International Electronic Journal of Algebra
Volume 32 (2022) 101-128
DOI: 10.24330/ieja.1096895

THE EFFECT OF S-ACCR ON INTERMEDIATE RINGS BETWEEN CERTAIN PAIRS OF RINGS

S. Visweswaran

Received: 16 June 2021; Revised: 22 December 2021; Accepted: 26 January 2022
Communicated by Abdullah Harmancı

Abstract. The rings considered in this article are commutative with identity and the modules are assumed to be unitary. If R is a subring of a ring T, then it is assumed that R contains the identity element of T. Let S be a multiplicatively closed subset (m.c. subset) of a ring R. In this paper, we consider the concept of S-accr, the generalization by Hamed and Hizem of the notion of (accr) in module theory given by Lu. We say that R satisfies (accr) if the increasing sequence of residuals of the form $(I :_R B) \subseteq (I :_R B^2) \subseteq (I :_R B^3) \subseteq \cdots$ is stationary for any ideal I of R and for any finitely generated ideal B of R. Focusing on certain pairs of rings $R \subseteq T$, the aim of this paper is to study whether S-accr on each intermediate ring A between R and T for a suitable m.c. subset S of A (depending on A) implies that A satisfies (accr) for each such A.

Mathematics Subject Classification (2020): 13A15, 13E99
Keywords: Accr pair, S-accr pair, Laskerian pair, strongly Laskerian pair, S-Noetherian pair

1. Introduction

The rings considered in this article are commutative with identity. Modules considered are assumed to be modules over commutative rings and are unitary. Let M be a module over a ring R. Recall from [10, Definition 1] that M is said to *satisfy* (*accr*) (respectively, (*accr**)) if the increasing sequence of residuals of the form $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ terminates for every submodule N of M and for every finitely generated (respectively, principal) ideal B of R. We use the abbreviation accr for ascending chain condition on residuals. A ring R is said to *satisfy* (*accr*) (respectively, (*accr**)) if it does as a module over itself. For interesting and inspiring theorems proved on rings and modules satisfying (accr), one can refer [10, 11]. If R is a subring of a ring T, then it is assumed that R contains the identity element of T. We say that (R, T) is an *accr pair* (respectively, *accr** *pair*) if A satisfies (accr) (respectively, (accr*)) for each subring A of T with

$R \subseteq A$. We use the abbreviation ACCRP (respectively, ACCR*P) for accr pair (respectively, accr* pair). It was proved in [10, Theorem 1] that for any module M over a ring R, the two properties (accr) and (accr*) are equivalent. Hence, for any subring R of a ring T, (R, T) is an ACCRP if and only if (R, T) is an ACCR*P.

Let M be module over a ring R. Let S be a multiplicatively closed subset (m.c. subset) of R. We use f.g. for finitely generated. Recall from [1] that M is said to be *S-finite* if there exist $s \in S$ and a f.g. submodule N of M such that $sM \subseteq N \subseteq M$. Recall that M is called *S-Noetherian* if every submodule of M is S-finite [1]. A ring R is said to be *S-Noetherian* if it is S-Noetherian as an R-module. In [1], Anderson and Dumitrescu stated and proved S-variant of Cohen's Theorem and Eakin-Nagata Theorem (see [1, Corollaries 5 and 7]). With certain suitable hypotheses, they also investigated the transfer of S-Noetherian property to the ring of polynomials and the ring of formal power series (see [1, Propositions 9 and 10]).

Let M be a module over a ring R. Let S be a m.c. subset of R. Recall from [6, Definition 2.1(1)] that an increasing sequence of submodules of M, $N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$ is said to be *S-stationary* if there exist $s \in S$ and $k \in \mathbb{N}$ such that $sN_j \subseteq N_k$ for all $j \geq k$. In [6], Hamed and Hizem generalized (accr) condition by introducing the definition of rings and modules satisfying S-accr condition. Recall from [6, Definition 3.1] that a module M is said to *satisfy S-accr* (respectively, *S-accr**) if the increasing sequence of residuals of the form $(N :_M B) \subseteq (N :_M B^2) \subseteq (N :_M B^3) \subseteq \cdots$ is S-stationary for any submodule N of M and for any f.g. (respectively, principal) ideal B of R. The ring R is said to *satisfy S-accr* (respectively, *S-accr**) if it does as an R-module. Several interesting results on modules satisfying S-accr were proved in [6]. It was shown in [6, Proposition 3.1] that for any R-module M, the properties S-accr and S-accr* are equivalent. Let N be a submodule of an R-module M. It was proved in [6, Theorem 3.2] that M satisfies S-accr if and only if N and $\frac{M}{N}$ satisfy S-accr. Let M be a f.g. module over R. If R satisfies S-accr, then it was shown in [6, Theorem 3.3] that M satisfies S-accr. If S is finite, then it was proved in [6, Theorem 3.4] that the polynomial ring $R[X]$ in one variable X over R satisfies S-accr if and only if R is an S-Noetherian ring.

Let $R \subseteq T$ be rings. In [15], certain pairs of rings $R \subseteq T$ were characterized such that (R, T) is an accr pair. The aim of this article is to determine whether A satisfies S-accr for each intermediate ring A between R and T for a suitable m.c. subset S of A (depending on A) implies that (R, T) is an accr pair.

Let M be a module over a ring R. Recall from [3, Exercise 23, page 295] that M is said to be a *Laskerian R-module* if M is a f.g. R-module and every proper submodule of M is a finite intersection of primary submodules of M. R is said to be a *Laskerian ring* if R is Laskerian as an R-module.

Let N be a $\mathfrak{p}$-primary submodule of an R-module M. N is said to be *strongly primary* if there exists a positive integer k such that $\mathfrak{p}^k M \subseteq N$. A f.g. R-module M is said to be a *strongly Laskerian R-module* if every proper submodule of M is a finite intersection of strongly primary submodules of M. R is said to be a *strongly Laskerian ring* if R is strongly Laskerian as an R-module [3, Exercise 28, page 298].

Several interesting and inspiring theorems were proved on Laskerian (respectively, strongly Laskerian) rings by Heinzer and Lantz in [8].

Let P be a property of rings. Let R be a subring of a ring T. We say that (R, T) is a P pair if A satisfies the property P for each intermediate ring A between R and T. Let S be a m.c. subset of R. We say that (R, T) is an S-P pair if A satisfies the property S-P for each intermediate ring A between R and T. We use the abbreviation NP (respectively, S-NP) for Noetherian pair (respectively, for S-Noetherian pair). The abbreviation LP (respectively, SLP) is used for Laskerian pair (respectively, for strongly Laskerian pair). We use the abbreviation S-ACCRP (respectively, S-ACCR*P) for S-accr pair (respectively, for S-accr* pair). By [6, Proposition 3.1], it follows that for any subring R of a ring T and any m.c. subset S of R, (R, T) is an S-ACCRP if and only if (R, T) is an S-ACCR*P.

For a ring R, let $Spec(R)$ denote the set of all prime ideals of R and let $Max(R)$ denote the set of all maximal ideals of R. If S is a m.c. subset of R, then we assume that $0 \notin S$. This article consists of five sections including the introduction. We denote by $R[X]$ (respectively, $R[[X]]$) the polynomial ring (respectively, the power series ring) in one variable X over R. In Section 2, we discuss the effect of S-accr on each intermediate ring A between R and $R[X]$ for a suitable m.c. subset S of A (depending on A) on the (accr) property of the intermediate rings between R and $R[X]$. It is proved in Theorem 2.7 that the following statements are equivalent: (1) $(R, R[X])$ is an SLP; (2)$(R, R[X])$ is an LP; (3) $(R, R[X])$ is an ACCRP; (4) For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R \backslash \mathfrak{p}$, the ring $R + (1 + rX)R[X]$ satisfies S_r-accr, where $S_r = \{r^n \mid n \in \mathbb{N} \cup \{0\}\}$; and (5) R is Artinian. For a m.c. subset S of a ring R, the concept of S-primary (respectively, S-strongly primary) ideals of R was introduced in [16] and also, the concept of S-Laskerian (respectively, S-strongly Laskerian) rings was introduced and studied in [16]. (The details of such a study is given in Remark 2.2 of this article.) The concept of S-strong accr* was introduced and studied in [17]. We use the abbreviation S-LP (respectively, S-SLP) for S-Laskerian pair (respectively, for S-strongly Laskerian pair). For a ring R, it is proved in Proposition 2.8 that the statements (1) to (5) of Theorem 2.7 are equivalent to each one of the following statements: ($1'$) For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R\backslash\mathfrak{p}$, the ring $R + (1 + rX)R[X]$ is S_r-strongly Laskerian; ($2'$) For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R\backslash\mathfrak{p}$, the ring $R + (1 + rX)R[X]$ is S_r-Laskerian; and ($3'$) For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R\backslash\mathfrak{p}$, the ring $R + (1 + rX)R[X]$ satisfies S_r-strong accr*.

Let R be a ring and let S be a m.c. subset of R. Let $f : R \rightarrow S^{-1}R$ denote the usual homomorphism of rings defined by $f(r) = \frac{r}{1}$. For any ideal I of R, the ideal $f^{-1}(S^{-1}I)$ is called the *saturation of I with respect to S* and is denoted by either $Sat_S(I)$ or $S(I)$. Let R be a subring of a ring T. We denote the collection $\{A \mid A \text{ is a subring of } T \text{ with } R \subseteq A\}$ by $[R, T]$.

We denote the nilradical of a ring R by $nil(R)$. A ring R is said to be *reduced* if $nil(R) = (0)$. Let S be a m.c. subset of a ring R. We use the abbreviation S-SACCR*P for S-strong accr* pair. It is shown in Theorem 2.9 that the following statements are equivalent: (1) $(R, R[X])$ is an S-SLP; (2) $(R, R[X])$ is an S-LP; (3) $(R, R[X])$ is an S-SACCR*P and for any $A \in [R, R[X]]$ and for any ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$; (4) $(R, R[X])$ is an S-ACCRP and for any $A \in [R, R[X]]$ and for any ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$; and (5) $S^{-1}R$ is Artinian and for any $A \in [R, R[X]]$ and for any ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$. Moreover, if R is reduced, then it is proved in Theorem 2.9 that the above statements (1) to (5) are equivalent to the following statement (6) $(R, R[X])$ is an S-NP.

Whenever a set A is a subset of a set B and $A \neq B$, we denote it by $A \subset B$. Let $R \subset T$ be rings. Let X be an indeterminate over T. Let $S = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$. It is clear that S is a m.c. subset of $R + XT[X]$ (respectively, $R + XT[[X]]$). In Section 3, necessary and sufficient conditions are determined in order that $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) to be an S-ACCRP. It is proved in Theorem 3.6 that the following statements are equivalent: (1) $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-SLP; (2) $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-LP; (3) $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-SACCR*P; (4) $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-ACCRP; (5) $T[X]$ (respectively, $T[[X]]$)

satisfies S-accr; (6) T is Noetherian; and (7) $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]]))$ is an S-NP.

Let $F_1 \subset F_2$ be fields. Let $n \geq 1$ and let for any field F, $F[X_1, \ldots, X_n]$ (respectively, $F[[X_1, \ldots, X_n]]$) denote the polynomial (respectively, the power series) ring in n variables $X_1, \ldots, X_n$ over F. It is convenient to denote $F_1[X_1, \ldots, X_n]$ by R and $F_2[X_1, \ldots, X_n]$ by T. In Section 4 of this article, for some m.c. subsets S of R, necessary and sufficient conditions are determined in order that an S-ACCRP (R, T) to be an ACCRP. Let $n = 1$. Let $f(X_1) \in R \backslash F_1$. Let $S = \{f(X_1)^m \mid m \in \mathbb{N} \cup \{0\}\}$. It is proved in Proposition 4.2 that the following statements are equivalent: (1) (R, T) is an LP; (2) (R, T) is an S-LP; (3) (R, T) is an ACCRP; (4) (R, T) is an S-ACCRP; (5) F_2 is an algebraic extension of F_1 ; and (6) T is integral over R. Let $n \geq 2$. Let $S = \{X_n^m \mid m \in \mathbb{N} \cup \{0\}\}$. It is shown in Proposition 4.3 that the following statements are equivalent: (1) (R, T) is an SLP; (2) (R, T) is an LP; (3) (R, T) is an ACCRP; (4) (R, T) is an S-ACCRP; (5) F_2 is a finite algebraic extension of F_1; and (6) (R, T) is an NP. It is proved in Proposition 4.4 that the statements (1) to (6) of Proposition 4.3 are equivalent to each one of the following statements: (1′) (R, T) is an S-SLP; (2′) (R, T) is an S-LP; and (3′) (R, T) is an S-SACCR*P. Let us denote $F_1[[X_1, \ldots, X_n]]$ by A and $F_2[[X_1, \ldots, X_n]]$ by B. Let $n = 1$. Let $S = \{X_1^m \mid m \in \mathbb{N} \cup \{0\}\}$. It is proved in Proposition 4.5 that the following statements are equivalent: (1) (A, B) is an LP; (2) (A, B) is an S-LP; (3) (A, B) is an ACCRP; (4) (A, B) is an S-ACCRP; (5) B is algebraic over A; and (6) B is integral over A. If $char(F_1) = 0$, then it is deduced in Corollary 4.6 that (A, B) is an ACCRP if and only if (A, B) is an NP. Let $n \geq 2$. Let $S = \{X_n^m \mid m \in \mathbb{N} \cup \{0\}\}$. It is shown in Theorem 4.7 that the following statements are equivalent: (1) (A, B) is an SLP; (2) (A, B) is an LP; (3) (A, B) is an ACCRP; (4) (A, B) is an S-ACCRP; (5) F_2 is a finite algebraic extension of F_1; and (6) (A, B) is an NP. It is proved in Proposition 4.8 that the statements (1) to (6) of Theorem 4.7 are equivalent to each one of the following statements: (1′) (A, B) is an S-SLP; (2′) (A, B) is an S-LP; and (3′) (A, B) is an S-SACCR*P.

Let R be a Noetherian domain which is not a field. Let S be a m.c. subset of R. It is proved in Theorem 5.1 that the following statements are equivalent: (1) $(R, S^{-1}R)$ is an SLP; (2) $(R, S^{-1}R)$ is an LP; (1) $(R, S^{-1}R)$ is an ACCRP; (4) $(R, S^{-1}R)$ is $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$-ACCRP for each $s \in S$ such that s is not a unit of R; (5) $S \subseteq C$, where C is the set of all elements of R which are contained in no $\mathfrak{m} \in Max(R)$ with $height\mathfrak{m} \geq 2$; and (6) $(R, S^{-1}R)$ is an NP. For a ring T, we denote the group of units of T by $U(T)$ and the set of all non-units of T by

$NU(T)$. With the same hypotheses as in the statement of Theorem 5.1, it is shown in Proposition 5.2 that the statements (1) to (6) of Theorem 5.1 are equivalent to each one of the following statements: (1′) $(R, S^{-1}R)$ is an $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$-SLP for each $s \in S \cap NU(R)$; (2′) $(R, S^{-1}R)$ is an $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$-LP for each $s \in S \cap NU(R)$; and (3′) $(R, S^{-1}R)$ is an $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$-SACCR*P for each $s \in S \cap NU(R)$.

2. The effect of S-accr on intermediate rings between R and $R[X]$

Let R be a ring. The aim of this section is to determine whether A satisfies S-accr for a suitable m.c. subset S of A (S depends on A) for each $A \in [R, R[X]]$ implies that $(R, R[X])$ is an ACCRP. Let $T = R[X]$ or $R[[X]]$. We first verify in Proposition 2.1 that T satisfies S_X-accr with $S_X = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$ if and only if R is Noetherian.

Proposition 2.1. *Let R be a ring and let $T = R[X]$ or $R[[X]]$. The following statements are equivalent:*

(1) *T is strongly Laskerian.*
(2) *T is Laskerian.*
(3) *T satisfies (accr).*
(4) *T satisfies S-accr for each m.c. subset S of T.*
(5) *T satisfies S_X-accr, where $S_X = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$.*
(6) *R is Noetherian.*

Proof. (1) $\Rightarrow$ (2) This is clear, since any strongly Laskerian ring is Laskerian.

(2) $\Rightarrow$ (3) As T is Laskerian, we obtain from [10, Proposition 3] that T satisfies (accr).

(3) $\Rightarrow$ (4) This is clear, since if a ring A satisfies (accr), then A satisfies S-accr for any m.c. subset S of A.

(4) $\Rightarrow$ (5) This is clear, since $S_X = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of T.

(5) $\Rightarrow$ (6) The proof proceeds along the same lines as the proof of [11, Theorem 2] except for some slight modifications. Let $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ be an increasing sequence of ideals of R. Let I be the ideal of $R[X]$ consisting of all $f(X) \in R[X]$ of the form $f(X) = \sum_{i=0}^{n} a_i X^i$ for some $n \geq 0$ with $a_i \in I_i$ for each $i \in \{0, \ldots, n\}$ in the case $T = R[X]$ and in the case $T = R[[X]]$, let I be the ideal of T consisting of all $f(X) \in R[[X]]$ of the form $f(X) = \sum_{i=0}^{\infty} a_i X^i$ with $a_i \in I_i$ for each $i \in \mathbb{N} \cup \{0\}$. By assumption T satisfies S_X-accr. Therefore, the increasing sequence of ideals of T, $(I :_T X) \subseteq (I :_T X^2) \subseteq (I :_T X^3) \subseteq \cdots$ is S_X-stationary. Hence, there

exist $s \in S_X$ and $k \in \mathbb{N}$ such that $s(I :_T X^j) \subseteq (I :_T X^k)$ for all $j \geq k$. We can assume that $s = X^n$ for some $n \in \mathbb{N}$. Thus $X^n(I :_T X^j) \subseteq (I :_T X^k)$ for all $j \geq k$. Let $m \geq 1$. Let $a \in I_{n+k+m}$. Then $a \in (I :_T X^{n+k+m})$ and from $X^n(I :_T X^{n+k+m}) \subseteq (I :_T X^k)$, we get that $aX^{n+k} \in I$. It follows from the definition of I that $a \in I_{n+k}$. This proves that $I_{n+k+m} \subseteq I_{n+k}$ and hence, $I_{n+k+m} = I_{n+k}$. This is true for all $m \geq 1$. This shows that any increasing sequence of ideals of R is stationary and therefore, R is Noetherian.

(6) $\Rightarrow$ (1) We are assuming that R is Noetherian. Hence, we obtain from Hilbert's Basis Theorem [2, Theorem 7.5] that $R[X]$ is Noetherian and we obtain from [9, Theorem 71] that $R[[X]]$ is Noetherian. Since any Noetherian ring is strongly Laskerian (see [2, Theorem 7.13 and Proposition 7.14]), we get that T is strongly Laskerian. $\square$

Remark 2.2. Let S be a m.c. subset of a ring R. Motivated by the work on S-prime ideals of a ring in [7], the concept of S-primary ideals of a ring was introduced and studied in [16]. Let $\mathfrak{q}$ be an ideal of R with $\mathfrak{q} \cap S = \emptyset$. Recall from [16] that $\mathfrak{q}$ is an *S-primary ideal* of R if the following condition holds: there exists $s \in S$ such that for all $a, b \in R$ with $ab \in \mathfrak{q}$, either $sa \in \mathfrak{q}$ or $sb \in \sqrt{\mathfrak{q}}$. If in addition, there exist $s' \in S$ and $n \in \mathbb{N}$ such that $s'(\sqrt{\mathfrak{q}}\,)^n \subseteq \mathfrak{q}$, then $\mathfrak{q}$ is said to be an *S-strongly primary* ideal of R. (In [16], an S-strongly primary ideal was referred to as a strongly S-primary ideal.) Some basic properties of S-primary (respectively, S-strongly primary) ideals of a ring were proved in [16]. Also, the concept of S-Laskerian rings was introduced and studied in [16]. Let I be an ideal of R such that $I \cap S = \emptyset$. Recall from [16, Introduction to Section 3] that I is said to be *S-decomposable* (respectively, *S-strongly decomposable*) if I is a finite intersection of S-primary (respectively, S-strongly primary) ideals of R. (In [16], an S-strongly decomposable ideal was referred to as a strongly S-decomposable ideal.) Also, recall from [16] that R is said to be *S-Laskerian* (respectively, *S-strongly Laskerian*) if each proper ideal I of R, either $I \cap S \neq \emptyset$ or there exists $s \in S$ such that $(I :_R s)$ is S-decomposable (respectively, S-strongly decomposable). (In [16], an S-strongly Laskerian ring was referred to as a strongly S-Laskerian ring.) It was verified in [16, Introduction to Section 3] that any Laskerian (respectively, strongly Laskerian) ring is S-Laskerian (respectively, S-strongly Laskerian).

Recall from [17] that a ring R is said to *satisfy strong accr** if for any ideal I of R and for any sequence $< a_n >$ of elements of R, the increasing sequence of residuals of the form $(I :_R a_1) \subseteq (I :_R a_1a_2) \subseteq (I :_R a_1a_2a_3) \subseteq \cdots$ terminates. Let S be a m.c. subset of R. Recall from [17] that R is said to *satisfy S-strong accr** if for any ideal I of R and for any sequence $< a_n >$ of elements of R, the increasing sequence of residuals of the form $(I :_R a_1) \subseteq (I :_R a_1a_2) \subseteq (I :_R a_1a_2a_3) \subseteq \cdots$ is S-stationary.

Proposition 2.3. *Let R, T be as in the statement of Proposition 2.1. The statements (1) to (6) of Proposition 2.1 are equivalent to each one of the following statements:*

(1′) *T is S_X-strongly Laskerian, where $S_X = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$.*

(2′) *T is S_X-Laskerian.*

(3′) *T satisfies S_X-strong accr*.*

Proof. If A is any Laskerian (respectively, strongly Laskerian) ring, then A is S-Laskerian (respectively, S-strongly Laskerian) for any m.c. subset S of A. Therefore, $(1) \Rightarrow (1')$ and $(2) \Rightarrow (2')$.

$(1') \Rightarrow (3')$ As T is S_X-strongly Laskerian by assumption, it follows from [16, Corollary 3.9(2)] that T satisfies S_X-strong accr*.

$(3') \Rightarrow (1)$ If a ring A satisfies S-strong accr* for some m.c. subset S of A, then it is clear that A satisfies S-accr* and so, A satisfies S-accr by [6, Proposition 3.1]. Hence, if $(3')$ holds, then it follows that T satisfies S_X-accr and so, (5) of Proposition 2.1 holds. Therefore, (1) of Proposition 2.1 holds.

$(2') \Rightarrow (2)$ As T is S_X-Laskerian, we obtain from [16, Corollary 3.9(1)] that T satisfies S_X-accr* and so, T satisfies S_X-accr. Therefore, (5) of Proposition 2.1 holds and so, (2) of Proposition 2.1 holds. □

Proposition 2.4. *Let r be a non-zero-divisor of a ring R. Let $S_r = \{r^n \mid n \in \mathbb{N} \cup \{0\}\}$. Consider the following statements:*

(1) *$(R, R[X])$ is an ACCRP.*

(2) *$(R, R[X])$ is an S_r-ACCRP.*

(3) *$A = R + (1 + rX)R[X]$ satisfies S_r-accr.*

(4) *$r \in U(R)$ and R is Noetherian.*

Then $(1) \Rightarrow (2) \Rightarrow (3) \Leftrightarrow (4)$.

Proof. It is clear that S_r is a m.c. subset of R and hence, for each $A \in [R, R[X]]$.

$(1) \Rightarrow (2)$ This is clear, since if a ring A satisfies (accr), then it satisfies S-accr for any m.c. subset S of A.

$(2) \Rightarrow (3)$ This is clear, since $A = R + (1 + rX)R[X] \in [R, R[X]]$.

$(3) \Rightarrow (4)$ We are assuming that $A = R + (1 + rX)R[X]$ satisfies S_r-accr. We prove $r \in U(R)$ by using the arguments found in the proof of [15, Proposition 1.3]. Let us denote the ideal $A(1+rX)$ of A by I. Since A satisfies S_r-accr, we obtain that the increasing sequence of ideals of A, $(I :_A r) \subseteq (I :_A r^2) \subseteq (I :_A r^3) \subseteq \cdots$ is S_r-stationary. Hence, it follows that there exist $s \in S_r$ and $k \in \mathbb{N}$ such that $s(I :_A r^j) \subseteq (I :_A r^k)$ for all $j \geq k$. We can assume that $s = r^n$ for some $n \in \mathbb{N}$. Now, as $I = A(1 + rX)$ and $(1 + rX)X^{n+k+1}, rX \in A$, we get that $(1+rX)X^{n+k+1} \in (I :_A r^{n+k+1})$. Therefore, $r^n(1+rX)X^{n+k+1} \in (I :_A r^k)$. This implies that $r^n(1 + rX)X^{n+k+1}r^k = (1 + rX)a$ for some $a \in A$. Since there is no non-zero $y \in R$ such that $(1 + rX)y = 0$, it follows from McCoy's Theorem [12, Theorem 2] that $1 + rX$ is a non-zero-divisor of $R[X]$. Hence, we obtain that $r^{n+k}X^{n+k+1} = a \in A$. As $r^{n+k-1}X^{n+k} = (1+rX)r^{n+k-1}X^{n+k} - r^{n+k}X^{n+k+1}$ and $(1 + rX)r^{n+k-1}X^{n+k}, r^{n+k}X^{n+k+1} \in A$, it follows that $r^{n+k-1}X^{n+k} \in A$. From $r^{n+k-2}X^{n+k-1} = (1 + rX)r^{n+k-2}X^{n+k-1} - r^{n+k-1}X^{n+k}$, we get that $r^{n+k-2}X^{n+k-1} \in A$. Proceeding like this, we obtain that $X \in A$. Therefore, $X = y + (1 + rX)f(X)$ for some $y \in R$ and $f(X) \in R[X]$. Since r is a non-zero-divisor of R, it follows that $f(X) \in R\backslash\{0\}$. By comparing the coefficient of X on both sides of $X = y + (1 + rX)f(X)$, we obtain that $1 \in Rr$ and so, $r \in U(R)$. In such a case, from $rX \in A$ and $r^{-1} \in A$, we get that $X \in A$. Therefore, $A = R[X]$ satisfies S_r-accr. It follows from $r \in U(R)$, the properties S_r-accr and (*accr*) are equivalent. Therefore, $R[X]$ satisfies (accr) and so, we obtain from [11, Theorem 2] that R is Noetherian.

(4) $\Rightarrow$ (3) By (4), $r \in U(R)$. It is observed in the proof of (3) $\Rightarrow$ (4) of this proposition that $A = R + (1 + rX)R[X] = R[X]$. By (4) R is Noetherian and so, $R[X]$ is Noetherian. Therefore, $R[X]$ satisfies (accr). As $r \in U(R)$, S_r-accr is the same as (accr). $\square$

We provide Example 2.5(1) to illustrate that (4) $\Rightarrow$ (3) of Proposition 2.4 can fail to hold if we drop the assumption that R is Noetherian in the statement (4). We provide Example 2.5(2) to illustrate that (3) $\Rightarrow$ (2) of Proposition 2.4 can fail to hold. We say that a ring R is *quasiocal* if R has a unique maximal ideal. A Noetherian quasilocal ring is referred to as a *local ring*.

Example 2.5. (1) Let V be an infinite dimensional vector space over a field K. Let $R = K(+)V$ be the ring obtained by using Nagata's principle of idealization. Then each non-zero-divisor of R is a unit in R but (3) of Proposition 2.4 does not hold.

(2) Let K be a field. Let $T = K[[X, Y]]$ and let $I = TX^2 + TXY$. Let $R = \frac{T}{I}$. Then each non-zero-divisor of R is a unit in R and the statement (3) of

Proposition 2.4 holds for R but the statement (2) of Proposition 2.4 does not hold for R.

Proof. (1) Let $\mathfrak{m} = (0)(+)V$. It is clear that $\mathfrak{m} \in Max(R)$. From $\mathfrak{m}^2$ is the zero ideal of R, it follows that R is quasilocal with $\mathfrak{m}$ as its unique maximal ideal. From $\mathfrak{m} = Z(R)$, we obtain that any non-zero-divisor of R is a unit in R. Let r be any non-zero-divisor of R. Let $A = R + (1 + rX)R[X]$. It is clear that $A = R[X]$. By assumption, V is an infinite dimensional vector space over K and so, R is not Noetherian. Therefore, we obtain from [11, Theorem 2] that $R[X]$ does not satisfy (accr). Since $r \in U(R)$, the property S_r-accr coincides with (accr). Hence, we get that $R[X]$ does not satisfy S_r-accr.
(2) It follows from [2, Exercise 5(iv), page 11] that $\mathfrak{m} = TX + TY$ is the unique maximal ideal of T. Hence, $\frac{\mathfrak{m}}{I}$ is the unique maximal ideal of $R = \frac{T}{I}$. Let $m \in \mathfrak{m}$. Then $m = t_1X + t_2Y$ for some $t_1, t_2 \in T$. Notice that $X \notin I = TX^2 + TXY$ and $(m + I)(X + I) = t_1X^2 + t_2XY + I = 0 + I$. This shows that $\frac{\mathfrak{m}}{I} \subseteq Z(R)$ and so, $Z(R) = \frac{\mathfrak{m}}{I}$. From $R\backslash(\frac{\mathfrak{m}}{I}) = U(R)$, we get that each non-zero divisor of R is a unit in R. We know from [9, Theorem 71] that T is Noetherian and so, $R = \frac{T}{I}$ is Noetherian. Let $R[Z]$ be the polynomial ring in one variable Z over R. Let r be any non-zero-divisor of R. Let $A = R + (1 + rZ)R[Z]$. Since $r \in U(R)$, it follows that $A = R[Z]$. Since R is Noetherian, we obtain that $R[Z]$ is Noetherian and so, $A = R[Z]$ satisfies (accr). Hence, A satisfies S_r-accr, where $S_r = \{r^n \mid n \in \mathbb{N} \cup \{0\}\}$. Thus the statement (3) of Proposition 2.4 holds. Notice that $\frac{TX}{I} \in Spec(R)\backslash Max(R)$ and so, we obtain from [2, Proposition 8.1] that R is not Artinian. Hence, we obtain from (1) $\Rightarrow$ (2) of [15, Theorem 1.1] that $(R, R[Z])$ is not an ACCRP. Equivalently, $(R, R[Z])$ is not an S_r-ACCRP. Therefore, the statement (2) of Proposition 2.4 does not hold. $\square$

Lemma 2.6. *Let R, T be rings. Let $\phi : R \to T$ be a homomorphism of rings such that ϕ is onto. If S is a m.c. subset of R such that $\phi(s) \neq 0$ for each $s \in S$, then $\overline{S} = \{\phi(s) \mid s \in S\}$ is a m.c. subset of T. Moreover, if R satisfies S-accr, then T satisfies $\overline{S}$-accr.*

Proof. It can be easily verified that $\overline{S}$ is a m.c. subset of T. Assume that R satisfies S-accr. As the properties S-accr and S-accr* are equivalent by [6, Proposition 3.1], it is enough to show that the increasing sequence of ideals of T of the form $(A :_T t) \subseteq (A :_T t^2) \subseteq (A :_T t^3) \subseteq \cdots$ is $\overline{S}$-stationary for any ideal A of T and for any element $t \in T$. Since ϕ is onto, there exist an ideal J of R with $J \supseteq ker(\phi)$ and $r \in R$ such that $A = \phi(J)$ and $t = \phi(r)$. As R satisfies S-accr, there exist $s \in S$ and

$k \in \mathbb{N}$ such that $s(J :_R r^j) \subseteq (J :_R r^k)$ for all $j \geq k$. Let $j \geq k$. Let $t' \in (A :_T t^j)$. Then $t' = \phi(r')$ for some $r' \in R$. From $t't^j \in \phi(J)$ and $J \supseteq ker(\phi)$, it follows that $r'r^j \in J$ and so, $sr' \in (J :_R r^k)$. Hence, $\phi(s)\phi(r')(\phi(r))^k \in \phi(J) = A$. This shows that $\phi(s)t' \in (A :_T t^k)$. Therefore, we obtain that there exist $\phi(s) \in \overline{S}$ and $k \in \mathbb{N}$ such that $\phi(s)(A :_T t^j) \subseteq (A :_T t^k)$ for all $j \geq k$. This proves that T satisfies $\overline{S}$-accr. $\square$

The Krull dimension of a ring R is simply referred to as the dimension of R and is denoted by $dimR$.

Theorem 2.7. *Let R be a ring. The following statements are equivalent:*

(1) *$(R, R[X])$ is an SLP.*
(2) *$(R, R[X])$ is an LP.*
(3) *$(R, R[X])$ is an ACCRP.*
(4) *For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R\backslash\mathfrak{p}$, the ring $R + (1 + rX)R[X]$ satisfies S_r-accr, where $S_r = \{r^n \mid n \in \mathbb{N} \cup \{0\}\}$.*
(5) *R is Artinian.*

Proof. (1) $\Rightarrow$ (2) This is clear, since any strongly Laskerian ring is Laskerian.

(2) $\Rightarrow$ (3) It follows from (2) and [10, Proposition 3] that $(R, R[X])$ is an ACCRP.

(3) $\Rightarrow$ (4) Let $\mathfrak{p} \in Spec(R)$ and let $r \in R\backslash\mathfrak{p}$. The ring $R+(1+rX)R[X] \in [R, R[X]]$ and so by (3), $R+(1+rX)R[X]$ satisfies (accr). Hence, $R+(1+rX)R[X]$ satisfies S_r-accr.

(4) $\Rightarrow$ (5) Let $\mathfrak{p} \in Spec(R)$. Notice that $1 \in R\backslash\mathfrak{p}$. By (4), $R + (1 + X)R[X]$ satisfies S_1-accr. It is clear that $R + (1 + X)R[X] = R[X]$. Therefore, we get that $R[X]$ satisfies S_1-accr. As S_1-accr and (accr) are equivalent, we obtain from [11, Theorem 2] that R is Noetherian. Let $\mathfrak{p} \in Spec(R)$ and let t be a non-zero element of $\frac{R}{\mathfrak{p}}$. Let $r \in R$ be such that $t = r + \mathfrak{p}$. From $t \neq 0 + \mathfrak{p}$, it follows that $r \notin \mathfrak{p}$. Let $\phi : R[X] \to \frac{R}{\mathfrak{p}}[X]$ be the homomorphism of rings defined by $\phi(\sum_{i=0}^{n} r_i X^i) = \sum_{i=0}^{n}(r_i + \mathfrak{p})X^i$. It is clear that ϕ is an onto homomorphism of rings. Let us denote the subring $R + (1 + rX)R[X]$ of $R[X]$ by A. By assumption, A satisfies S_r-accr. For an element $a \in R$, we denote $a + \mathfrak{p}$ by $\overline{a}$. From $\phi(A) = \frac{R}{\mathfrak{p}} + (\overline{1} + \overline{r}X)\frac{R}{\mathfrak{p}}[X]$ and $\phi(S_r) = \{\overline{r}^n \mid n \in \mathbb{N} \cup \{0\}\} = S_{\overline{r}}$. As A satisfies S_r-accr, we obtain from Lemma 2.6 that $\phi(A)$ satisfies $S_{\overline{r}}$-accr. Since $\frac{R}{\mathfrak{p}}$ is an integral domain, it follows that $t = \overline{r}$ is a non-zero-divisor of $\frac{R}{\mathfrak{p}}$. From $\phi(A)$ satisfies $S_{\overline{r}}$-accr, we obtain from (3) $\Rightarrow$ (4) of Proposition 2.4 that $t = \overline{r}$ is a unit in $\frac{R}{\mathfrak{p}}$. This shows that each non-zero element of $\frac{R}{\mathfrak{p}}$ is a unit in $\frac{R}{\mathfrak{p}}$ and so, $\frac{R}{\mathfrak{p}}$ is a field. Therefore, $\mathfrak{p} \in Max(R)$. Hence,

$Spec(R) = Max(R)$ and so, $dim R = 0$. Thus R is Noetherian and $dim R = 0$. Therefore, we obtain from [2, Theorem 8.5] that R is Artinian.

$(5) \Rightarrow (1)$ This is $(2) \Rightarrow (3)$ of [15, Theorem 1.1]. □

Proposition 2.8. *For any ring R, the statements* (1) *to* (5) *of Theorem 2.7 are equivalent to each one of the following statements:*

- $(1')$ *For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R \backslash \mathfrak{p}$, the ring $R + (1 + rX)R[X]$ is S_r-strongly Laskerian, where $S_r = \{r^n \mid n \in \mathbb{N} \cup \{0\}\}$.*
- $(2')$ *For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R \backslash \mathfrak{p}$, the ring $R + (1 + rX)R[X]$ is S_r-Laskerian.*
- $(3')$ *For any $\mathfrak{p} \in Spec(R)$ and for any $r \in R \backslash \mathfrak{p}$, the ring $R + (1 + rX)R[X]$ satisfies S_r-strong accr*.*

Proof. This proposition can be proved using arguments similar to those that are used in the proof of Proposition 2.3. □

Let R be a subring of a ring T. Let S be a m.c. subset of R. We say that (R, T) is an *S-Laskerian pair* (respectively, *S-strongly Laskerian pair*) if A is S-Laskerian (respectively, S-strongly Laskerian) for each $A \in [R, T]$. We use the abbreviation S-LP (respectively, S-SLP) for S-Laskerian pair (respectively, for S-strongly Laskerian pair). We use the abbreviation S-SACCR$^*P$ for S-strong accr* pair.

Theorem 2.9. *Let S be a m.c. subset of a ring R. The following statements are equivalent:*

1. *$(R, R[X])$ is an S-SLP.*
2. *$(R, R[X])$ is an S-LP.*
3. *$(R, R[X])$ is an S-SACCR$^*P$ and for any $A \in [R, R[X]]$ and for any given ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$.*
4. *$(R, R[X])$ is an S-ACCRP and for any $A \in [R, R[X]]$ and for any given ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$.*
5. *$S^{-1}R$ is Artinian and for any $A \in [R, R[X]]$ and for any given ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$.*

   Moreover, if R is reduced, then the above equivalent statements (1) *to* (5) *are equivalent to the following statement:*
6. *$(R, R[X])$ is an S-NP.*

Proof. $(1) \Rightarrow (2)$ This is clear, since any S-strongly Laskerian ring is S-Laskerian.

(1) $\Rightarrow$ (3) We know from [16, Corollary 3.9(2)] that any S-strongly Laskerian ring satisfies S-strong accr*. Thus if (1) holds, then $(R, R[X])$ is an S-SACCR*P. Let $A \in [R, R[X]]$. Let I be an ideal of A. If $I \cap S \neq \emptyset$, then $S(I) = A = (I :_A s)$ for any $s \in I \cap S$. If $I \cap S = \emptyset$, then it follows from (1) $\Rightarrow$ (2) of [16, Proposition 3.2] that there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$.

(3) $\Rightarrow$ (4) Let $A \in [R, R[X]]$. As A satisfies S-strong accr*, it follows that A satisfies S-accr* and so, A satisfies S-accr by [6, Proposition 3.1]. Therefore, $(R, R[X])$ is an S-ACCRP. The rest of the assertion stated in (4) follows immediately from (3).

(4) $\Rightarrow$ (5) Let A be any subring of $R[X]$ such that $R \subseteq A$. By hypothesis, A satisfies S-accr. Hence, it is clear that A satisfies S-accr*. As A satisfies S-accr*, we obtain from [6, Example 3.1(3)] that $S^{-1}A$ satisfies (accr*). Let B be a subring of $S^{-1}R[X] = (S^{-1}R)[X]$ such that $S^{-1}R \subseteq B$. Then $B = S^{-1}A$ for some subring A of $R[X]$ such that $R \subseteq A$. Hence, $B = S^{-1}A$ satisfies (accr*). This shows that $(S^{-1}R, (S^{-1}R)[X])$ is an ACCRP. Therefore, it follows from (1) $\Rightarrow$ (2) of [15, Theorem 1.1] that $S^{-1}R$ is Artinian. Let $A \in [R, R[X]]$. By (4), for any given ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$.

(5) $\Rightarrow$ (1) Since $S^{-1}R$ is Artinian, we know from (2) $\Rightarrow$ (3) of [15, Theorem 1.1] that $(S^{-1}R, (S^{-1}R)[X])$ is an SLP. Let $A \in [R, R[X]]$. From $S^{-1}(R[X]) = (S^{-1}R)[X]$, we get that $S^{-1}A \in [S^{-1}R, (S^{-1}R)[X]]$. Hence, $S^{-1}A$ is strongly Laskerian. By (5), for any ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_R s)$. Hence, we obtain from (2) $\Rightarrow$ (1) of [16, Proposition 3.2] that A is S-strongly Laskerian. This proves that $(R, R[X])$ is an S-SLP.

(2) $\Rightarrow$ (4) This can be proved using arguments similar to those that are used in the proof of (1) $\Rightarrow$ (3) of this theorem with the help of [16, Corollary 3.9(1)], [6, Proposition 3.1], and (1) $\Rightarrow$ (2) of [16, Proposition 3.2].

Assume that R is reduced and the statement (1) holds. We know from (1) $\Rightarrow$ (5) of this theorem that $S^{-1}R$ is Artinian. As R is reduced by assumption, it follows that $S^{-1}R$ is reduced. Since a local Artinian reduced ring is a field, we obtain from [2, Theorem 8.7] that $S^{-1}R$ is isomorphic to a finite direct product of fields. It follows from (3) $\Rightarrow$ (2) of [5, Theorem 2.3] that $(S^{-1}R, (S^{-1}R)[X])$ is an NP. Let $A \in [R, R[X]]$. As $S^{-1}(R[X]) = (S^{-1}R)[X]$, we get that $S^{-1}A \in [S^{-1}R, (S^{-1}R)[X]]$ and so, $S^{-1}A$ is Noetherian. It is already noted in the proof of (1) $\Rightarrow$ (3) of this theorem that for any $A \in [R, R[X]]$ and for any ideal I of A, there exists $s \in S$ (depending on I) such that $S(I) = (I :_A s)$. Hence, we obtain from [1, Proposition 2(f)] that A is S-Noetherian. This shows that $(R, R[X])$ is an S-NP. This proves (1) $\Rightarrow$ (6).

$(6) \Rightarrow (1)$ Assume that (6) holds. Let $A \in [R, R[X]]$. Then A is S-Noetherian. We know from [16, Corollary 3.3] that any S-Noetherian ring (it can be non-reduced) is S-strongly Laskerian. Therefore, A is S-strongly Laskerian. Hence, we obtain that $(R, R[X])$ is an S-SLP. Notice that the proof of $(6) \Rightarrow (1)$ does not need the assumption that R is reduced. $\square$

In Example 2.10, we provide an integral domain D and a m.c. subset S of D such that $S^{-1}D$ is Artinian but $(D, D[X])$ is not an S-ACCRP.

Example 2.10. Let p be a prime number and let $D = \mathbb{Z}_{p\mathbb{Z}}$. Let $S = \{p^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then $S^{-1}D = \mathbb{Q}$ but $(D, D[X])$ is not an S-ACCRP.

Proof. It is well-known that $D = \mathbb{Z}_{p\mathbb{Z}}$ is a local one-dimensional domain with pD as its unique maximal ideal. Notice that $S = \{p^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of D and $S^{-1}D = \mathbb{Q}$. Thus $S^{-1}D$ is a field and hence, it is Artinian. Let $A = D + (1 + pX)D[X]$. Since p is a non-zero-divisor of D but not a unit of D, we obtain from $(3) \Rightarrow (4)$ of Proposition 2.4 that A does not satisfy S-accr. Therefore, $(D, D[X])$ is not an S-ACCRP. $\square$

We provide Example 2.11 to illustrate that $(1) \Rightarrow (6)$ of Theorem 2.9 can fail to hold if R is not reduced.

Example 2.11. Let $R = \mathbb{Z}_4$ and let $S = \{1\}$. Then $(R, R[X])$ is an S-SLP but $(R, R[X])$ is not an S-NP.

Proof. Notice that $R = \mathbb{Z}_4$ is Artinian. Hence, we obtain from $(2) \Rightarrow (3)$ of [15, Theorem 1.1] that $(R, R[X])$ is an SLP. It is clear that $A = \mathbb{Z}_4 + 2\mathbb{Z}_4[X] \in [R, R[X]]$. If the ideal $2\mathbb{Z}_4[X]$ of A is a f.g. ideal of A, then from $2^2 = 0$, it follows that $2\mathbb{Z}_4[X]$ is a f.g. $\mathbb{Z}_4$-module. This is not true and so, $2\mathbb{Z}_4[X]$ is not a f.g. ideal of A. Therefore, A is not Noetherian and so, $(R, R[X])$ is not an NP. As $S = \{1\}$, we get that S-SLP agrees with SLP and S-NP agrees with NP. Hence, $(R, R[X])$ is an S-SLP but it is not an S-NP. $\square$

3. The effect of S-accr on the intermediate rings between $R + XT[X]$ (respectively, $R + XT[[X]]$) and $T[X]$ (respectively, $T[[X]]$)

Let $R \subset T$ be rings. Let X be an indeterminate over T. Let $S = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$. The aim of this section is to study the effect of S-accr on all the intermediate rings between $R + XT[X]$ (respectively, $R + XT[[X]]$) and $T[X]$ (respectively, $T[[X]]$).

Lemma 3.1. *Let $A \subseteq B$ be rings. Let C be a non-zero proper ideal of B such that $C \subset A$. Let S be a m.c. subset of A such that $C \cap S \neq \emptyset$. The following statements are equivalent:*

(1) *A satisfies S-accr.*
(2) *B satisfies S-accr.*
(3) *(A, B) is an S-ACCRP.*

Proof. (1) $\Rightarrow$ (2) In view of [6, Proposition 3.1], it is enough to show that B satisfies S-accr*. Let J be any ideal of B and let b be any element of B. We claim that the increasing sequence of ideals of B, $(J :_B b) \subseteq (J :_B b^2) \subseteq (J :_B b^3) \subseteq \cdots$ is S-stationary. Let $c \in C \cap S$. Notice that $J \cap A$ is an ideal of A and $bc \in C \subset A$. Since A satisfies S-accr, the increasing sequence of ideals of A, $(J \cap A :_A bc) \subseteq (J \cap A :_A b^2c^2) \subseteq (J \cap A :_A b^3c^3) \subseteq \cdots$ is S-stationary. Hence, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(J \cap A :_A b^jc^j) \subseteq (J \cap A :_A b^kc^k)$ for all $j \geq k$. As $s, c^{2k} \in S$, it follows that $sc^{2k} \in S$. We verify that $sc^{2k}(J :_B b^j) \subseteq (J :_B b^k)$ for all $j \geq k$. Let $j \geq k$ and let $x \in (J :_B b^j)$. Then $xc^k \in C \subset A$ is such that $(xc^k)(b^jc^j) \in J \cap A$. Hence, $s(xc^k) \in (J \cap A :_A c^kb^k)$. This implies that $sc^{2k}xb^k \in J \cap A \subseteq J$ and so, $sc^{2k}x \in (J :_B b^k)$. This proves that $sc^{2k}(J :_B b^j) \subseteq (J :_B b^k)$ for all $j \geq k$. Thus the increasing sequence of ideals of B of the form $(J :_B b) \subseteq (J :_B b^2) \subseteq (J :_B b^3) \subseteq \cdots$ is S-stationary, where J is any ideal of B and b is any element of B. Therefore, we obtain that B satisfies S-accr.

(2) $\Rightarrow$ (1) In view of [6, Proposition 3.1], it is enough to show that A satisfies S-accr*. Let I be any ideal of A and let a be any element of A. We claim that the increasing sequence of ideals of A, $(I :_A a) \subseteq (I :_A a^2) \subseteq (I :_A a^3) \subseteq \cdots$ is S-stationary. Since B satisfies S-accr, the increasing sequence of ideals of B, $(IB :_B a) \subseteq (IB :_B a^2) \subseteq (IB :_B a^3) \subseteq \cdots$ is S-stationary. Hence, there exist $s \in S$ and $k \in \mathbb{N}$ such that $s(IB :_B a^j) \subseteq (IB :_B a^k)$ for all $j \geq k$. Let $c \in C \cap S$. As $s, c \in S$, it follows that $sc \in S$. We verify that $sc(I :_A a^j) \subseteq (I :_A a^k)$ for all $j \geq k$. Let $j \geq k$ and let $x \in (I :_A a^j)$. As $(I :_A a^j) \subseteq (IB :_B a^j)$, it follows that $x \in (IB :_B a^j)$ and so, $sx \in (IB :_B a^k)$. This implies that $(sc)xa^k \in cIB = (cB)I \subseteq CI \subseteq I$. This shows that $sc(I :_A a^j) \subseteq (I :_A a^k)$ for all $j \geq k$. Thus the increasing sequence of ideals of A of the form $(I :_A a) \subseteq (I :_A a^2) \subseteq (I :_A a^3) \subseteq \cdots$ is S-stationary for any ideal I of A and for any $a \in A$. Therefore, we get that A satisfies S-accr.

(1) $\Rightarrow$ (3) Let $R \in [A, B]$. Notice that C is an ideal common to both R and A, and S is a m.c. subset of A with $C \cap S \neq \emptyset$. Hence, it follows from (1) $\Rightarrow$ (2) of this lemma that R satisfies S-accr. This proves that (A, B) is an S-ACCRP.

(3) $\Rightarrow$ (1) is clear. $\square$

Example 3.2 given below illustrates that (2) $\Rightarrow$ (1) of Lemma 3.1 can fail to hold if the hypothesis that $C \cap S \neq \emptyset$ is omitted.

Example 3.2. Let $B = \mathbb{Z}[X]$ and let $A = \mathbb{Z} + (1+2X)\mathbb{Z}[X]$. Let $S = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then B satisfies S-accr but A does not satisfy S-accr.

Proof. Notice that $(1+2X)B$ is a non-zero proper ideal of both B and A. Observe that $S = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of $\mathbb{Z}$. Since 2 is not a zero-divisor of $\mathbb{Z}$ and $2 \notin U(\mathbb{Z})$, it follows from (3) $\Rightarrow$ (4) of Proposition 2.4 that A does not satisfy S-accr. Notice that $B = \mathbb{Z}[X]$ is Noetherian and so, B satisfies S-accr. Thus this example illustrates that (2) $\Rightarrow$ (1) of Lemma 3.1 can fail to hold if the hypothesis that $C \cap S \neq \emptyset$ is omitted. $\square$

Example 3.4 given below illustrates that (1) $\Rightarrow$ (2) of Lemma 3.1 can fail to hold if the hypothesis that $C \cap S \neq \emptyset$ is omitted. We use Lemma 3.3 in the verification of Example 3.4.

Lemma 3.3. *Let A be a ring and let a, b be non-units of A such that $Aa + Ab = A$. The following statements are equivalent:*

(1) *A satisfies (accr).*
(2) *A satisfies both S_a-accr and S_b-accr, where $S_a = \{a^n \mid n \in \mathbb{N} \cup \{0\}\}$ and $S_b = \{b^n \mid n \in \mathbb{N} \cup \{0\}\}$.*

Proof. By hypothesis, $a, b \in NU(A)$ are such that $Aa + Ab = A$. For any $n \in \mathbb{N}$, $a^n, b^n \in NU(A)$ and $Aa^n + Ab^n = A$. Hence, $a, b \notin nil(A)$. Now, $S_a = \{a^n \mid n \in \mathbb{N} \cup \{0\}\}$ (respectively, $S_b = \{b^n \mid n \in \mathbb{N} \cup \{0\}\}$) is a m.c. subset of A.
(1) $\Rightarrow$ (2) As A satisfies (accr), it follows that A satisfies both S_a-accr and S_b-accr.
(2) $\Rightarrow$ (1) By hypothesis, $a, b \in NU(A)$ are such that $Aa + Ab = A$ and A satisfies both S_a-accr and S_b-accr. We claim that A satisfies (accr). In view of [10, Theorem 1], it is enough to show that A satisfies (accr*). Consider the increasing sequence of ideals of A of the form $(I :_A x) \subseteq (I :_A x^2) \subseteq (I :_A x^3) \subseteq \cdots$, where I is an ideal of A and $x \in A$. By assumption, A satisfies S_a-accr and S_b-accr. Therefore, there exist $n, k \in \mathbb{N}$ such that $a^n(I :_A x^j) \subseteq (I :_A x^k)$ and $b^n(I :_A x^j) \subseteq (I :_A x^k)$ for all $j \geq k$. From $Aa + Ab = A$, we obtain that $Aa^n + Ab^n = A$. Hence, there exist $\lambda, \mu \in A$ such that $\lambda a^n + \mu b^n = 1$. Let $j \geq k$. It is clear that $(I :_A x^j) = 1(I :_A x^j) = (\lambda a^n + \mu b^n)(I :_A x^j) \subseteq (I :_A x^k) \subseteq (I :_A x^j)$. Therefore, $(I :_A x^j) = (I :_A x^k)$ for all $j \geq k$. This proves that the increasing sequence of ideals of A of the form

$(I :_A x) \subseteq (I :_A x^2) \subseteq (I :_A x^3) \subseteq \cdots$ is stationary for any ideal I of A and for any $x \in A$. This proves that A satisfies (accr). $\square$

Example 3.4. Let K be a field. Let B be the subring of the polynomial ring $K[X,Y]$ given by $B = K[X] + (1+XY)K[X,Y]$. Let A be the subring of B given by $A = K+(1+XY)K[X,Y]$. Let $S_{XY} = \{(XY)^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then A satisfies S_{XY}-accr but B does not satisfy S_{XY}-accr.

Proof. Notice that $C = (1+XY)K[X,Y]$ is a non-zero proper ideal of both A and B. It is clear that $S_{XY} = \{(XY)^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of A. We know from (3) $\Rightarrow$ (4) of [15, Lemma 5.9] that A is strongly Laskerian and so, we obtain from [10, Proposition 3] that A satisfies (accr). Hence, A satisfies S-accr for any m.c. subset S of R. In particular, A satisfies S_{XY}-accr. We claim that B does not satisfy S_{XY}-accr. Suppose that B satisfies S_{XY}-accr. Let $S_{1+XY} = \{(1+XY)^n \mid n \in \mathbb{N} \cup \{0\}\}$. Notice that S_{1+XY} is also a m.c. subset of A and $C \cap S_{1+XY} \neq \emptyset$. As A satisfies S_{1+XY}-accr, it follows from (1) $\Rightarrow$ (2) of Lemma 3.1 that B satisfies S_{1+XY}-accr. As X is a non-zero-divisor of $K[X]$ but not a unit of $K[X]$, we obtain from (3) $\Rightarrow$ (4) of Proposition 2.4 that B does not satisfy S_X-accr, where $S_X = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$. Hence, B does not satisfy (accr). It is clear that $Aa + Ab = A$ with $a = XY$ and $b = 1 + XY$. Hence, $Ba + Bb = B$. Notice that B satisfies S_{1+XY}-accr. As B does not satisfy (accr), it follows from (2) $\Rightarrow$ (1) of Lemma 3.3 that B does not satisfy S_{XY}-accr. $\square$

Lemma 3.5. *Let A, B, C and S be as in the statement of Lemma 3.1. The following statements are equivalent:*

(1) *A is S-Noetherian.*
(2) *B is S-Noetherian.*
(3) *(A, B) is an S-NP.*

Proof. (1) $\Rightarrow$ (2) This is (1) $\Rightarrow$ (2) of [16, Corollary 3.7].

(2) $\Rightarrow$ (1) This is (2) $\Rightarrow$ (1) of [16, Corollary 3.7].

(1) $\Rightarrow$ (3) This can be proved using arguments similar to those that are used in the proof of (1) $\Rightarrow$ (3) of Lemma 3.1.

(3) $\Rightarrow$ (1) This is clear. $\square$

Theorem 3.6. *Let R be a subring of a ring T. Let X be an indeterminate over T. Let $S = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$. The following statements are equivalent:*

(1) *$(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-SLP.*
(2) *$(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-LP.*

(3) $(R+XT[X], T[X])$ *(respectively,* $(R+XT[[X]], T[[X]])$*) is an* S*-SACCR***P.*

(4) $(R+XT[X], T[X])$ *(respectively,* $(R+XT[X]], T[[X]])$*) is an* S*-ACCRP.*

(5) $T[X]$ *(respectively,* $T[[X]]$*) satisfies* S*-accr.*

(6) T *is Noetherian.*

(7) $(R+XT[X], T[X])$ *(respectively,* $(R+XT[[X]], T[[X]])$*) is an* S*-NP.*

Proof. Notice that $S = \{X^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of $R + XT[X]$ (respectively, $R + XT[[X]]$) and the non-zero proper ideal $XT[X]$ (respectively, $XT[[X]]$) of $T[X]$ (respectively, $T[[X]]$) is also an ideal of $R+XT[X]$ (respectively, $R + XT[[X]]$). The m.c. subset S of $R + XT[X]$ (respectively. $R + XT[[X]]$) is such that $XT[X] \cap S \neq \emptyset$ (respectively, $XT[[X]] \cap S \neq \emptyset$).

(1) $\Rightarrow$ (2) This is clear, since any S-strongly Laskerian ring is S-Laskerian.

(1) $\Rightarrow$ (3) This follows from [16, Corollary 3.9(2)].

(3) $\Rightarrow$ (4) This is clear, since S-strong accr* implies S-accr.

(4) $\Rightarrow$ (5) Notice that $T[X] \in [R + XT[X], T[X]]$ (respectively, $T[[X]] \in [R + XT[[X]], T[[X]]]$). Hence, $T[X]$ (respectively, $T[[X]]$) satisfies S-accr.

(5) $\Rightarrow$ (6) By assumption, $T[X]$ (respectively, $T[[X]]$) satisfies S-accr. Hence, we obtain from (5) $\Rightarrow$ (6) of Proposition 2.1 that T is Noetherian.

(6) $\Rightarrow$ (7) We are assuming that T is Noetherian. Hence, we obtain that $T[X]$ is Noetherian. We know from [9, Theorem 71] that $T[[X]]$ is Noetherian. Therefore, $T[X]$ (respectively, $T[[X]]$) is S-Noetherian. Hence, we obtain from (2) $\Rightarrow$ (3) of Lemma 3.5 that $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-NP.

(7) $\Rightarrow$ (1) We know from [16, Corollary 3.3] that any S-Noetherian ring is S-strongly Laskerian. Therefore, we obtain from (7) that $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$ is an S-SLP.

(2) $\Rightarrow$ (4) Since any S-Laskerian ring satisfies S-accr, it follows from (2) that $(R + XT[X], T[X])$ (respectively, $(R + XT[[X]], T[[X]])$) is an S-ACCRP. □

Example 3.7. Let $R = K[X]$ and let $T = K[X, Y]$, where $T = K[X, Y]$ is the polynomial ring in two variables X, Y over a field K. Let Z be an indeterminate over T. Let $S = \{Z^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then $(R + ZT[Z], T[Z])$ is an S-SLP but it is not an LP.

Proof. It is well-known that $T = K[X, Y]$ is Noetherian. Hence, we obtain from (6) $\Rightarrow$ (1) of Theorem 3.6 that $(R + ZT[Z], T[Z])$ is an S-SLP. Let us denote

$ZT[Z]$ by $\mathfrak{p}$. It is clear that $\mathfrak{p} \in Spec(T[Z])$ and as $\mathfrak{p} \subset R + ZT[Z]$, it follows that $\mathfrak{p} \in Spec(R + ZT[Z])$. Notice that $\frac{R+ZT[Z]}{\mathfrak{p}} \cong R = K[X]$ as rings and $\frac{T[Z]}{\mathfrak{p}} \cong T = K[X,Y]$ as rings. If $(R+ZT[Z], T[Z])$ is an LP, then we obtain from [14, Lemma 1.2(1)] that $(\frac{R+ZT[Z]}{\mathfrak{p}}, \frac{T[Z]}{\mathfrak{p}})$ is an LP. This implies that $(K[X], K[X,Y])$ is an LP. Observe that $B = K[X] + (1+XY)K[X,Y] \in [K[X], K[X,Y]]$. It is already verified in Example 3.4 that the ring B does not satisfy S_{XY}-accr. Hence, B is not S_{XY}-Laskerian and so, B is not Laskerian. Therefore, we obtain that $(R + ZT[Z], T[Z])$ is not an LP. $\square$

4. The effect of S-accr on the intermediate rings between $F_1[X_1,\ldots,X_n]$ (respectively, $F_1[[X_1,\ldots,X_n]]$) and $F_2[X_1,\ldots,X_n]$ (respectively, $F_2[[X_1,\ldots,X_n]]$)

Let $F_1 \subset F_2$ be fields. Let $n \geq 1$. Let us denote $F_1[X_1,\ldots,X_n]$ by R and $F_2[X_1,\ldots,X_n]$ by T. Let us denote $F_1[[X_1,\ldots,X_n]]$ by A and $F_2[[X_1,\ldots,X_n]]$ by B. Focusing on some m.c. subsets S of R (respectively, A), the aim of this section is to study whether S-accr on all intermediate rings between R and T (respectively, A and B) implies that each intermediate ring between R and T (respectively, A and B) has a ring-theoretic property stronger than S-accr. Throughout this section, unless otherwise specified, the symbols n, F_1, F_2, R, T, A, B have the above meanings.

Lemma 4.1. *Let $n = 1$ and let $f(X_1) \in R$ be such that $f(X_1) \notin F_1$. Let $S = \{f(X_1)^m \mid m \in \mathbb{N} \cup \{0\}\}$. If (R,T) is an S-ACCRP, then F_2 is algebraic over F_1.*

Proof. We are assuming that (R,T) is an S-ACCRP. First, we verify that T is algebraic over R. Let $t \in T$. Suppose that t is not algebraic over R. Let $C = R + (1 + tf(X_1))R[t]$. Notice that $f(X_1)$ is a non-zero-divisor of R and $C \in [R,T]$. Since C satisfies S-accr, we obtain from (3) $\Rightarrow$ (4) of Proposition 2.4 that $f(X_1) \in U(R)$. This is impossible, since we know from [2, Exercise 2(i), page 11] that $U(R) = F_1 \backslash \{0\}$. Therefore, T is algebraic over R.

Let $\beta \in F_2 \backslash \{0\}$. Since T is algebraic over R, there exist $k \in \mathbb{N}$, $f_i(X_1) \in R$ for each $i \in \{0,\ldots,k\}$ with $f_k(X_1) \neq 0$ such that $\sum_{i=0}^{k} f_i(X_1)\beta^i = 0$. Let $j \geq 0$ be least with the property that the coefficient of X_1^j in $f_k(X_1)$ is not equal to 0. For each $i \in \{0,\ldots,k\}$, let $\alpha_{ij} \in F_1$ be the coefficient of X_1^j in $f_i(X_1)$. Notice that $\alpha_{kj} \neq 0$. By comparing the coefficient of X_1^j on both sides of $\sum_{i=0}^{k} f_i(X_1)\beta^i = 0$, we get that $\sum_{i=0}^{k} \alpha_{ij}\beta^i = 0$. This implies that β is algebraic over F_1. This is true for any $\beta \in F_2$. Hence, F_2 is algebraic over F_1. $\square$

Proposition 4.2. *Let $n = 1$. Let $f(X_1) \in R \backslash F_1$. The following statements are equivalent:*

(1) *(R, T) is an LP.*
(2) *(R, T) is an S-LP, where $S = \{f(X_1)^m \mid m \in \mathbb{N} \cup \{0\}\}$.*
(3) *(R, T) is an ACCRP.*
(4) *(R, T) is an S-ACCRP, where $S = \{f(X_1)^m \mid m \in \mathbb{N} \cup \{0\}\}$.*
(5) *F_2 is algebraic over F_1.*
(6) *T is integral over R.*

Proof. (1) $\Rightarrow$ (2) This is clear, since any Laskerian ring is S-Laskerian.

(1) $\Rightarrow$ (3) This is clear, since we know from [10, Proposition 3] that any Laskerian ring satisfies (accr).

(3) $\Rightarrow$ (4) This is clear, since (accr) implies S-accr.

(2) $\Rightarrow$ (4) This is clear, since any S-Laskerian ring satisfies S-accr.

(4) $\Rightarrow$ (5) This follows from Lemma 4.1.

(5) $\Rightarrow$ (6) Since F_2 is algebraic over F_1 and F_1 is a field, we get that F_2 is integral over F_1. Let $t \in T \backslash \{0\}$ be such that $deg(t) > 0$. Now, there exist $k \in \mathbb{N}$ and $\beta_i \in F_2$ for each $i \in \{0, \ldots, k\}$ such that $t = \sum_{i=0}^{k} \beta_i X_1^i$ with $\beta_k \neq 0$. Notice that for each $i \in \{0, \ldots, k\}$, $\beta_i X_1^i$ is integral over R. Therefore, we obtain from [2, Corollary 5.2] that $C = R[\beta_0, \beta_1 X_1, \ldots, \beta_k X_1^k]$ is a finitely generated R-module. It is clear that $R[t]$ is a subring of C. Hence, we obtain from $(iii) \Rightarrow (i)$ of [2, Proposition 5.1] that t is integral over R. This proves that T is integral over R.

(6) $\Rightarrow$ (1) This is (3) $\Rightarrow$ (4) of [15, Proposition 3.4]. $\square$

Proposition 4.3. *Let $n \geq 2$. The following statements are equivalent:*

(1) *(R, T) is an SLP.*
(2) *(R, T) is an LP.*
(3) *(R, T) is an ACCRP.*
(4) *(R, T) is an S-ACCRP, where S is the m.c. subset of R given by $S = \{X_n^m \mid m \in \mathbb{N} \cup \{0\}\}$.*
(5) *F_2 is a finite algebraic extension of F_1.*
(6) *(R, T) is an NP.*

Proof. (1) $\Rightarrow$ (2) This is clear, since any strongly Laskerian ring is Laskerian.

(2) $\Rightarrow$ (3) This is clear, any Laskerian ring satisfies (accr).

(3) $\Rightarrow$ (4) This follows immediately, since (accr) implies S-accr.

(4) $\Rightarrow$ (5) Let $C \in [F_1[X_1, \ldots, X_{n-1}], F_2[X_1, \ldots, X_{n-1}]]$. It is clear that $C[X_n] \in [R, T]$. By hypothesis, $C[X_n]$ satisfies S-accr. Hence, we obtain from (5) $\Rightarrow$ (6) of Proposition 2.1 that C is Noetherian. Therefore, $(F_1[X_1, \ldots, X_{n-1}], F_2[X_1, \ldots, X_{n-1}])$ is an NP. It now follows as in the proof of (2) $\Rightarrow$ (3) of [15, Proposition 3.6] that F_2 is a finite algebraic extension of F_1.

(5) $\Rightarrow$ (6) This is (3) $\Rightarrow$ (4) of [15, Proposition 3.6].

(6) $\Rightarrow$ (1) This is clear, since any Noetherian ring is strongly Laskerian. □

Proposition 4.4. *Let $n \geq 2$ and let $S = \{X_n^m \mid m \in \mathbb{N} \cup \{0\}\}$. Then the statements (1) to (6) of Proposition 4.3 are equivalent to each one of the following statements:*

- (1′) *(R, T) is an S-SLP.*
- (2′) *(R, T) is an S-LP.*
- (3′) *(R, T) is an S-SACCR*P.*

Proof. This can be proved using arguments similar to those that are used in the proof of Proposition 2.3. □

If $n = 1$, then we show in Proposition 4.5, that (A, B) is an ACCRP if and only if (A, B) is an S-ACCRP, where $S = \{X_1^m \mid m \in \mathbb{N} \cup \{0\}\}$.

Proposition 4.5. *Let $n = 1$. Let $S = \{X_1^m \mid m \in \mathbb{N} \cup \{0\}\}$. The following statements are equivalent:*

1. *(A, B) is an LP.*
2. *(A, B) is an S-LP.*
3. *(A, B) is an ACCRP.*
4. *(A, B) is an S-ACCRP.*
5. *B is algebraic over A.*
6. *B is integral over A.*

Proof. (1) $\Rightarrow$ (2) This is clear, since any Laskerian ring is S-Laskerian.

(1) $\Rightarrow$ (3) This is clear, since any Laskerian ring satisfies (accr) by [10, Proposition 3].

(3) $\Rightarrow$ (4) This is clear, since (accr) implies S-accr.

(4) $\Rightarrow$ (5) Let $f(X_1) \in B$. Suppose that $f(X_1)$ is transcendental over A. Let C be the subring of $A[f(X_1)]$ given by $C = A + (1 + X_1 f(X_1))A[f(X_1)]$. It is clear that $C \in [A, B]$. By assumption, C satisfies S-accr. As X_1 is a non-zero-divisor of A, we obtain from (3) $\Rightarrow$ (4) of Proposition 2.4 that $X_1 \in U(A)$. This is a contradiction. Therefore, B is algebraic over A.

(5) $\Rightarrow$ (6) From B is algebraic over A, it can be shown as in the proof of Lemma 4.1 that F_2 is algebraic over F_1. As B is algebraic over A, we obtain from [4, Corollary 5.2] that B is integral over A.

(6) $\Rightarrow$ (1) This is (2) $\Rightarrow$ (3) of [15, Proposition 3.5].

(2) $\Rightarrow$ (4) This is clear, since any S-Laskerian ring satisfies S-accr by [16, Corollary 3.9(1)] and [6, Proposition 3.1]. □

We denote the characteristic of a field F by $char(F)$.

Corollary 4.6. *Let $n = 1$. Let $char(F_1) = 0$. Let $S = \{X_1^m \mid m \in \mathbb{N} \cup \{0\}\}$. The following statements are equivalent:*

(1) *(A, B) is an S-ACCRP.*
(2) *(A, B) is an NP.*
(3) *(A, B) is an SLP.*
(4) *(A, B) is an S-SLP.*
(5) *(A, B) is an S-SACCR*P.*
(6) *(A, B) is an LP.*
(7) *(A, B) is an S-LP.*

Proof. (1) $\Rightarrow$ (2) It follows from (4) $\Rightarrow$ (5) of Proposition 4.5 that B is algebraic over A. In such a case, it is observed in the proof of (5) $\Rightarrow$ (6) of Proposition 4.5 that F_2 is algebraic over F_1. By hypothesis, $char(F_1) = 0$. Therefore, F_2 is a separable extension of F_1. As B is algebraic over A, we obtain from [4, Corollary 4.2] that F_2 is a finite extension of F_1. Therefore, B is a finitely generated A-module. It is well-known that A is Noetherian. Hence, it follows as in the proof of (3) $\Rightarrow$ (4) of [15, Proposition 3.6] that (A, B) is an NP.

(2) $\Rightarrow$ (3) This is clear, since any Noetherian ring is strongly Laskerian.

(3) $\Rightarrow$ (4) This is clear, since strongly Laskerian ring is S-strongly Laskerian.

(4) $\Rightarrow$ (5) This is clear, since any S-strongly Laskerian ring satisfies S-strong accr*.

(5) $\Rightarrow$ (1) This is clear, since S-strong accr* implies S-accr.

Since any strongly Laskerian ring is Laskerian and any Laskerian ring is S-Laskerian, (3) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are clear.

(7) $\Rightarrow$ (1) This is clear, since any S-Laskerian ring satisfies S-accr. □

Theorem 4.7. *Let $n \geq 2$. Let S be the m.c. subset of A given by $S = \{X_n^m \mid m \in \mathbb{N} \cup \{0\}\}$. The following statements are equivalent:*

(1) *(A, B) is an SLP.*

(2) (A, B) *is an LP.*
(3) (A, B) *is an ACCRP.*
(4) (A, B) *is an S-ACCRP.*
(5) F_2 *is a finite extension of* F_1.
(6) (A, B) *is an NP.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (5) $\Rightarrow$ (6), and (6) $\Rightarrow$ (1) can be proved as in the proof of $(1) \Rightarrow (2), (2) \Rightarrow (3), (3) \Rightarrow (4), (5) \Rightarrow (6)$, and (6) $\Rightarrow$ (1) of Proposition 4.3. We need only to prove (4) $\Rightarrow$ (5).
(4) $\Rightarrow$ (5) Let C be a subring of $F_2[[X_1, \ldots, X_{n-1}]]$ such that $C \supseteq F_1[[X_1, \ldots, X_{n-1}]]$. Then $C[[X_n]] \in [A, B]$. By hypothesis, $C[[X_n]]$ satisfies S-accr. Hence, we obtain from (5) $\Rightarrow$ (6) of Proposition 2.1 that C is Noetherian. Therefore, we get that $(F_1[[X_1, \ldots, X_{n-1}]], F_2[[X_1, \ldots, X_{n-1}]])$ is an NP. This implies that $(F_1[[X_1]], F_2[[X_1]])$ is an NP. Now, it follows from the Remark following Theorem 2 of [18] that F_2 is a finite extension of F_1. □

Proposition 4.8. *Let n, A, B, S be as in the statement of Theorem 4.7. Then the statements* (1) *to* (6) *of Theorem 4.7 are equivalent to each one of the following statements:*

(1′) (A, B) *is an S-SLP.*
(2′) (A, B) *is an S-LP.*
(3′) (A, B) *is an S-SACCR*P.*

Proof. This can be proved using arguments similar to those that are used in the proof of Proposition 2.3. □

5. The effect of S-accr on the intermediate rings between R and $S^{-1}R$, where R is a Noetherian domain

Let T be an integral domain. Recall from [18] that $\mathfrak{m} \in Max(T)$ is said to be a *low maximal* if *height* $\mathfrak{m} = 1$ and $\mathfrak{m}$ is said to be a *high maximal* if *height* $\mathfrak{m} > 1$.

Let R be a Noetherian domain which is not a field. Let S be a m.c. subset of R. We prove in Theorem 5.1 that if A satisfies $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$-accr for each $A \in [R, S^{-1}R]$ and for each $s \in S \cap NU(R)$, then $(R, S^{-1}R)$ is an NP.

Theorem 5.1. *Let S be a m.c. subset of a Noetherian integral domain R which is not a field. The following statements are equivalent:*

(1) $(R, S^{-1}R)$ *is an SLP.*
(2) $(R, S^{-1}R)$ *is an LP.*

(3) $(R, S^{-1}R)$ *is an ACCRP.*

(4) $(R, S^{-1}R)$ *is an* $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$*-ACCRP for each* $s \in S \cap NU(R)$.

(5) $S \subseteq C$*, where* C *is the set of all elements of* R *which are contained in no high maximal ideal of* R.

(6) $(R, S^{-1}R)$ *is an NP.*

Proof. It is clear that for any $s \in S \cap NU(R)$, $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of R.

(1) $\Rightarrow$ (2) This is clear, since any strongly Laskerian ring is Laskerian.

(2) $\Rightarrow$ (3) This is clear, since any Laskerian ring satisfies (accr).

(3) $\Rightarrow$ (4) This is clear, since if a ring A satisfies (accr), then it satisfies S_1-accr for any m.c. subset S_1 of A.

(4) $\Rightarrow$ (5) We use some arguments found in the proof of [15, Proposition 4.3]. Let $s \in S$. We claim that $s \in C$. This is clear if $s \in U(R)$. Suppose that $s \in NU(R)$. Let $\mathfrak{m} \in Max(R)$ such that $s \in \mathfrak{m}$. Let $\mathfrak{p} \in Spec(R) \backslash \{(0)\}$ be such that $\mathfrak{p} \subseteq \mathfrak{m}$. Let $x \in \mathfrak{p} \backslash \{0\}$. Let A be the subring of $S^{-1}R$ given by $A = R + \frac{x}{s} R[\frac{1}{s}]$. Let $S_1 = \{s^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then S_1 is a m.c. subset of R. Let us denote the ideal $\frac{x}{s}A$ of A by I. By assumption, A satisfies S_1-accr. Hence, the increasing sequence of ideals of A, $(I :_A s) \subseteq (I :_A s^2) \subseteq (I :_A s^3) \subseteq \cdots$ is S_1-stationary. Therefore, there exist $n, k \in \mathbb{N}$ such that $s^n(I :_A s^j) \subseteq (I :_A s^k)$ for all $j \geq k$. As $s^{n+k+1} \frac{x}{s^{n+k+2}} = \frac{x}{s} \in I$, we get that $s^n \frac{x}{s^{n+k+2}} \in s^n(I :_A s^{n+k+1}) \subseteq (I :_A s^k)$. Hence, we obtain that $s^n \frac{x}{s^{n+k+2}} s^k \in I$. This implies that $\frac{x}{s^2} = \frac{x}{s} a$ for some $a \in A$ and so, it follows that $\frac{1}{s} = a \in A$. Hence, $\frac{1}{s} = r + \frac{xr_1}{s^{m+1}}$ for some $r, r_1 \in R$ and $m \in \mathbb{N}$. Therefore, $s^m = rs^{m+1} + xr_1$ and so, $s^m(1 - rs) = xr_1 \in \mathfrak{p}$. We claim that $1 - rs \notin \mathfrak{p}$. If $1 - rs \in \mathfrak{p}$, then as $\mathfrak{p} \subseteq \mathfrak{m}$, it follows that $1 - rs \in \mathfrak{m}$. From $s \in \mathfrak{m}$, we obtain that $rs \in \mathfrak{m}$ and so, $1 = 1 - rs + rs \in \mathfrak{m}$. This is impossible. Hence, $1 - rs \notin \mathfrak{p}$. It follows from $s^m(1 - rs) \in \mathfrak{p}$ and $\mathfrak{p} \in Spec(R)$ that $s^m \in \mathfrak{p}$ and so, $s \in \mathfrak{p}$. This shows that $s \in \mathfrak{p}$ for any non-zero $\mathfrak{p} \in Spec(R)$ with $\mathfrak{p} \subseteq \mathfrak{m}$. We claim that $height\mathfrak{m} = 1$. If $height\mathfrak{m} > 1$, then since R is Noetherian, it follows from [9, Theorem 144] that there exist infinitely many $\mathfrak{p} \in Spec(R)$ with $\mathfrak{p} \subset \mathfrak{m}$ and $height\mathfrak{p} = 1$. Thus s belongs to infinitely many height one prime ideals of R. This is impossible in view of [2, Theorem 7.13] and [2, Proposition 4.6]. Therefore, $height\mathfrak{m} = 1$. This shows that $S \subseteq C$.

(5) $\Rightarrow$ (6) Since R is a Noetherian domain and $S \subseteq C$, we obtain from [18, Corollary 12] that $(R, S^{-1}R)$ is an NP.

(6) $\Rightarrow$ (1) This is clear, since any Noetherian ring is strongly Laskerian. $\square$

Proposition 5.2. *Let S, R be as in the statement of Theorem 5.1. Then the statements* (1) *to* (6) *of Theorem 5.1 are equivalent to each one of the following statements:*

(1′) $(R, S^{-1}R)$ *is an* $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$*-SLP for each* $s \in S \cap NU(R)$.
(2′) $(R, S^{-1}R)$ *is an* $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$*-LP for each* $s \in S \cap NU(R)$.
(3′) $(R, S^{-1}R)$ *is an* $\{s^n \mid n \in \mathbb{N} \cup \{0\}\}$*-SACCR*P for each* $s \in S \cap NU(R)$.

Proof. This can be proved using arguments similar to those that are used in the proof of Proposition 2.3. □

In Example 5.5, we illustrate that (5) ⇒ (3) of Theorem 5.1 can fail to hold if we omit the hypothesis that R is Noetherian in the statement of Theorem 5.1. We use Lemma 5.3 in the verification of Examples 5.4 and 5.5.

Lemma 5.3. *Let V be a discrete valuation ring with $\mathfrak{m} = V\pi$ as its unique maximal ideal. Let $R = V[X]$. Then $\mathfrak{p} = (1 + \pi X)V[X] \in Max(R)$ and $height\,\mathfrak{p} = 1$.*

Proof. Let us denote the quotient field of V by K. Let $\phi : V[X] \to K$ be the ring homomorphism defined by $\phi(f(X)) = f(\frac{-1}{\pi})$ for any $f(X) \in V[X]$. Let $\alpha \in K$. It is clear that α can be expressed in the form $\alpha = \frac{v}{\pi^{2m}}$ for some $v \in V$ and $m \in \mathbb{N}$. As $\phi(vX^{2m}) = \frac{v}{\pi^{2m}} = \alpha$, we obtain that ϕ is onto. Since $\phi(1 + \pi X) = 0$, we get that $(1 + \pi X)V[X] \subseteq ker(\phi)$. Let $f(X) \in ker(\phi)$. Then $f(\frac{-1}{\pi}) = 0$. Since $(1 + \pi X)K[X] \in Max(K[X])$, it follows that $f(X) = (1 + \pi X)g(X)$ for some $g(X) \in K[X]$. By comparing the coefficients of powers of X on both sides of $f(X) = (1+\pi X)g(X)$, we obtain that $g(X) \in V[X]$. Therefore, $f(X) \in (1+\pi X)V[X]$. This proves that $ker(\phi) \subseteq (1+\pi X)V[X]$ and so, $\mathfrak{p} = (1+\pi X)V[X] = ker(\phi)$. Thus ϕ is a homomorphism of rings from $R = V[X]$ onto K with $ker(\phi) = \mathfrak{p}$. Hence, it follows from the fundamental theorem of homomorphism of rings that $\frac{R}{\mathfrak{p}} \cong K$ as rings. Since K is a field, we obtain from [2, page 3] that $\mathfrak{p} \in Max(R)$. Let $S = V\backslash\{0\}$. Then S is a m.c. subset of V and $S^{-1}R = K[X]$. Since $\mathfrak{p} \cap S = \emptyset$, it follows from [2, Proposition 3.11(iv)] that $S^{-1}\mathfrak{p} \in Spec(K[X])\backslash\{(0)\} = Max(K[X])$. Therefore, $height\,S^{-1}\mathfrak{p} = 1$ and hence, we obtain from [2, Proposition 3.11(iv)] that $height\,\mathfrak{p} = 1$. □

Example 5.4. Let p be a prime number and let $V = \mathbb{Z}_{p\mathbb{Z}}$. Let $R = V[X]$. Let $S = \{(1 + pX)^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then S is a m.c. subset of R and $(R, S^{-1}R)$ is an NP.

Proof. It is well-known that V is a discrete valuation ring (see [2, Example (1), page 94]). Notice that pV is the unique maximal ideal of V. As V is Noetherian,

$R = V[X]$ is Noetherian. We know from Lemma 5.3 that $\mathfrak{p} = (1+pX)R \in Max(R)$ and $height\mathfrak{p} = 1$. It is clear that $S = \{(1+pX)^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of R and $S \subseteq C$, where C is the set of all elements of R which are contained in no high maximal ideal of R. Therefore, we obtain from [18, Corollary 12] that $(R, S^{-1}R)$ is an NP. $\square$

Example 5.5. Let $K(X_1, X_2)$ be the field of rational functions in two variables X_1, X_2 over a field K. Let Y be an indeterminate over $K(X_1, X_2)$ and let $A = K(X_1, X_2)[Y]$. Let $V = A_{YA}$. Let $T = V[Z]$, where Z is an indeterminate over V. Let $R = K + (1 + YZ)V[Z]$. Let $S = \{(1 + YZ)^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then S is a m.c. subset of R, $S \subseteq C$, where C is the set of all elements of R which are contained in no high maximal ideal of R and the ring $B = K[X_1] + (1 + X_1X_2)K[X_1, X_2] + (1+YZ)V[Z]$ is such that it does not satisfy $S_{X_1X_2}$-accr, where $S_{X_1X_2} = \{(X_1X_2)^n \mid n \in \mathbb{N} \cup \{0\}\}$.

Proof. As A is a principal ideal domain (PID) and $YA \in Max(A)$, it follows that $V = A_{YA}$ is a discrete valuation ring with VY as its unique maximal ideal. Let us denote the ideal $(1+YZ)V[Z]$ by $\mathfrak{p}$. We know from Lemma 5.3 that $\mathfrak{p} \in Max(V[Z])$ and $height\mathfrak{p} = 1$ in $T = V[Z]$. It is clear that $\mathfrak{p}$ is also an ideal of $R = K + \mathfrak{p}$. It follows from $\frac{R}{\mathfrak{p}} \cong K$ as rings and K being a field that $\mathfrak{p} \in Max(R)$. We claim that $height\mathfrak{p} = 1$ in R. Suppose that $height\mathfrak{p} > 1$ in R. Then there exists $\mathfrak{q} \in Spec(R)\backslash\{(0)\}$ such that $\mathfrak{q} \subset \mathfrak{p}$. Hence, $\mathfrak{q} \not\supseteq \mathfrak{p}$. Since $\mathfrak{p}$ is an ideal common to both R and T, we obtain from [13, Lemma 6] that there exists $\mathfrak{q}' \in Spec(T)$ such that $\mathfrak{q}' \cap R = \mathfrak{q}$. As $\mathfrak{q} \neq (0)$, it follows that $\mathfrak{q}' \neq (0)$. From $height\mathfrak{p} = 1$ in T, we obtain that $\mathfrak{q}' \not\subseteq \mathfrak{p}$. Since $\mathfrak{p} \in Max(T)$, it follows that $\mathfrak{q}' + \mathfrak{p} = T$ and so, there exist $t \in \mathfrak{q}'$ and $x \in \mathfrak{p}$ such that $t + x = 1$. Now, $t = 1 - x \in \mathfrak{q}' \cap R = \mathfrak{q} \subset \mathfrak{p}$. This implies that $1 = t + x \in \mathfrak{p}$. This is in contradiction to the fact that $\mathfrak{p} \neq T$. Therefore, $height\mathfrak{p} = 1$ in R.

It is clear that $S = \{(1 + YZ)^n \mid n \in \mathbb{N} \cup \{0\}\}$ is a m.c. subset of R. We verify that $S \subseteq C$, where C is the set of all elements of R which are contained in no high maximal ideal of R. First, we show that $\mathfrak{p} = \sqrt{(1 + YZ)R}$. From $1 + YZ \in \mathfrak{p}$, it follows that $\sqrt{(1 + YZ)R} \subseteq \mathfrak{p}$. If $p \in \mathfrak{p}$, then $p^2 \in (1 + YZ)R$ and so, $\mathfrak{p} \subseteq \sqrt{(1 + YZ)R}$. This proves that $\mathfrak{p} = \sqrt{(1 + YZ)R}$. From $height\mathfrak{p} = 1$, $\mathfrak{p} \in Max(R)$, and $\mathfrak{p} = \sqrt{(1 + YZ)R}$, we obtain that if s is any element of S, then s does not belong to any high maximal ideal of R. This proves that $S \subseteq C$. We know from Example 3.4 that $K[X_1]+(1+X_1X_2)K[X_1, X_2]$ does not satisfy $S_{X_1X_2}$-accr. Since $\mathfrak{p}$ is an ideal common to both R and T and $\mathfrak{p} \cap S \neq \emptyset$, it follows that $S^{-1}R = S^{-1}T$. It is clear that $R \subset B = K[X_1] + (1 + X_1X_2)K[X_1, X_2] + (1 +$

$YZ)V[Z] \subset T = V[Z] \subset S^{-1}T = S^{-1}R$. It follows from $\mathfrak{p} \cap K[X_1, X_2] = (0)$ that $\frac{B}{\mathfrak{p}} \cong K[X_1] + (1 + X_1X_2)K[X_1, X_2]$ as rings. It is clear that $S_{X_1X_2}$ is a m.c. subset of B and $s \notin \mathfrak{p}$ for each $s \in S_{X_1X_2}$. From $\frac{B}{\mathfrak{p}} \cong K[X_1] + (1 + X_1X_2)K[X_1, X_2]$ as rings, it follows that $\frac{B}{\mathfrak{p}}$ does not satisfy $\overline{S_{X_1X_2}} = \{s + \mathfrak{p} \mid s \in S_{X_1X_2}\}$-accr. Therefore, we obtain from Lemma 2.6 that B does not satisfy $S_{X_1X_2}$-accr. Hence, $(R, S^{-1}R)$ is not an ACCRP. □

Acknowledgements

I am very much thankful to the referees for their carefully reading the manuscript and their useful and helpful suggestions. I am also very much thankful to Professor Abdullah Harmanci for his support.

References

[1] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Comm. Algebra, 30(9) (2002), 4407-4416.

[2] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Massachusetts, 1969.

[3] N. Bourbaki, Commutative Algebra, Addison-Wesley, Reading, Massachusetts, 1972.

[4] R. Gilmer, *Integral dependence in power series rings*, J. Algebra, 11 (1969), 488-502.

[5] R. Gilmer and W. Heinzer, *Finitely generated intermediate rings*, J. Pure Appl. Algebra, 37(3) (1985), 237-264.

[6] A. Hamed and S. Hizem, *Modules satisfying S-Noetherian property and S-ACCR*, Comm. Algebra, 44(5) (2016), 1941-1951.

[7] A. Hamed and A. Malek, *S-prime ideals of a commutative ring*, Beitr. Algebra Geom., 61(3) (2020), 533-542.

[8] W. Heinzer and D. Lantz, *The Laskerian property in commutative rings*, J. Algebra, 72(1) (1981), 101-114.

[9] I. Kaplansky, Commutative Rings, The University of Chicago Press, Chicago, 1974.

[10] C. P. Lu, *Modules satisfying ACC on a certain type of colons*, Pacific J. Math., 131(2) (1988), 303-318.

[11] C. P. Lu, *Modules and rings satisfying (accr)*, Proc. Amer. Math. Soc., 117(1) (1993), 5-10.

[12] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly, 49 (1942), 286-295.

[13] N. Radu, *Sur les Anneaux Laskeriens*, in: Proceedings of the Week of Algebraic Geometry, Bucharest, (1980), 158-163.

[14] S. Visweswaran, *Laskerian pairs*, J. Pure Appl. Algebra, 59(1) (1989), 87-110.

[15] S. Visweswaran, *ACCR pairs*, J. Pure Appl. Algebra, 81(3) (1992), 313-334.

[16] S. Visweswaran, *Some results on S-primary ideals of a commutative ring*, Beitr Algebra Geom., (2021) https://doi.org/10.1007/s13366-021-00580-5.

[17] S. Visweswaran and P. T. Lalchandani, *Some results on modules satisfying S-strong accr**, Arab J. Math. Sci., 25(2) (2019), 145-155.

[18] A. R. Wadsworth, *Pairs of domains where all intermediate domains are Noetherian*, Trans. Amer. Math. Soc., 195 (1974), 201-211.

S. Visweswaran
Retired Faculty
Department of Mathematics
Saurashtra University
Rajkot, India, 360005
e-mail: s_visweswaran2006@yahoo.co.in