Thus we have $(P :_{\mathbb{Z}} M) = 4\mathbb{Z}$ and $2.\overline{2} \in (\overline{0})$. Since for every $s \in S$, $2s \notin (P :_{\mathbb{Z}} M)$ and $s.\overline{2} \notin P$, P is not an S-prime submodule of M. Put $s = 1$ and let $k\overline{a} = \overline{0}$. If $\overline{a} = \overline{0}$, there is nothing to prove. Thus assume that $\overline{a} \neq \overline{0}$. Then $k = 2k'$ for some $k' \in \mathbb{Z}$. This implies that $k \in \sqrt{(P :_{\mathbb{Z}} M)}$. Therefore, P is an S-primary submodule of M.

Remark 2.6. Let S be a multiplicatively closed subset of R. Recall that the saturation S^* of S is defined as

$$S^* = \{x \in R \mid \frac{x}{1} \text{ is a unit of } S^{-1}R\}.$$

It is obvious that S^* is a multiplicatively closed subset of R containing S [6].

Proposition 2.7. *Let S be a multiplicatively closed subset of R and M be an R-module. Then we have the following.*

(a) *If $S_1 \subseteq S_2$ are multiplicatively closed subsets of R and P is an S_1-primary submodule of M, then P is an S_2-primary submodule of M in case $(P :_R M) \cap S_2 = \emptyset$.*

(b) *P is an S-primary submodule of M if and only if P is an S^*-primary submodule of M.*

(c) *If P is an S-primary submodule of M, then $S^{-1}P$ is a primary submodule of $S^{-1}R$-module $S^{-1}M$.*

Proof. (a) It is clear.

(b) Assume that P is an S-primary submodule of M. We need to prove that $(P :_R M)$ and S^* are disjoint. Suppose there exists $x \in (P :_R M) \cap S^*$. As $x \in S^*$, $\frac{x}{1}$ is a unit of $S^{-1}R$ and so $(\frac{x}{1})(\frac{a}{s}) = 1$ for some $a \in R$ and $s \in S$. This yields that $us = uxa$ for some $u \in S$. Now we have that $us = uxa \in (P :_R M) \cap S$, a contradiction. Thus $(P :_R M) \cap S^* = \emptyset$. Now as $S \subseteq S^*$, by part (a), P is an S^*-primary submodule of M. Conversely, assume that P is an S^*-primary submodule of M. Let $rm \in P$, where $r \in R$ and $m \in M$. Then there exists $x \in S^*$ such that $xr \in \sqrt{(P :_R M)}$ or $xm \in P$. As $\frac{x}{1}$ is a unit of $S^{-1}R$, there exist $u, s \in S$ and $a \in R$ such that $us = uxa$. Put $us = s' \in S$. Then note that $s'r = (us)r = uaxr \in \sqrt{(P :_R M)}$ or $s'm \in P$. Therefore, P is an S-primary submodule of M.

(c) Let $(\frac{r}{s})(\frac{m}{t}) \in S^{-1}P$, where $\frac{r}{s} \in S^{-1}R$ and $\frac{m}{t} \in S^{-1}M$. Then $urm \in P$ for some $u \in S$. Since P is an S-primary submodule of M, there is an $s' \in S$ so that $s'ur \in \sqrt{(P :_R M)}$ or $s'm \in P$. This yields that $\frac{r}{s} = \frac{s'ur}{s'us} \in S^{-1}\sqrt{(P :_R M)} \subseteq \sqrt{(S^{-1}P :_{S^{-1}R} S^{-1}M)}$ or $\frac{m}{t} = \frac{s'm}{s't} \in S^{-1}P$. Hence, $S^{-1}P$ is a primary submodule of $S^{-1}M$. $\square$

The following example shows that the converse of part (c) of Proposition 2.7 is not true in general.

Example 2.8. Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. Take the submodule $N = \mathbb{Z}$ and the multiplicatively closed subset $S = \mathbb{Z} - \{0\}$ of $\mathbb{Z}$. Then $(N :_{\mathbb{Z}} M) = (0)$. Let s be an arbitrary element of S. Choose a prime number p with $gcd(p, s) = 1$. Then note that $p\frac{1}{p} = 1 \in N$. But $sp \notin \sqrt{(N :_{\mathbb{Z}} M)}$ and $\frac{s}{p} \notin N$, it follows that N is not an S-primary submodule of M. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$ is a field, $S^{-1}(\mathbb{Q})$ is a vector space. Therefore the proper submodule $S^{-1}N$ is a primary submodule of $S^{-1}\mathbb{Q}$.

Proposition 2.9. *Suppose $f : M \to M'$ is an R-homomorphism. Then we have the following.*

(a) *If P' is an S-primary submodule of M' provided that $(f^{-1}(P') :_R M) \cap S = \emptyset$, then $f^{-1}(P')$ is an S-primary submodule of M.*

(b) *If f is an epimorphism and P is an S-primary submodule of M with $ker(f) \subseteq P$, then $f(P)$ is an S-primary submodule of M'.*

Proof. (a) Let $rm \in f^{-1}(P')$ for some $r \in R$ and $m \in M$. This yields that $f(rm) = rf(m) \in P'$. Since P' is an S-primary submodule of M', there is an $s \in S$ so that $sr \in \sqrt{(P' :_R M')}$ or $sf(m) \in P'$. Now we will show that $(P' :_R M') \subseteq (f^{-1}(P') :_R M)$. Take $x \in (P' :_R M')$. Then we have $xM' \subseteq P'$. Since $f(M) \subseteq M'$, we conclude that $f(xM) = xf(M) \subseteq xM' \subseteq P'$. This implies that $xM \subseteq f^{-1}(f(M)) \subseteq f^{-1}(p')$ and thus $x \in (f^{-1}(P') :_R M)$. Hence we have $sr \in \sqrt{(f^{-1}(P') :_R M)}$ or $sm \in f^{-1}(P')$. It follows that $f^{-1}(P')$ is an S-primary submodule of M.

(b) First note that $(f(P) :_R M') \cap S = \emptyset$. Otherwise there would be an $s \in (f(P) :_R M') \cap S$. Since $s \in (f(P) :_R M')$, $sM' \subseteq f(P)$, but then

$f(sM) = sf(M) = sM' \subseteq f(P)$. By taking their inverse images under f, we have

$$sM \subseteq sM + ker(f) \subseteq f^{-1}(f(P)) = P + ker(f) = P.$$

That means $s \in (P :_R M)$, which is a contradiction. Now take $r \in R$ and $m' \in M'$ with $rm' \in f(P)$. As f is an epimorphism, there is an $m \in M$ such that $m' = f(m)$. Then $rm' = rf(m) = f(rm) \in f(P)$. Since $Ker(f)$ is a subset of P, we get $rm \in P$. As P is an S-primary submodule of M, there is an $s \in S$ so that $sr \in \sqrt{(P :_R M)}$ or $sm \in P$. Since $\sqrt{(P :_R M)} \subseteq \sqrt{(f(P) :_R M')}$, we have $sr \in \sqrt{(f(P) :_R M')}$ or $f(sm) = sf(m) = sm' \in f(P)$. Accordingly, $f(P)$ is an S-primary submodule of M'. □

Corollary 2.10. *Let S be a multiplicatively closed subset of R and take a submodule L of M. Then we have the following.*

(a) *If P' is an S-primary submodule of M with $(P' :_R L) \cap S = \emptyset$, then $L \cap P'$ is an S-primary submodule of L.*

(b) *Suppose that P is a submodule of M with $L \subseteq P$. Then P is an S-primary submodule of M if and only if P/L is an S-primary submodule of M/L.*

Proof. (a) Consider the injection $i : L \to M$ defined by $i(m) = m$ for all $m \in L$. Then note that $i^{-1}(P') = L \cap P'$. Now we will show that $(i^{-1}(P') :_R L) \cap S = \emptyset$. Assume that $s \in (i^{-1}(P') :_R L) \cap S$. Then we have $sL \subseteq i^{-1}(P') = L \cap P' \subseteq P'$. This implies that $s \in (P' :_R L) \cap S$, a contradiction. The rest follows from Proposition 2.9 (a).

(b) Assume that P is an S-primary submodule of M. Then consider the canonical homomorphism $\pi : M \to M/L$ defined by $\pi(m) = m + L$ for all $m \in M$. By Proposition 2.9 (b), P/L is an S-primary submodule of M/L. Conversely, assume that P/L is an S-primary submodule of M/L. Let $rm \in P$ for some $r \in R$ and $m \in M$. This yields that $r(m + L) = rm + L \in P/L$. As P/L is an S-primary submodule of M/L, there is an $s \in S$ so that $sr \in \sqrt{(P/L :_R M/L)} = \sqrt{(P :_R M)}$ or $s(m + L) = sm + L \in P/L$. Therefore, we have $sr \in \sqrt{(P :_R M)}$ or $sm \in P$. Hence, P is an S-primary submodule of M. □

An R-module M is said to be a multiplication module if for every submodule N of M there exists an ideal I of R such that $N = IM$ [4].

Proposition 2.11. *Let M be an R-module and S be a multiplicatively closed subset of R. The following statements hold.*

(a) *If P is an S-primary submodule of M, then $(P :_R M)$ is an S-primary ideal of R.*

(b) *If M is a multiplication module and $(P :_R M)$ is an S-primary ideal of R, then P is an S-primary submodule of M.*

Proof. (a) Let $xy \in (P :_R M)$ for some $x, y \in R$. Then $xym \in P$ for all $m \in M$. As P is an S-primary submodule, there exists $s \in S$ such that $sx \in \sqrt{(P :_R M)}$ or $sym \in P$ for all $m \in M$. If $sx \in \sqrt{(P :_R M)}$, there is nothing to prove. Suppose that $sx \notin \sqrt{(P :_R M)}$. Then $sym \in P$ for all $m \in M$ so that $sy \in (P :_R M)$. Therefore, $(P :_R M)$ is an S-primary ideal of R.

(b) Let J be an ideal of R and N a submodule of M with $JN \subseteq P$. Then we can conclude that $J(N :_R M) \subseteq (JN :_R M) \subseteq (P :_R M)$. As $(P :_R M)$ is an S-primary ideal of R, there is an $s \in S$ so that $s(N :_R M) \subseteq (P :_R M)$ or $sJ \subseteq \sqrt{(P :_R M)}$. Thus, we can conclude that $sN = s(N :_R M)M \subseteq (P :_R M)M = P$ or $sJ \subseteq \sqrt{(P :_R M)}$. Therefore, by Theorem 2.2 (d), P is an S-primary submodule of M. □

Remark 2.12. (a) Assume that M is a multiplication R-module and K, L are two submodules of M. The product of K and L is defined as $KL = (K :_R M)(L :_R M)M$ [1].

(b) Let M be an R-module and N a submodule of M. The radical of N, denoted by $rad(N)$, is the intersection of all prime submodules of M containing N; that is, $rad(N) = \bigcap\{P \mid N \subseteq P,\ P \in Spec(M)\}$ [8].

As an immediate consequence of the Proposition 2.11 and Theorem 2.2 (d), we have the following explicit result.

Corollary 2.13. *Suppose that M is a multiplication R-module and P a submodule of M provided that $(P :_R M) \cap S = \emptyset$, where S is a multiplicatively closed subset of R. Then the following are equivalent:*

(a) *P is an S-primary submodule of M;*

(b) *There exists $s \in S$ such that whenever $LN \subseteq P$, where L and N are submodules of M, then $s(L :_R M) \subseteq \sqrt{(P :_R M)}$ or $sN \subseteq P$.*

Corollary 2.14. *Suppose that M is a finitely generated multiplication R-module and P is a submodule of M provided that $(P :_R M) \cap S = \emptyset$, where S is a multiplicatively closed subset of R. Then the following are equivalent:*

(a) *P is an S-primary submodule of M;*

(b) *There exists $s \in S$ such that whenever $LN \subseteq P$, where L and N are submodules of M, then $sL \subseteq rad(P)$ or $sN \subseteq P$.*

Proof. (a)⇒(b) Assume that $LN \subseteq P$, where L and N are submodules of M. By Remark 2.12 (a), $LN = (L :_R M)N \subseteq P$. Then there exists $s \in S$ so that $s(L :_R M) \subseteq \sqrt{(P :_R M)}$ or $sN \subseteq P$ by Theorem 2.2 (d). Since M is multiplication, by [4, Theorem 2.12], we have $s(L :_R M)M = sL \subseteq \sqrt{(P :_R M)}M = rad(P)$ or $sN \subseteq P$.

(b)⇒(a) Assume that $JN \subseteq P$, where N is a submodule of M and J is an ideal of R. Set $K := JM$. As M is a multiplication module, Then we have

$$KN = (K :_R M)(N :_R M)M = J(N :_R M)M = JN \subseteq P.$$

By assumption, there exists $s \in S$ so that $sK \subseteq rad(P)$ or $sN \subseteq P$. As M is finitely generated, by [9, Thoerem 4.4], $sK \subseteq rad(P)$ implies that

$$sJ \subseteq (sK :_R M) \subseteq (rad(P) :_R M) = \sqrt{(P :_R M)}.$$

Therefore P is an S-primary submodule of M by Corollary 2.13. □

Remark 2.15. (a) Let M be an R-module and p be a maximal ideal of R. In [4], $T_p(M)$ is defined as follows

$$T_p(M) = \{\ m \in M |\ (1 - r)m = 0 \text{ for some } r \in p\}.$$

Clearly $T_p(M)$ is a submodule of M. An R-module M is said to be p-*cyclic* provided there exist $q \in p$ and $m \in M$ such that $(1 - q)M \subseteq Rm$ [4].

(b) Let M be an R-module. Then M is a multiplication R-module if and only if for every maximal ideal p of R either $M = T_p(M)$ or M is p-cyclic [4, Theorem 1.2].

Lemma 2.16. *Let S be a multiplicatively closed subset of R, p be an S-primary (resp. S-prime) ideal of R and M be a faithful multiplication R-module. Then there exists an $s \in S$ such that whenever $am \in pM$, where $a \in R$ and $m \in M$, then $sa \in \sqrt{p}$ (resp. $sa \in p$) or $sm \in pM$.*

Proof. It is enough to prove it for S-primary submodules. The technique is similar for S-prime. As p is an S-primary ideal, there exists $s \in S$, whenever $rr' \in p$, where $r, r' \in R$, then $sr \in \sqrt{p}$ or $sr' \in p$. Let $a \in R$ and $m \in M$ satisfy $am \in pM$. Suppose $sa \notin \sqrt{p}$. Set $K := (pM :_R sm)$. Assume that $K \neq R$. Then there exists a maximal ideal Q of R so that $K \subseteq Q$. $m \notin T_Q(M)$, since otherwise, there exists $q \in Q$ such that $(1 - q)m = 0$ and so $(1 - q)sm = 0$. This implies that $(1 - q) \in K \subseteq Q$, a contradiction. Since M is Q-cyclic, by [4, Theorem 1.2], there exist $m' \in M$ and

$q \in Q$ such that $(1-q)M \subseteq Rm'$. In particular, $(1-q)m = s'm'$, $(1-q)am = p'm'$ for some $s' \in R$ and $p' \in p$. Thus $(as' - p')m' = 0$. Now $(1-q)(Ann_R(m'))M \subseteq (Ann_R(m'))Rm' = \mathbf{0}$ implies $(1-q)Ann_R(m) \subseteq Ann_R(M) = \mathbf{0}$, because M is faithful, and hence $(1-q)as' = (1-q)p' \in p$. As p is an S-primary ideal, $ss' \in p$ or $sa \in \sqrt{p}$ or $s(1-q)^n \in p$ for some $n \in \mathbb{N}$. But $p \subseteq K \subseteq Q$ so that in each case, we have a contradiction. It follows that $K = R$ and $sm \in pM$, as required. □

In the following, the Theorem 2.11 in [11] will be extended by removing the condition "finitely generated".

Theorem 2.17. *Let M be a multiplication R-module and P a submodule of M provided that $(P :_R M) \cap S = \emptyset$, where S is a multiplicatively closed subset of R. Then the following are equivalent:*

(a) *P is an S-primary (resp. S-prime) submodule of M.*
(b) *$(P :_R M)$ is an S-primary (resp. S-prime) ideal of R.*
(c) *$P = IM$ for some S-primary (resp. S-prime) ideal I of R with $Ann(M) \subseteq I$.*

Proof. (a)⇒(b) It is clear from Proposition 2.11 (a).

(b)⇒(c) It is clear.

(c)⇒(a) As M is a faithful multiplication $R/Ann_R(M)$-module, by Corollary 2.10 (b), $I/Ann_R(M)$ is an S-primary (resp. S-prime) ideal of $R/Ann_R(M)$. Hence $P = IM$ is an S-primary (resp. S-prime) submodule of $R/Ann_R(M)$-module M by Lemma 2.16. Therefore, P is an S-primary (resp. S-prime) submodule of R-module M, as required. □

Proposition 2.18. *Let P be an S-primary submodule of multiplication R-module M. Suppose that $N \cap L \subseteq P$ for some submodules N and L of M. Then $sN \subseteq P$ or $sL \subseteq rad(P)$ for some $s \in S$.*

Proof. Since P is an S-primary submodule, there exists $s \in S$ such that for every $r \in R$ and $m \in M$, if $rm \in P$, then $sr \in \sqrt{(P :_R M)}$ or $sm \in P$. Let $sN \nsubseteq P$. Then $sm' \notin P$ for some $m' \in N$. Take an element $a \in (L :_R M)$. This yields that $am' \in (L :_R M)N \subseteq L \cap N \subseteq P$. As P is an S-primary submodule of M and $sm' \notin P$, we can conclude that $sa \in \sqrt{(P :_R M)}$ so that $s(L :_R M) \subseteq \sqrt{(P :_R M)}$. As M is a multiplication module, by [4, Theorem 2.12], we have

$$sL = s(L :_R M)M \subseteq \sqrt{(P :_R M)}M = rad(P).$$ □

Lemma 2.19. *Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$ where S_i is a multiplicatively closed subset of R_i. Suppose $p = p_1 \times p_2$ is an ideal of R. Then the following are equivalent:*

(a) *p is an S-primary ideal of R.*

(b) *p_1 is an S_1-primary ideal of R_1 and $p_2 \cap S_2 \neq \emptyset$ or p_2 is an S_2-primary ideal of R_2 and $p_1 \cap S_1 \neq \emptyset$.*

Proof. (a)$\Rightarrow$(b) Since $(1,0)(0,1) = (0,0) \in p$, there exists $s = (s_1, s_2) \in S$ so that $s(1,0) = (s_1, 0) \in \sqrt{p}$ or $s(0,1) = (0, s_2) \in p$. Thus $p_1 \cap S_1 \neq \emptyset$ or $p_2 \cap S_2 \neq \emptyset$. We may assume that $p_1 \cap S_1 \neq \emptyset$. As $P \cap S = \emptyset$, we have $p_2 \cap S_2 = \emptyset$. Let $xy \in p_2$ for some $x, y \in R_2$. Since $(0,x)(0,y) \in p$ and p is an S-primary ideal of R. We get either $s(0,x) = (0, s_2x) \in \sqrt{p}$ or $s(0,y) = (0, s_2y) \in p$ and this yields $s_2x \in \sqrt{p_2}$ or $s_2y \in p_2$. Therefore, p_2 is an S-primary ideal of R_2. In the other case, one can easily show that p_1 is an S-primary ideal of R_1.

(b)$\Rightarrow$(a) Assume that $p_1 \cap S_1 \neq \emptyset$ and p_2 is an S-primary ideal of R_2. Then there exists $s_1 \in p_1 \cap S_1$. Let $(a,b)(c,d) = (ac, bd) \in p$ for some $a, c \in R_1$ and $b, d \in R_2$. This yields that $bd \in p_2$ and thus there exists $s_2 \in S_2$ so that $s_2b \in \sqrt{p_2}$ or $s_2d \in p_2$. Put $s = (s_1, s_2) \in S$. Then note that $s(a,b) = (s_1a, s_2b) \in \sqrt{p}$ or $s(c,d) \in p$. Therefore, p is an S-primary ideal of R. In other case, one can similarly prove that p is an S-primary ideal of R. □

Theorem 2.20. *Suppose that $M = M_1 \times M_2$ and $R = R_1 \times R_2$-module and $S = S_1 \times S_2$ is a multiplicatively closed subset of R, where M_i is a R_i-module and S_i is a multiplicatively closed subset of R_i for each $i = 1, 2$. Assume $P = P_1 \times P_2$ is a submodule of M. Then the following are equivalent:*

(a) *P is an S-primary submodule of M.*

(b) *P_1 is an S_1-primary submodule of M_1 and $(P_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ or P_2 is an S_2-primary submodule of M_2 and $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$.*

Proof. (a)$\Rightarrow$(b) By Proposition 2.11, $(P :_R M) = (P_1 :_{R_1} M_1) \times (P_2 :_{R_2} M_2)$ is an S-primary ideal of R and so by Lemma 2.19, either $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ or $(P_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$. We may assume that $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$. Now we will show that P_2 is an S_2-primary submodule of M_2. Let $rm \in P_2$ for some $r \in R_2$ and $m \in M_2$. Then $(1,r)(0,m) = (0, rm) \in P$. As P is an S-primary, there is an $s = (s_1, s_2) \in S$ so that $s(1,r) = (s_1, s_2r) \in \sqrt{(P :_R M)}$ or $s(0 :_R m) = (0, s_2m) \in P$. This implies that $s_2r \in \sqrt{(P_2 :_{R_2} M_2)}$ or $s_2m \in P_2$. Therefore, P_2 is an S_2 is an S_2-primary submodule of M_2. In the other case, it can be similarly show that P_1 is an S_1-primary submodule of M_1.

(b)$\Rightarrow$(a) Assume that $(P_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and P_2 is an S_2-primary submodule of M_2. Then there exists $s_1 \in (P_1 :_{R_1} M_1) \cap S_1$. Let $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2) \in P$ for some $r_i \in R_i$ and $m_i \in M_i$, where $i = 1, 2$. Then $r_2m_2 \in P_2$. As P_2 is an S_2-primary submodule of M_2, there is an $s_2 \in S_2$ so that $s_2r_2 \in \sqrt{(P_2 :_{R_2} M_2)}$ or

$s_2m_2 \in P_2$. Now put $s = (s_1, s_2) \in S$. Then note that $s(r_1, r_2) = (s_1r_1, s_2r_2) \in \sqrt{(P :_R M)}$ or $s(m_1, m_2) = (s_1m_1, s_2m_2) \in P_1 \times P_2 = P$. Therefore, P is an S-primary submodule of M. Similarly, one can show that if P_1 is an S_1-primary submodule of M_1 and $(P_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$, then P is an S-primary submodule of M. □

Theorem 2.21. *Let $M = M_1 \times M_2 \times \cdots \times M_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$-module and $S = S_1 \times S_2 \times \cdots \times S_n$ is a multiplicatively closed subset of R, where M_i is an R_i-module and S_i is a multiplicatively closed subset of R_i for each $i = 1, 2, \ldots, n$. Assume $P = P_1 \times P_2 \times \cdots \times P_n$ is a submodule of M. Then the following are equivalent:*

(a) *P is an S-primary submodule of M.*
(b) *P_i is an S_i-primary submodule of M_i for some $i \in \{1, 2, \ldots, n\}$ and $(P_j :_{R_j} M_j) \cap S_j \neq \emptyset$ for all $j \in \{1, 2, \ldots, n\} - \{i\}$.*

Proof. We apply induction on n. For $n = 1$, the result is true. If $n = 2$, then (a)⇔(b) follows from Theorem 2.20. Assume that (a) and (b) are equivalent when $k < n$. Now, we shall prove (a)⇔(b) when $k = n$. Let $P = P_1 \times P_2 \times \cdots \times P_n$. Put $P' = P_1 \times P_2 \times \cdots \times P_{n-1}$ and $S' = S_1 \times S_2 \times \cdots \times S_{n-1}$. Then by Theorem 2.20, the necessary and sufficient condition for $P = P' \times P_n$ is an S-primary submodule of M is that P' is an S-primary submodule of M' and $(P_n :_{R_n} M_n) \cap S_n \neq \emptyset$ or P_n is an S-primary submodule of M_n and $(P' :_{R'} M') \cap S' \neq \emptyset$, where $M' = M_1 \times M_2 \times \cdots \times M_{n-1}$ and $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$. The rest follows from the induction hypothesis. □

Lemma 2.22. *Suppose that P is an S-primary submodule of M. Then the following statements hold for some $s \in S$.*

(a) *$(P :_M s') \subseteq (P :_M s)$ for all $s' \in S$.*
(b) *$((P :_R M) :_R s') \subseteq ((P :_R M) :_R s)$ for all $s' \in S$.*

Proof.
(a) Take an element $m' \in (P :_M s')$, where $s' \in S$. Then $s'm' \in P$. Since P is an S-primary submodule of M, there exists $s \in S$ such that $ss' \in \sqrt{(P :_R M)}$ or $sm' \in P$. As $(P :_R M) \cap S = \emptyset$, we get $sm' \in P$, namely $m' \in (P :_M s)$.
(b) Follows from part (a). □

Proposition 2.23. *Suppose that M is a finitely generated R-module, S is a multiplicatively closed subset of R, and P is a submodule of M satisfying $(P :_R M) \cap S = \emptyset$. Then the following are equivalent:*

(a) *P is an S-primary submodule of M.*

(b) *$S^{-1}P$ is a primary submodule of $S^{-1}M$ and there is an $s \in S$ satisfying $(P :_M s') \subseteq (P :_M s)$ for all $s' \in S$.*

Proof. (a)⇒(b) It is clear from Proposition 2.7 (c) and Lemma 2.22.

(b)⇒(a) Take $a \in R$ and $m \in M$ with $am \in P$. Then $\frac{a}{1}.\frac{m}{1} \in S^{-1}P$. Since $S^{-1}P$ is a primary submodule of $S^{-1}M$ and M is finitely generated, we can conclude that $\frac{a}{1} \in \sqrt{(S^{-1}P :_{S^{-1}R} S^{-1}M)} = \sqrt{S^{-1}(P :_R M)}$ or $\frac{m}{1} \in S^{-1}P$. Then $ua \in \sqrt{(P :_R M)}$ or $u'm \in P$ for some $u, u' \in S$. By assumption, there is an $s \in S$ so that $(P :_R s') \subseteq (P :_R s)$ for all $s' \in S$. If $ua \in \sqrt{(P :_R M)}$, then $a^n M \subseteq (P :_M u^n) \subseteq (P :_R s)$ for some $n \in \mathbb{N}$ and thus $sa \in \sqrt{(P :_R M)}$. If $u'm \in P$, a similar argument shows that $sm \in P$. Therefore, P is an S-primary submodule of M. □

Theorem 2.24. *Suppose that P is a submodule of M provided $(P :_R M) \cap S = \emptyset$. Then P is an S-primary submodule of M if and only if $(P :_M s)$ is a primary submodule of M for some $s \in S$.*

Proof. Assume $(P :_M s)$ is a primary submodule of M for some $s \in S$. Let $am \in P$, where $a \in R$ and $m \in M$. As $am \in (P :_M s)$, we get $a \in \sqrt{((P :_M s) :_R M)}$ or $m \in (P :_M s)$. This yields that $as \in \sqrt{(P :_R M)}$ or $sm \in P$. Conversely, assume that P is an S-primary submodule of M. Then there exists $s \in S$ such that whenever $am \in P$, where $a \in R$ and $m \in M$, then $sa \in \sqrt{(P :_R M)}$ or $sm \in P$. Now we prove that $(P :_M s)$ is primary. Take $r \in R$ and $m \in M$ with $rm \in (P :_M s)$. Then $srm \in P$. As P is S-primary, we get $s^2 r \in \sqrt{(P :_R M)}$ or $sm \in P$. If $sm \in P$, then there is nothing to show. Assume that $sm \notin P$. Then $s^2 r \in \sqrt{(P :_R M)}$ and hence $sr \in \sqrt{(P :_R M)}$. Thus $r^n \in ((P :_R M) :_R s^n) \subseteq ((P :_R M) :_R s)$ for some $n \in \mathbb{N}$, by Lemma 2.22. Thus, we can conclude that $r^n \in ((P :_M s) :_R M)$, namely $r \in \sqrt{((P :_M s) :_R M)}$. Hence $(P :_M s)$ is a prime submodule of M. □

Theorem 2.25. *Suppose that P is a submodule of M provided $(P :_R M) \subseteq Jac(R)$, where $Jac(R)$ is the Jacobson radical of R. Then the following statements are equivalent:*

(a) *P is a primary submodule of M.*

(b) *$(P :_R M)$ is a primary ideal of R and P is an $(R - \mathfrak{m})$-primary submodule of M for each $\mathfrak{m} \in Max(R)$.*

Proof. (a)⇒(b) Since $(P :_R M) \subseteq Jac(R)$, $(P :_R M) \subseteq \mathfrak{m}$ for each $\mathfrak{m} \in Max(R)$ and hence $(P :_R M) \cap (R - \mathfrak{m}) = \emptyset$. The rest follows from Lemma 2.3 (a).

(b)⇒(a) Let $am \in P$ with $a \notin (P :_R M)$ for some $a \in R$ and $m \in M$. Let $\mathfrak{m} \in Max(R)$. As P is an $(R - \mathfrak{m})$-primary submodule of M, there exists $s_{\mathfrak{m}} \notin \mathfrak{m}$ such that $as_{\mathfrak{m}} \in \sqrt{(P :_R M)}$ or $s_{\mathfrak{m}} m \in P$. As $(P :_R M)$ is a primary ideal of R and

$s_{\mathfrak{m}} \notin \sqrt{(P :_R M)}$, we have $as_{\mathfrak{m}} \notin (P :_R M)$ and so $s_{\mathfrak{m}}m \in P$. Now consider the set $\Omega = \{s_{\mathfrak{m}} \mid \exists\ \mathfrak{m} \in Max(R),\ s_{\mathfrak{m}} \notin \mathfrak{m} \text{ and } s_{\mathfrak{m}}m \in P\}$. Then note that $(\Omega) = R$. To see this, take any maximal ideal $\mathfrak{m}'$ containing Ω. Then the definition of Ω requires that there exists $s_{\mathfrak{m}'} \in \Omega$ and $s_{\mathfrak{m}'} \notin \mathfrak{m}'$. As $\Omega \subseteq \mathfrak{m}'$, we have $s_{\mathfrak{m}'} \in \Omega \subseteq \mathfrak{m}'$, a contradiction. Thus $(\Omega) = R$ and this yields $1 = r_1 s_{\mathfrak{m}_1} + r_2 s_{\mathfrak{m}_2} + \cdots + r_n s_{\mathfrak{m}_n}$ for some $r_i \in R$ and $s_{\mathfrak{m}_i} \notin \mathfrak{m}_i$ with $s_{\mathfrak{m}_i}m \in P$, where $\mathfrak{m}_i \in Max(R)$ for each $i = 1, 2, ..., n$. This yields that $m = r_1 s_{\mathfrak{m}_1}m + r_2 s_{\mathfrak{m}_2}m + \cdots + r_n s_{\mathfrak{m}_n}m \in P$. Therefore, P is a primary submodule of M. □

Now we determine all primary submodules of a module over a quasi-local ring in terms of S-primary submodules.

Corollary 2.26. *Suppose M is a module over a quasi-local ring $(R, \mathfrak{m})$. Then the following statements are equivalent:*

(a) *P is a primary submodule of M.*

(b) *$(P :_R M)$ is a primary ideal of R and P is an $(R - \mathfrak{m})$-primary submodule of M for each $\mathfrak{m} \in Max(R)$.*

Proof. This is clear from Theorem 2.25. □

Remark 2.27. (a) Suppose that M is an R-module. The idealization $R(+)M = \{(a, m) \mid a \in R,\ m \in M\}$ of M is a commutative ring whose addition is component-wise and whose multiplication is defined as $(a, m)(b, m') = (ab, am' + bm)$ for each a, $b \in R$ and m, $m' \in M$. If S is a multiplicatively closed subset of R and P is a submodule of M, then $S(+)P = \{(s, p) \mid s \in S,\ p \in P\}$ is a multiplicatively closed subset of $R(+)M$ [2, 10].

(b) Radical ideals of $R(+)M$ have the form $I(+)M$, where I is a radical ideal of R. If J is an ideal of $R(+)M$, then $\sqrt{J} = \sqrt{I}(+)M$. In particular, if I is an ideal of R and N is a submodule of M, then $\sqrt{I(+)N} = \sqrt{I}(+)M$ [2, Theorem 3.2 (3)].

Proposition 2.28. *Let M be an R-module and p be an ideal of R such that $p \subseteq Ann(M)$. Then the following are equivalent:*

(a) *p is a primary ideal of R.*

(b) *$p(+)M$ is a primary ideal of $R(+)M$.*

Proof. This is straightforward. □

Theorem 2.29. *Let S be a multiplicatively closed subset of R, p be an ideal of R provided $p \cap S = \emptyset$ and M be an R-module. Then the following are equivalent:*

(a) *p is an S-primary ideal of R.*

(b) *$p(+)M$ is an $S(+)0$-primary ideal of $R(+)M$.*

(c) *$p(+)M$ is an $S(+)M$-primary ideal of $R(+)M$.*

Proof. (a)⇒(b) Let $(x,m)(y,m') = (xy, xm' + ym) \in p(+)M$, where $x, y \in R$ and $m, m' \in M$. Then we get $xy \in p$. As p is S-primary, there exists $s \in S$ such that $sx \in \sqrt{p}$ or $sy \in p$. Now put $s' = (s,0) \in S(+)0$. Then we have $s'(x,m) = (sx, sm) \in \sqrt{p}(+)M = \sqrt{p(+)M}$ or $s'(y,m') = (sy, sm') \in p(+)M$. Therefore, $p(+)M$ is an $S(+)0$-primary ideal of $R(+)M$.

(b)⇒(c) It is clear from Proposition 2.7.

(c)⇒(a) Let $xy \in p$ for some $x, y \in R$. Then $(x,0)(y,0) \in p(+)M$. Since $p(+)M$ is $S(+)M$-primary, there exists $s = (s_1, m_1) \in S(+)M$ such that $s(x,0) = (s_1x, xm_1) \in \sqrt{p(+)M} = \sqrt{p}(+)M$ or $s(y,0) = (s_1y, ym_1) \in p(+)M$ and hence we get $s_1x \in \sqrt{p}$ or $s_1y \in p$. Therefore p is an S-primary ideal of R. □

Remark 2.30. Let M be an R-module and let S be a multiplicatively closed subset of R such that $Ann_R(M) \cap S = \emptyset$. We say that M is an *S-torsion-free module* in the case that there is an $s \in S$ such that if $rm = 0$, where $r \in R$ and $m \in M$, then $sm = 0$ or $sr = 0$ [11, Definition 2.23].

Proposition 2.31. *Let M be an R-module. Assume that P is a submodule of M and S is a multiplicatively closed subset of R such that $Ann_R(M) \cap S = \emptyset$. Then P is an S-primary submodule of M if and only if the factor module M/P is a $\pi(S)$-torsion-free $R/\sqrt{(P :_R M)}$-module, where $\pi : R \to R/\sqrt{(P :_R M)}$ is the canonical homomorphism.*

Proof. Suppose that P is an S-primary submodule of M. Let $\overline{am} = 0_{M/P}$, where $\overline{a} = a + \sqrt{(P :_R M)}$ and $\overline{m} = m + P$ for some $a \in R$ and $m \in M$. This yields that $am \in P$. As P is S-primary, there exists $s \in S$ such that $sa \in \sqrt{(P :_R M)}$ or $sm \in P$. Then we can conclude that $\pi(s)\overline{a} = 0_{R/\sqrt{(P:_RM)}}$ or $\pi(s)\overline{m} = 0_{M/P}$. Therefore, M/P is a $\pi(S)$-torsion-free $R/\sqrt{(P :_R M)}$-module. For the other direction, suppose that M/P is a $\pi(S)$-torsion-free $R/\sqrt{(P :_R M)}$-module. Let $am \in P$,where $a \in R$ and $m \in M$. Put $\overline{a} = a + \sqrt{(P :_R M)}$ and $\overline{m} = m + P$. Then note that $\overline{am} = 0_{M/P}$. As M/P is a $\pi(S)$-torsion-free $R/\sqrt{(P :_R M)}$-module, there exists $s \in S$ such that $\pi(s)\overline{a} = 0_{R/\sqrt{(P:_RM)}}$ or $\pi(s)\overline{m} = 0_{M/P}$. This yields that $sa \in \sqrt{(P :_R M)}$ or $sm \in P$. Accordingly, P is an S-primary submodule of M. □

Definition 2.32. Let M be an R-module and let S be a multiplicatively closed subset of R such that $Ann_R(M) \cap S = \emptyset$. We say that M is a *quasi S-torsion-free module*, if there exists $s \in S$ such that whenever $rm = 0$, where $r \in R$ and $m \in M$, then $sm = 0$ or $(sr)^t = 0$ for some $t \in \mathbb{N}$.

According to Definition 2.32, Proposition 2.31 can be expressed as follows.

Proposition 2.33. *Let M be an R-module. Assume that P is a submodule of M and S is a multiplicatively closed subset of R such that $Ann_R(M) \cap S = \emptyset$. Then P is an S-primary submodule of M if and only if the factor module M/P is a quasi $\pi'(S)$-torsion-free $R/(P :_R M)$-module, where $\pi' : R \to R/(P :_R M)$ is the canonical homomorphism.*

Theorem 2.34. *Let M be a module over an integral domain R. The following are equivalent:*

(a) *M is a torsion-free module;*
(b) *M is a quasi $(R - p)$-torsion-free module for each $p \in Spec(R)$;*
(c) *M is a quasi $(R - \mathfrak{m})$-torsion-free module for each $\mathfrak{m} \in Max(R)$.*

Proof. (a)⇒(b) It is clear.

(b)⇒(c) It is clear.

(c)⇒(a) Assume that $a \neq 0$. Take $\mathfrak{m} \in Max(R)$. As M is quasi $(R-\mathfrak{m})$-torsion-free, there exists $s_m \neq \mathfrak{m}$ so that $s_m m = 0$ or $(s_m a)^t = 0$ for some $t \in \mathbb{N}$. As R is an integral domain, $(s_m a)^t \neq 0$. Now, put $\Omega = \{\, s_m \in R \,|\, \exists \mathfrak{m} \in Max(R), s_m \notin \mathfrak{m}$ and $s_m m = 0\}$. A similar argument in the proof of Theorem 2.25 shows that $\Omega = R$. Then we have $(s_{m_1}) + (s_{m_2}) + \cdots + (s_{m_n}) = R$ for some $(s_{m_i}) \in \Omega$. This implies that $Rm = \sum_{i=1}^{n} (s_{m_i})m = (0)$ and hence $m = 0$. This means M is a torsion-free module. □

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