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DERIVING SOME PROPERTIES OF STANLEY-REISNER RINGS FROM THEIR SQUAREFREE ZERO-DIVISOR GRAPHS

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Abstract. Let Δ be a simplicial complex, I_Δ its Stanley-Reisner ideal and $R = K[\Delta]$ its Stanley-Reisner ring over a field K. In 2018, the author introduced the squarefree zero-divisor graph of R, denoted by $\Gamma_{\mathrm{sf}}(R)$, and proved that if Δ and Δ' are two simplicial complexes, then the graphs $\Gamma_{\mathrm{sf}}(K[\Delta])$ and $\Gamma_{\mathrm{sf}}(K[\Delta'])$ are isomorphic if and only if the rings $K[\Delta]$ and $K[\Delta']$ are isomorphic. Here we derive some algebraic properties of R using combinatorial properties of $\Gamma_{\mathrm{sf}}(R)$. In particular, we state combinatorial conditions on $\Gamma_{\mathrm{sf}}(R)$ which are necessary or sufficient for R to be Cohen-Macaulay. Moreover, we investigate when $\Gamma_{\mathrm{sf}}(R)$ is in some well-known classes of graphs and show that in these cases, I_Δ has a linear resolution or is componentwise linear. Also we study the diameter and girth of $\Gamma_{\mathrm{sf}}(R)$ and their algebraic interpretations.

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1. Introduction

In this paper all rings are commutative with identity and K is a field. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over K. By a squarefree monomial ideal of S we mean an ideal generated by a set of squarefree monomials of S. In the last few decades, the study of squarefree monomial ideals has got a large attention (for example, see [3, 7–9, 11–14, 16]). This is because of the fact that if we know algebraic properties of squarefree monomial ideals well, then we can understand many algebraic properties of much larger classes of ideals such as graded ideals of S (see [7]).

There are strong relations between squarefree monomial ideals and several combinatorial objects (see, for instance, [8, 11, 12] and Part III of [7]). Here we use two related combinatorial objects. The first one is the concept of simplicial complexes.

Recall that a *simplicial complex* Δ on $[n] = \{1, \ldots, n\}$ is a family of subsets of $[n]$, called *faces* of Δ, with the following properties:

(i) if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$;
(ii) $\{i\} \in \Delta$ for all $i \in [n]$.

For every subset $F \subseteq [n]$ we set $x_F = \prod_{i \in F} x_i$. Then the ideal $I_\Delta = \langle x_F | F \subseteq [n], F \notin \Delta \rangle$ of S is called the *Stanley-Reisner ideal* of Δ and the ring $K[\Delta] = S/I_\Delta$ is called the *Stanley-Reisner ring of* Δ *over* K. Note that there is a one-to-one correspondence between simplicial complexes on $[n]$ and squarefree monomial ideals of S in $\langle x_1, \ldots, x_n \rangle^2$.

On the other hand, recently several authors have defined some graphs based on the structure of rings and used them to study the algebraic properties of these rings. One of the first defined and most studied of such graphs is the zero-divisor graph, see for example [1, 6, 10, 15] and the references therein. It is quite usual that these graphs, which are defined for general commutative rings, are isomorphic for non-isomorphic rings, see for example Theorem 2.2 and Corollary 2.4 of [12]. In [12], the author used the ideas in the definition of zero-divisor graphs, in the special case of Stanley-Reisner rings and introduced squarefree zero-divisor graphs of Stanley-Reisner rings as a “stronger version” of the zero-divisor graphs of such rings. Suppose that I is a squarefree monomial ideal of S and $R = S/I$. Let V be image in R of the set of all squarefree monomials of S which are not in I. By the *squarefree zero-divisor graph of* R, we mean the graph on vertex set V, in which, two vertices u and v are adjacent if and only if $uv = 0$ in R. We denote this graph by $\Gamma_{\mathrm{sf}}(R)$. If $I = I_\Delta$ for a simplicial complex Δ, we also use $\Gamma_{\mathrm{sf}}(\Delta)$ instead of $\Gamma_{\mathrm{sf}}(R)$. Note that $\Gamma_{\mathrm{sf}}(\Delta)$ can be described using Δ: the vertices of $\Gamma_{\mathrm{sf}}(\Delta)$ correspond to nonempty faces of Δ and two vertices F and G are adjacent if and only if $F \cup G \notin \Delta$. Theorem 3.4 of [12] shows that $\Gamma_{\mathrm{sf}}(\Delta) \cong \Gamma_{\mathrm{sf}}(\Delta')$ if and only if $K[\Delta] \cong K[\Delta']$ for every pair of simplicial complexes Δ and Δ'.

The main advantage of squarefree zero-divisor graphs, is that, against simplicial complexes, they are graphs. So they let us to use the rich theory of graphs for studying squarefree monomial ideals. The main aim of this study is to see what we can find about algebraic properties of R by knowing combinatorial properties of $\Gamma_{\mathrm{sf}}(R)$. In Section 3, we find some combinatorial properties that $\Gamma_{\mathrm{sf}}(R)$ must have, when R is Cohen-Macaulay. Then in Section 3, we study diameter and girth of $\Gamma_{\mathrm{sf}}(R)$ and their algebraic interpretations. Finally in Section 4, we investigate when $\Gamma_{\mathrm{sf}}(R)$ is in some well-known classes of graphs, such as complete, bipartite, complete multipartite, regular or chordal graphs. In these cases, we answer the questions when R is (sequentially) Cohen-Macaulay or I has a linear resolution (or is componentwise linear) in terms of combinatorial properties of $\Gamma_{\rm sf}(R)$.

In the sequel, all graphs are finite, simple and undirected. If u and v are vertices of a graph, by $v \sim u$ we mean that u and v are adjacent. We denote the set of vertices of a graph G by $\mathrm{V}(G)$. Any undefined graph theoretic notation is as in [17]. Also recall that the maximal faces of a simplicial complex Δ are called *facets* and by $\Delta = \langle F_1, \ldots, F_t\rangle$ we mean that $F_1, \ldots, F_t$ are all facets of Δ. Moreover, $\dim \Delta = \max\{|F|-1|F \in \Delta\}$. For more on (squarefree) monomial ideals, simplicial complexes, and the related algebraic concepts, such as linear resolutions or Cohen-Macaulayness, see [2, 7].

2. Combinatorial properties necessary for Cohen-Macaulayness

One of the important topics of research in commutative algebra is finding characterizations of Cohen-Macaulay commutative rings. In particular, many have tried to give combinatorial characterizations for Cohen-Macaulayness of certain classes of squarefree monomial ideals (see for example [7, 8]). It is well-known that if I_Δ is Cohen-Macaulay and $\dim \Delta > 0$, then Δ is *pure* (that is, all of its facets have the same size) and *connected* (which means there is a sequence of faces $F_1, \ldots, F_t$ between any two nonempty faces F_1 and F_t, such that $F_i \cap F_{i+1} \neq \emptyset$). Here we study when Δ is pure or connected in terms of $\Gamma_{\rm sf}(\Delta)$. In the following, by $\overline{G}$ we mean the complement of a graph G. Also we always assume that Δ is a simplicial complex on $[n]$, unless stated otherwise explicitly.

Theorem 2.1. *Suppose that Δ is a simplicial complex. Then Δ is connected if and only if $\overline{\Gamma_{\rm sf}(\Delta)}$ is connected.*

Proof. Suppose that Δ is connected. If x_F and x_G are two vertices of $\Gamma_{\rm sf}(\Delta)$, then there is a sequence $F = F_1, \ldots, F_t = G$ of faces of Δ such that $F_i \cap F_{i+1} \neq \emptyset$, for each i. Now in $\overline{\Gamma_{\rm sf}(\Delta)}$, we have the following path between x_F and x_G:

$$x_{F_1} \sim x_{F_1\cap F_2} \sim x_{F_2} \sim x_{F_2\cap F_3} \sim \cdots \sim x_{F_{t-1}\cap F_t} \sim x_{F_t}.$$

Conversely, assume that $\overline{\Gamma_{\rm sf}(\Delta)}$ is connected. If Δ is not connected, then there is a partition $[n] = V_1 \cup V_2$ such that no face of Δ has vertices from both V_1 and V_2. Let $A_1 = \{x_F|F \subseteq V_1\}$ and $A_2 = \{x_F|F \subseteq V_2\}$. Then $A_1 \cup A_2 = \mathrm{V}(\Gamma_{\rm sf}(\Delta))$. Since in $\Gamma_{\rm sf}(\Delta)$ we have $u_1 \sim u_2$ for each pair $u_i \in A_i$, thus in $\overline{\Gamma_{\rm sf}(\Delta)}$ there is no edge between A_1 and A_2 which means $\overline{\Gamma_{\rm sf}(\Delta)}$ is not connected. From this contradiction, we deduce that Δ is connected. $\square$

Theorem 2.2. *Suppose that Δ is a simplicial complex. Then Δ is pure if and only if there is a nonnegative integer r such that each vertex of $\Gamma_{\mathrm{sf}}(\Delta)$ which is in a largest clique has degree r.*

Proof. First suppose that F is a facet of Δ and A is an arbitrary face. Then $A \subseteq F$ if and only if $A \cup F \in \Delta$ if and only if $x_A x_F \neq 0$ in $K[\Delta]$ if and only if $x_A \not\sim x_F$. Therefore the set all vertices not adjacent to x_F correspond exactly to all nonempty subsets of F, the number of which is $2^{|F|} - 1$. Therefore $\deg(x_F) = |\mathrm{V}(\Gamma_{\mathrm{sf}}(\Delta))| - 2^{|F|} + 1$. Let r_F denote this number.

Now assume that x_G is a vertex of $\Gamma_{\mathrm{sf}}(\Delta)$ which is in a largest clique $\mathcal{A}$. If there are two facets F_1 and F_2 with $G \subseteq F_1, F_2$, then $(\mathcal{A} \setminus \{x_G\}) \cup \{x_{F_1}, x_{F_2}\}$ is a clique larger that $\mathcal{A}$, a contradiction. So G is contained in exactly one facet F of Δ. Now

$$x_A \sim x_G \Leftrightarrow A \cup G \notin \Delta \Leftrightarrow A \nsubseteq F \Leftrightarrow x_A \sim x_F.$$

Therefore $\deg(x_G) = r_F$. Thus each vertex of $\Gamma_{\mathrm{sf}}(\Delta)$ which is in a largest clique has degree r_F for some facet F. But as r_F just depends on $|F|$, Δ is pure if and only if all r_F's are equal and the result follows. $\square$

Remark 2.3. The proof of Theorem 2.2, shows that largest cliques of $\Gamma_{\mathrm{sf}}(\Delta)$ correspond to the set of all facets of Δ. More concretely, if for each facet F of Δ, we consider a face $G_F \subseteq F$ which is not contained in any other facet, then the set of all x_{G_F} for all facets F is a largest clique and every largest clique has this form. Also note that, by the proof of Theorem 2.2, x_{G_F} and x_F have the same neighborhoods.

As instant corollaries of the above results we get the following.

Corollary 2.4. *Suppose that I is a squarefree monomial ideal and $R = S/I$. If R is Cohen-Macaulay, then either $\Gamma_{\mathrm{sf}}(R)$ is a complete graph or $\overline{\Gamma_{\mathrm{sf}}(R)}$ is connected and the degrees of all vertices of $\Gamma_{\mathrm{sf}}(R)$ contained in a largest clique are equal.*

Proof. If $I \nsubseteq \langle x_1, \ldots, x_n \rangle^2$, say $x_r, \ldots, x_n \in I$ for some $1 \leq r \leq n$, then $R \cong S'/I'$ for $S' = K[x_1, \ldots, x_{r-1}]$ and a squarefree monomial ideal I' of S' contained in $\langle x_1, \ldots, x_{r-1} \rangle^2$. Clearly $\Gamma_{\mathrm{sf}}(R)$ does not differ whether we view R as S/I or S'/I'. Therefore we can assume that $I \subseteq \langle x_1, \ldots, x_n \rangle^2$, hence $I = I_\Delta$ for a simplicial complex Δ. If $\dim \Delta = 0$, then $\Gamma_{\mathrm{sf}}(\Delta)$ is a complete graph. Else, since R is Cohen-Macaulay, Δ is pure and connected and the result follows from Theorem 2.1 and Theorem 2.2. $\square$

Corollary 2.5. *Suppose that I is a squarefree monomial ideal and $R = S/I$. If the maximum size of an independent set of $\Gamma_{\mathrm{sf}}(R)$ which meets a largest clique is 3, then R is Cohen-Macaulay if and only if $\overline{\Gamma_{\mathrm{sf}}(R)}$ is connected.*

Proof. By [12, Corollary 3.6], $3 = 2^{\dim R} - 1 = 2^{\dim \Delta + 1} - 1$. By [2, Exercise 5.1.26(c)], when $\dim \Delta = 1$, then $K[\Delta]$ is Cohen-Macaulay if and only if Δ is connected. $\square$

3. Diameter and girth of $\Gamma_{\mathrm{sf}}(R)$

Next we study the diameter of squarefree zero-divisor graphs. Recall that the *diameter* of a graph G, denoted $\operatorname{diam} G$, is the maximum distance of two vertices in G. Let $R = K[\Delta]$ for the simplicial complex $\Delta = \langle F_1, \dots, F_t \rangle$. Note that the vertices of $\Gamma_{\mathrm{sf}}(R)$ corresponding to the faces F with $F \subseteq \cap_{i=1}^{t} F_i$ are isolated vertices. Thus if such faces exist, $\operatorname{diam} \Gamma_{\mathrm{sf}}(R) = \infty$. Assume that $\mathrm{Z}(R)$ denote the set of zero-divisors of R. Then the graph $\Gamma_{\mathrm{sf}}(R)$ is a disjoint union of $2^a - 1$ isolated vertices which are not in $\mathrm{Z}(R)$ (where $a = |\cap_{i=1}^{t} F_i|$) and the induced subgraph of $\Gamma_{\mathrm{sf}}(R)$ on $\mathrm{V}(\Gamma_{\mathrm{sf}}(R)) \cap \mathrm{Z}(R)$. Let's denote the latter graph by $\hat{\Gamma}_{\mathrm{sf}}(R)$. If Δ has exactly one facet, then R is a polynomial ring over K and $\Gamma_{\mathrm{sf}}(R)$ is just a set of isolated vertices. In other words, $\hat{\Gamma}_{\mathrm{sf}}(R)$ is empty in this case. The following result shows that $\hat{\Gamma}_{\mathrm{sf}}(R)$ is the "main part" of $\Gamma_{\mathrm{sf}}(R)$, in the sense that if we know $\hat{\Gamma}_{\mathrm{sf}}(R)$, then we can reconstruct $\Gamma_{\mathrm{sf}}(R)$, unless $\hat{\Gamma}_{\mathrm{sf}}(R) = \emptyset$.

Proposition 3.1. *Assume that R is a Stanley-Reisner ring with nonempty $\hat{\Gamma}_{\mathrm{sf}}(R)$. Suppose that α equals the size of a maximal independent set of $\hat{\Gamma}_{\mathrm{sf}}(R)$ meeting a largest clique. If $\alpha = 2^a p$ where p is odd, then $\Gamma_{\mathrm{sf}}(R)$ is a disjoint union of $\hat{\Gamma}_{\mathrm{sf}}(R)$ and $2^a - 1$ isolated vertices. Hence, if S is another Stanley-Reisner ring, then $R \cong S$ if and only if $\hat{\Gamma}_{\mathrm{sf}}(R) \cong \hat{\Gamma}_{\mathrm{sf}}(S)$.*

Proof. Let $R = K[\Delta]$ and $\Delta = \langle F_1, \dots, F_t \rangle$. We know that $\Gamma_{\mathrm{sf}}(R)$ is a disjoint union of $\hat{\Gamma}_{\mathrm{sf}}(R)$ and a set $\mathcal{J}$ of $2^{|F_0|} - 1$ isolated vertices, where $F_0 = \cap_{i=1}^{t} F_i$. Suppose that $\mathcal{I}$ is a maximal independent set of $\hat{\Gamma}_{\mathrm{sf}}(R)$ meeting a largest clique of $\hat{\Gamma}_{\mathrm{sf}}(R)$. Then this clique is also a largest clique in $\Gamma_{\mathrm{sf}}(R)$ and hence $\mathcal{I}' = \mathcal{I} \cup \mathcal{J}$ is a maximal independent set of $\Gamma_{\mathrm{sf}}(R)$ meeting a largest clique of $\Gamma_{\mathrm{sf}}(R)$. According to [12, Corollary 3.6], $|\mathcal{I}'| = 2^{\dim R} - 1$. Therefore, $2^a p = |\mathcal{I}| = 2^{\dim R} - 1 - (2^{|F_0|} - 1) = 2^{|F_0|}(2^{\dim R - |F_0|} - 1)$ and hence $a = |F_0|$, as required. The final statement follows from [12, Theorem 3.4]. $\square$

We can see $\hat{\Gamma}_{\mathrm{sf}}(R)$ as the zero-divisor graph of a semigroup. The zero-divisor graph $\Gamma(E)$ of a commutative semigroup E with a zero was introduced and studied in [4, 5]. The vertices of this graph are nonzero zero-divisors of E and two vertices x and y are connected when $xy = 0$. Set $E = \mathrm{V}(\hat{\Gamma}_{\mathrm{sf}}(R)) \cup \{0\}$ and define $0 \cdot x = 0$ for all $x \in E$ and

$$x_F \cdot x_{F'} = \begin{cases} x_{F \cup F'} & \text{if } x_F x_{F'} \neq 0 \\ 0 & \text{else} \end{cases},$$

for each $x_F, x_{F'} \in \mathrm{V}(\hat{\Gamma}_{\mathrm{sf}}(R))$. Then $(E, \cdot)$ is a commutative semigroup with zero and $\Gamma(E) = \hat{\Gamma}_{\mathrm{sf}}(R)$. We get the following as an immediate corollary of this fact.

Corollary 3.2. *Let R be a Stanley-Reisner ring with $\hat{\Gamma}_{\mathrm{sf}}(R) \neq \emptyset$. Then $\hat{\Gamma}_{\mathrm{sf}}(R)$ is connected and* $\operatorname{diam} \hat{\Gamma}_{\mathrm{sf}}(R) \leq 3$.

Proof. This holds for the zero-divisor graph of any semigroup by [4, Theorem 1.2]. □

In the sequel, we denote the unique minimal generating set of a monomial ideal I by $\mathrm{G}(I)$ and call the elements of $\mathrm{G}(I)$ the minimal generators of I. Also by P_F, where $F \subseteq [n]$, we mean the prime ideal generated by $\{x_i | i \in F\}$. By [7, Lemma 1.5.4] every minimal prime ideal of a squarefree monomial ideal has this form.

Theorem 3.3. *Let Δ be a simplicial complex on $[n]$ and $R = K[\Delta]$. Set $I = I_\Delta$ and assume that $\hat{\Gamma}_{\mathrm{sf}}(R) \neq \emptyset$.*

- **(i)** $\operatorname{diam} \hat{\Gamma}_{\mathrm{sf}}(R) = 1$ *if and only if $R \cong K[x_1, \ldots, x_n]/I$ where $I = \langle x_i x_j | 1 \leq i \neq j \leq n \rangle$ and $n = |\mathrm{V}(\Gamma_{\mathrm{sf}}(R))|$.*
- **(ii)** $\operatorname{diam} \hat{\Gamma}_{\mathrm{sf}}(R) \leq 2$ *if and only if for each pair of distinct indeterminates x_i and x_j each of which appears in some minimal generator of I with the property that $x_i x_j \notin I$, we have $(I : x_i) \cap (I : x_j) \not\subseteq I + \langle x_i, x_j \rangle$.*

Proof. (i) By Proposition 3.1, $\Gamma_{\mathrm{sf}}(R)$ is complete. It is easy to see that for the specified ring R, $\Gamma_{\mathrm{sf}}(R)$ is a complete graph on n vertices. According to [12, Theorem 3.4], $\Gamma_{\mathrm{sf}}(R) \cong \Gamma_{\mathrm{sf}}(R')$ if and only if $R \cong R'$, when R and R' are Stanley-Reisner rings over K, hence the result follows.

(ii) Assume that $\operatorname{diam} \hat{\Gamma}_{\mathrm{sf}}(R) \leq 2$ and x_i and x_j are distinct indeterminates appearing in some minimal generators of I such that $x_i x_j \notin I$. Since x_i and x_j appear in some minimal generators of I, $x_i, x_j \in \mathrm{Z}(R)$ and hence $x_i, x_j \in \hat{\Gamma}_{\mathrm{sf}}(R)$. Since $x_i x_j \neq 0$, these vertices are not adjacent and as $\operatorname{diam} \hat{\Gamma}_{\mathrm{sf}}(R) \leq 2$, there is a vertex $x_F \in \hat{\Gamma}_{\mathrm{sf}}(R)$ such that $x_i x_F = x_j x_F = 0$, in particular, $i, j \in \bar{F} = [n] \setminus F$. By replacing F with a facet of Δ containing F, we can assume that F is a facet of Δ.

According to [7, Lemma 1.5.4], $P_{\bar{F}}$ is a minimal prime of I and $x_i, x_j \in P_{\bar{F}}$. Suppose that $(I : x_i) \cap (I : x_j) \subseteq I + \langle x_i, x_j \rangle \subseteq P_{\bar{F}}$. Then either $(I : x_i) \subseteq P_{\bar{F}}$ or $(I : x_j) \subseteq P_{\bar{F}}$. Without loss of generality, assume that the former holds. Let $P' = P_{\bar{F} \setminus \{i\}}$ which is strictly contained in $P_{\bar{F}}$. So by the minimality of $P_{\bar{F}}$, we have $I \nsubseteq P'$. To complete the proof we get a contradiction by showing that $I \subseteq P'$. Let $u \in \mathrm{G}(I)$. If $x_i \nmid u$, then as $u \in P_{\bar{F}}$, $x_r | u$ for some $i \neq r \in \bar{F}$ and hence $u \in P'$. If $x_i | u$, then $u/x_i \in (I : x_i) \subseteq P_{\bar{F}}$ and $x_i \nmid (u/x_i)$. Thus as above, $u/x_i \in P'$ and hence $u \in P'$. This shows that $I \subseteq P'$ as required.

Conversely, assume that for each pair of distinct indeterminates x_i and x_j appearing in some minimal generator of I such that $x_i x_j \notin I$, we have $(I : x_i) \cap (I : x_j) \nsubseteq I + \langle x_i, x_j \rangle$. Let $x_{F_1} \neq x_{F_2}$ be two nonadjacent vertices of $\hat{\Gamma}_{\mathrm{sf}}(R)$. We must find a vertex adjacent to both x_{F_1} and x_{F_2}. If $F_1 \subseteq F_2$, then every vertex adjacent to x_{F_1} is also adjacent to x_{F_2}. So in this case, since $\hat{\Gamma}_{\mathrm{sf}}(R)$ has not isolated vertices, we are done. Thus we assume that F_1 and F_2 are incomparable. If $i \in F_1 \setminus F_2$ and $j \in F_2 \setminus F_1$, then x_i and x_j are nonadjacent and every vertex adjacent to both x_i and x_j is also adjacent to both x_{F_1} and x_{F_2}. So we can assume that $F_1 = \{i\}$ and $F_2 = \{j\}$.

By assumption, $(I : x_i) \cap (I : x_j) \nsubseteq I + \langle x_i, x_j \rangle$ and as $I + \langle x_i, x_j \rangle$ is squarefree monomial and hence a radical ideal, there is a minimal prime P_F of $I + \langle x_i, x_j \rangle$ such that $(I : x_i) \cap (I : x_j) \nsubseteq P_F$. Let P_G be a minimal prime of I contained in P_F. Then $(I : x_i) \nsubseteq P_G$ and $(I : x_j) \nsubseteq P_G$. Let $u \in (I : x_i) \setminus P_G$, then $x_i u \in I \subseteq P_G$ and hence $x_i \in P_G$, that is, $i \in G$. Similarly $j \in G$. According to [7, Lemma 1.5.4], $\bar{G} = [n] \setminus G$ is a facet of Δ and as $i, j \notin \bar{G}$, we have $x_i x_{\bar{G}} = x_j x_{\bar{G}} = 0$, that is, $x_{\bar{G}}$ is a neighbor of both x_i and x_j, as required. □

Recall that the *girth* of a graph is defined as the length of the smallest cycle of the graph. Girth of a squarefree zero-divisor graph can be easily found.

Proposition 3.4. *If the simplicial complex Δ has at least three different facets, then* $\operatorname{girth} \Gamma_{\mathrm{sf}}(\Delta) = 3$. *If Δ has exactly one facet, then* $\operatorname{girth} \Gamma_{\mathrm{sf}}(\Delta) = \infty$. *If Δ has two facets F_1, F_2 with $|F_1| \leq |F_2|$, then if $|F_1| = 1$, we have* $\operatorname{girth} \Gamma_{\mathrm{sf}}(\Delta) = \infty$, *and if $|F_1| > 1$, we have* $\operatorname{girth} \Gamma_{\mathrm{sf}}(\Delta) = 4$.

Proof. If Δ has at least three different facets F_1, F_2, F_3, then $x_{F_1} \sim x_{F_2} \sim x_{F_3} \sim x_{F_1}$ is a cycle with length 3. If Δ has one facet, then $\Gamma_{\mathrm{sf}}(\Delta)$ is a set of isolated vertices. If Δ has two facets F_1, F_2, then $\Gamma_{\mathrm{sf}}(\Delta)$ is disjoint union of a set of $2^{|F_1 \cap F_2|} - 1$ isolated vertices and a complete bipartite graph with partition sizes $2^{|F_1|} - 2^{|F_1 \cap F_2|}$ and $2^{|F_2|} - 2^{|F_1 \cap F_2|}$. From this, the result easily follows. □

4. When $\Gamma_{\mathrm{sf}}(R)$ is in a well-known class of graphs

In the final section of this article, we investigate when $\Gamma_{\mathrm{sf}}(R)$ is in some of the well-known classes of graph. Recall that a graph is *complete r-partite*, when its set of vertices can be partitioned to r subsets such that two vertices are adjacent if and only if they are from different subsets. Moreover, a graph is called *regular*, when all of its vertices have the same degree. Also a graph G is said to be *chordal*, when for each $S \subseteq \mathrm{V}(G)$, the induced subgraph of G on S is not a cycle, unless $|S| = 3$.

Theorem 4.1. *Let R be a Stanley-Reisner ring over a field K.*

- **(i)** $\Gamma_{\mathrm{sf}}(R)$ *is complete, if and only if* $R \cong K[x_1, \ldots, x_n]/I$ *where* $I = \langle x_i x_j | 1 \leq i \neq j \leq n\rangle$ *for some positive integer n.*
- **(ii)** $\Gamma_{\mathrm{sf}}(R)$ *is bipartite, if and only if* $R \cong K[x_1, \ldots, x_{n+m+a}]/I$ *where* $I = \langle x_i x_j | 1 \leq i \leq n, n+1 \leq j \leq n+m\rangle$*, for some nonnegative integers* a, n, m.
- **(iii)** $\Gamma_{\mathrm{sf}}(R)$ *is a complete r-partite graph, if and only if there are positive integers* $n_1, \ldots, n_r$ *such that* $R \cong K[x_{i,j} | 1 \leq i \leq r, 1 \leq j \leq n_i]/I$ *where* $I = \langle x_{i,j} x_{l,m} | i \neq l\rangle$.
- **(iv)** $\Gamma_{\mathrm{sf}}(R)$ *is regular, if and only if there are positive integers n, r such that* $R \cong K[x_{i,j} | 1 \leq i \leq r, 1 \leq j \leq n]/I$*, where* $I = \langle x_{i,j} x_{l,m} | i \neq l\rangle$.
- **(v)** $\Gamma_{\mathrm{sf}}(R)$ *is chordal, if and only if there is an integer* $0 \leq r \leq n$ *with* $0 < n$ *such that* $R \cong K[x_1, \ldots, x_n]/I$ *where either*

$$I = \langle x_i x_j | 1 \leq i \neq j \leq n, i \leq r\rangle \qquad \text{or}$$

$$I = \langle x_i x_j | 1 \leq i \neq j \leq n, i \leq r\rangle + \langle x_i x_j x_k | r < i < j < k \leq n\rangle.$$

Proof. Throughout the proof we assume that $R = K[\Delta]$ where $\Delta = \langle F_1, \ldots, F_t\rangle$.

(i) Indeed, this was proved in the proof of Theorem 3.3(i).

(ii) Suppose $\Gamma_{\mathrm{sf}}(R)$ is bipartite. Then $t = |\mathrm{Ass}(R)|$ equals the size of the largest clique of $\Gamma_{\mathrm{sf}}(R)$ by [12, Corollary 3.5]. Since $\Gamma_{\mathrm{sf}}(R)$ is bipartite, it follows that $t \leq 2$. If $t = 1$, then R has the claimed form with $a = |F_1|$ and $n = m = 0$. Assume that $t = 2$. Let $F_1 \setminus F_2 = \{1, \ldots, n\}$, $F_2 \setminus F_1 = \{n+1, \ldots, n+m\}$ and $F_1 \cap F_2 = \{n+m+1, \ldots, n+m+a\}$. Then it is easy to see that the ideal $I = I_\Delta$ has the claimed form. The converse is clear.

(iii) Suppose that $\Gamma_{\mathrm{sf}}(\Delta)$ is complete r-partite. Since the number of facets of Δ is $|\mathrm{Ass}(R)|$ and according to [12, Corollary 3.5], $t = r$. Suppose that $F_i \cap F_j \neq \emptyset$ for some $i \neq j$. Then $x_{F_j \cap F_i}$ is not adjacent to x_{F_i} and x_{F_j}. So these three vertices should be in one part which contradicts $x_{F_i} \sim x_{F_j}$. Consequently, the facets of Δ

are mutually disjoint. Thus the minimal non-faces of Δ are of the form $\{u, v\}$ with u and v from different facets. Hence if we assume $F_i = \{(i, j) | 1 \leq j \leq n_i\}$, then R is the ring claimed in the statement. Conversely, suppose that R has the specified form. Then it is easy to see that in $\Gamma_{\mathrm{sf}}(R)$ the set of vertices $P_i = \big\{x_F | F \subseteq \{(i, 1), \ldots, (i, n_i)\}\big\}$ is independent for each i and if $i \neq j$ then every vertex of P_i is adjacent to every vertex of P_j. Therefore, $\Gamma_{\mathrm{sf}}(R)$ is complete r-partite.

(iv) If R is the specified ring, then by part (iii), $\Gamma_{\mathrm{sf}}(R)$ is a complete r-partite graph in which all parts have the same size. Thus $\Gamma_{\mathrm{sf}}(R)$ is regular. Conversely, assume that $\Gamma_{\mathrm{sf}}(R)$ is regular. If $F_i \neq F_j$ are facets of Δ and $F_i \cap F_j \neq \emptyset$, then x_{F_i} is adjacent to all neighbors of $x_{F_i \cap F_j}$. Since $x_{F_j} \nsim x_{F_i \cap F_j}$ but $x_{F_i} \sim x_{F_j}$, $\deg(x_{F_i \cap F_j}) < \deg(x_{F_i})$ which contradicts regularity of $\Gamma_{\mathrm{sf}}(R)$. Therefore, as in the proof of the previous part, $\Gamma_{\mathrm{sf}}(R)$ is complete r-partite with part sizes $2^{|F_i|} - 1$. A complete r-partite graph is regular, only if all parts have the same size and from this the result follows.

(v) Suppose that R is any of the two rings mentioned in this part. It is routine to see that vertices of $\Gamma_{\mathrm{sf}}(R)$ can be partitioned into two sets V_1 and V_2, such that V_1 is a clique and V_2 is an independent set of $\Gamma_{\mathrm{sf}}(R)$. Therefore any induces cycle of $\Gamma_{\mathrm{sf}}(R)$ with length at least 4, has at most 2 vertices in V_1. Therefore, such a cycle must have at leas two adjacent vertices in V_2, a contradiction. Hence $\Gamma_{\mathrm{sf}}(R)$ is chordal.

Now assume that $\Gamma_{\mathrm{sf}}(R)$ is chordal. Let $V_1 = \{i \in [n] | \forall j \neq i \quad x_i x_j \in I\}$ and $V_2 = [n] \setminus V_1$. We can assume that $V_1 = [r]$ for some $r \leq n$. Suppose that $I \neq I_0 = \langle x_i x_j | 1 \leq i \neq j \leq n, i \leq r \rangle$. Thus there is a squarefree monomial $u \in \mathrm{G}(I) \setminus I_0$. If $\deg(u) = 2$, then $u = x_i x_j$ for some $r < i \neq j \leq n$. Since $i, j > r$, there are $i', j' \in [n]$ such that $x_i x_{i'}, x_j x_{j'} \notin I$. If $x_{i'} x_j \notin I$, then $x_{i'} x_i \sim x_j \sim x_i \sim x_{i'} x_j \sim x_{i'} x_i$ is an induced cycle of length 4 in $\Gamma_{\mathrm{sf}}(R)$. Therefore $x_{i'} x_j \in I$. But now $x_{i'} \sim x_j \sim x_i \sim x_j x_{j'} \sim x_{i'}$ is an induced cycle of length 4. This contradiction shows that $\deg(u) > 2$.

Suppose that $\deg(u) \geq 4$. Then $u = x_i x_j x_k x_l v$ for some squarefree monomial v. Now we have the induced cycle $x_i x_j \sim x_k x_l v \sim x_i x_j x_l \sim x_j x_k x_l v \sim x_i x_j$ with length 4, a contradiction. So $\deg(u) = 3$. Now we show that if $x_i x_j x_k \in \mathrm{G}(I)$ (for mutually different i, j, k), then $x_F \in \mathrm{G}(I)$ for each $F \subseteq \{r+1, \ldots, n\}$ such that $|F \cap \{i, j, k\}| = 2$. We just need to show that $x_F \in I$, for such sets F. Suppose $x_F \notin I$ for some such F, say $F = \{i, j, l\}$ with $l \notin \{i, j, k\}$. Then we get the following induced cycle of length 4 which is a contradiction: $x_k \sim x_i x_j \sim x_j x_k \sim x_i x_j x_l \sim x_k$. Now assume that $G = \{a, b, c\}$ is an arbitrary 3-subset of $\{r+1, \ldots, n\}$. If

$|G \cap \{i,j,k\}| = 1$, say $a = i$, then by the above argument $x_i x_j x_c \in I$ and hence again applying the above argument with $\{i,j,c\}$ instead of $\{i,j,k\}$ we see that $x_G \in \mathrm{G}(I)$. If $G \cap \{i,j,k\} = \emptyset$, then by a similar argument we see that $x_G \in \mathrm{G}(I)$. Thus

$$I = \langle x_i x_j | 1 \leq i \neq j \leq n, i \leq r\rangle + \langle x_i x_j x_k | r < i < j < k \leq n\rangle. \qquad \square$$

Recall that a graded ideal I of S has a linear resolution if it can be generated in degree d and its Castelnuovo-Mumford regularity (see definition (2) on [7, p. 48]) is equal to d. Also it is said that I is componentwise linear if for all j, the ideal generated by all homogenous polynomials of I with degree j has a linear resolution. Moreover, a squarefree monomial ideal I is called squarefree stable if for all monomials $u \in I$ (or equivalently, $u \in \mathrm{G}(I)$) and for all $j < m(u)$ such that x_j does not divide u, one has $x_j(u/x_{m(u)}) \in I$, where $m(u)$ denotes the largest index of an indeterminate which divides u.

Corollary 4.2. *Suppose that $R = S/I$ is a Stanley-Reisner ring. If $\Gamma_{\mathrm{sf}}(R)$ is complete or bipartite or complete r-partite or regular, then I has a linear resolution. If $\Gamma_{\mathrm{sf}}(R)$ is chordal, then R is componentwise linear.*

Proof. Suppose that $\Gamma_{\mathrm{sf}}(R)$ is complete or bipartite or complete r-partite or regular. Let G be the graph in which vertices denote the indeterminates of S and two vertices are adjacent when their product is in I. Using Theorem 4.1, it is easy to see that $\overline{G}$ is a chordal graph and it follows from [7, Theorem 9.2.12], that I has a linear resolution.

Now assume that $\Gamma_{\mathrm{sf}}(R)$ is chordal. According to [7, Proposition 8.2.17], to show that I is componentwise linear, we just need to show that $I_{[j]}$ has a linear resolution for each j, where $I_{[j]}$ denotes the ideal generated by the squarefree monomials of I with degree j. But it follows Theorem 4.1(v), that for all j, $I_{[j]}$ is squarefree stable. Thus by [7, Corollary 7.4.2] that $I_{[j]}$ has a linear resolution. $\square$

Suppose that $R = K[\Delta]$ and let $\Delta^{[i]} = \langle F \in \Delta | |F| = i+1\rangle$ be the *pure i skeleton* of Δ. Then R is called sequentially Cohen-Macaulay if $K[\Delta^{[i]}]$ is Cohen-Macaulay for each $i \leq \dim \Delta$. It is well-known that R is Cohen-Macaulay if and only if it is sequentially Cohen-Macaulay and Δ is pure.

Corollary 4.3. *Suppose that R is a Stanley-Reisner ring.*

- **(i)** *Suppose that $\Gamma_{\mathrm{sf}}(R)$ is complete. Then R is Cohen-Macaulay.*
- **(ii)** *Suppose that $\Gamma_{\mathrm{sf}}(R)$ is bipartite. Then R is Cohen-Macaulay if and only if $\Gamma_{\mathrm{sf}}(R)$ has at most one edge. Also R is sequentially Cohen-Macaulay if and only if $\Gamma_{\mathrm{sf}}(R)$ is a union of a star and a set of isolated vertices.*

(iii) *Suppose that $\Gamma_{\mathrm{sf}}(R)$ is complete r-partite. Then R is Cohen-Macaulay if and only if either $\Gamma_{\mathrm{sf}}(R)$ or $\overline{\Gamma_{\mathrm{sf}}(R)}$ is complete. Also R is sequentially Cohen-Macaulay if and only if the vertices of $\Gamma_{\mathrm{sf}}(R)$ can be partitioned into a clique V_1 and an independent set V_2, such that every vertex of V_1 is adjacent to every vertex of V_2.*

(iv) *Suppose that $\Gamma_{\mathrm{sf}}(R)$ is regular. Then R is Cohen-Macaulay if and only if R is sequentially Cohen-Macaulay if and only if either $\Gamma_{\mathrm{sf}}(R)$ or $\overline{\Gamma_{\mathrm{sf}}(R)}$ is complete.*

(v) *Suppose that $\Gamma_{\mathrm{sf}}(R)$ is chordal. Then R is sequentially Cohen-Macaulay. Also R is Cohen-Macaulay if and only if either $\Gamma_{\mathrm{sf}}(R)$ or $\overline{\Gamma_{\mathrm{sf}}(R)}$ is complete or the set of vertices of $\Gamma_{\mathrm{sf}}(R)$ can be partitioned into an independent set V_1 with size n and a clique V_2 with size $\binom{n}{2}$ such that for each vertex v of V_1, $\deg(v) = \binom{n}{2} - (n-1)$.*

Proof. (i) If $R = K[\Delta]$, then by Theorem 4.1(i), Δ is zero-dimensional and hence Cohen-Macaulay.

(iii) Let G be the graph in which vertices denote the indeterminates of S and two vertices are adjacent when their product is in I. Then it follows Theorem 4.1(iii) that G is a complete r-partite graph. Suppose that part i has size n_i with $1 \le n_1 \le n_2 \cdots \le n_r$. By [16, Theorem 2.17], R is sequentially Cohen-Macaulay if and only if $n_{r-1} = 1$. In this case, if we let V_1 to be the r-th part and V_2 to be the set of all other vertices, then V_1 and V_2 have the required properties. Note that since each part of G is a facet of Δ, thus if $R = K[\Delta]$, then Δ is pure, if and only if $n_r = n_1$. So R is Cohen-Macaulay if and only if either $r = 1$ or $r > 1$ and all n_i's are one. In the former case, $\Gamma_{\mathrm{sf}}(R)$ is a set of isolated vertices and in the latter case, $\Gamma_{\mathrm{sf}}(R)$ is a complete graph. (It should be noted that in [16, Theorem 2.12], which characterises Cohen-Macaulay complete t-partite graphs, it seems that the authors have assumed $t > 1$ without mentioning this.)

(ii) and (iv) By parts (ii) and (iv) of Theorem 4.1, if $\Gamma_{\mathrm{sf}}(R)$ is bipartite or regular, then it is a complete r-partite graph with possibly some extra isolated vertices. Noting that isolated vertices denote indeterminates not appeared in $\mathrm{G}(I)$ and using the fact that R is Cohen-Macaulay if and only if $R[x]$ is so, the result follows from part (iii).

(v) Suppose that $R = K[\Delta]$. Then by Theorem 4.1(v) either $\Delta = \langle \{i\} | 1 \le i \le r \rangle \cup \langle \{r+1, \dots, n\} \rangle$ or $\Delta = \langle \{i\} | 1 \le i \le r \rangle \cup \langle \{i,j\} | r < i < j \le n \rangle$. In both cases and for all $0 < i \le \dim \Delta$, $\Delta^{[i]}$ is the pure i skeleton of a simplex on $n - r$ vertices and hence is Cohen-Macaulay. Therefore R is sequentially Cohen-Macaulay. Thus

R is Cohen-Macaulay if and only if Δ is pure if and only if either $r = 0$ or $r = n$. If $r = n$, then $\Gamma_{\mathrm{sf}}(R)$ is complete. If $r = 0$, then either Δ has exactly one facet and $\Gamma_{\mathrm{sf}}(R)$ is a set of isolated vertices or $\Delta = \langle \{i, j\} | 1 \le i < j \le n \rangle$. In the latter case, $\mathrm{V}(\Gamma_{\mathrm{sf}}(R)) = V_1 \cup V_2$ with $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{x_F | F \subseteq [n], |F| = 2\}$. Also V_1 is an independent set and V_2 is a clique and each vertex x_i of V_1 is adjacent to all $x_F \in V_2$ with $i \notin F$. So $\deg(x_i) = |V_2| - (n - 1)$.

Conversely, if such a partition of $\mathrm{V}(\Gamma_{\mathrm{sf}}(R))$ exists, then $\Gamma_{\mathrm{sf}}(R)$ is chordal and Δ is one of the aforementioned complexes. Because there does not exist any vertex adjacent to all other vertices, we must have $r = 0$ and Δ is pure. From this, the claim follows. $\square$

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