International Electronic Journal of Algebra
Volume 33 (2023) 213-225
DOI: 10.24330/ieja.1226320

ON THE IRREDUCIBLE REPRESENTATIONS OF THE JORDAN TRIPLE SYSTEM OF $p \times q$ MATRICES

Hader A. Elgendy

Received: 3 August 2022; Revised: 13 October 2022; Accepted: 15 October 2022
Communicated by Burcu Üngör

Dedicated to the memory of Professor Edmund R. Puczyłowski

Abstract. Let $\mathcal{J}_{\mathbb{F}}$ be the Jordan triple system of all $p \times q$ $(p \neq q;\ p, q > 1)$ rectangular matrices over a field $\mathbb{F}$ of characteristic 0 with the triple product $\{x, y, z\} = xy^t z + zy^t x$, where y^t is the transpose of y. We study the universal associative envelope $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ of $\mathcal{J}_{\mathbb{F}}$ and show that $\mathcal{U}(\mathcal{J}_{\mathbb{F}}) \cong M_{p+q\times p+q}(\mathbb{F})$, where $M_{p+q\times p+q}(\mathbb{F})$ is the ordinary associative algebra of all $(p+q)\times(p+q)$ matrices over $\mathbb{F}$. It follows that there exists only one nontrivial irreducible representation of $\mathcal{J}_{\mathbb{F}}$. The center of $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ is deduced.

Mathematics Subject Classification (2020): 17C50, 17B35, 16W10, 16G60
Keywords: Jordan triple system, rectangular matrix, universal associative envelope, representation theory

1. Introduction

A vector space V over a field $\mathbb{F}$ of characteristic 0 equipped with a triple product $\{a, b, c\}$ is called a *Jordan triple system* if

$$\{x, y, z\} = \{z, y, x\},$$

$$\{u, v, \{x, y, z\}\} = \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\} + \{x, y, \{u, v, z\}\},$$

for all $x, y, z, u, v \in V$.

Jordan structures appeared in many areas of mathematics like Lie Theory, differential geometry and analysis [1,12,13,14,15,22]. In addition to that Jordan triple systems have been used to find several solutions of the Yang-Baxter equation [20]. The linkages between Jordan structures, Lie algebras, and projective geometries are given in [6]. Jordan structures play also an important role in theoretical physics. They are appeared in the theory of superstrings [2,5,7,8,11,21], and in the theory of colour and confinement [9], in supersymmetry [10]. The description of some of these applications has been given in the survey [16]. More information about

Jordan triple systems can be found in [18,19]. It is well known that to every associative algebra A one can relate a Jordan triple system J with the triple product $\{x, y, z\} = xyz + zyx$. A Jordan triple system is *special* if it can be imbedded as a subtriple of some J, otherwise it is *exceptional*. A *representation* of a Jordan triple system J is a Jordan triple homomorphism $\Theta : J \to (EndV)_-$, where $EndV$ is the space of endomorphisms of a vector space V to itself. A representation ρ is called *irreducible* if the only invariant subspaces of V under ρ are the trivial ones, $\{0\}$ and V. It is known that any Jordan algebra gives rise to a Jordan triple system. One of the most important examples of a Jordan triple system which doesn't come from a bilinear product is the rectangular matrices $M_{p\times q}(\mathbb{F})$ with the triple product $xy^tz + zy^tx$; if $p \neq q$ there is no natural way to multiply two $p \times q$ matrices to get a third $p \times q$ matrix, see [17]. This example shows the necessity of a ternary product. This example is a special Jordan triple system (see the map Θ of Corollary 3.3 (of the present paper)).

The problem of the classification of the representations of a special Jordan triple system can be converted into a problem of an associative algebra by passage to the universal associative envelope of the Jordan triple system. In [12], it was shown that the universal (associative) envelope of any Jordan triple system of finite dimension is finite-dimensional. In [3], we showed that the universal (associative) envelope of the Jordan triple system J of all n by n ($n \geq 2$) matrices over a field $\mathbb{F}$ of characteristic 0 (with respect to the product xyz + zyx) is isomorphic to $M_{n\times n}(\mathbb{F}) \oplus M_{n\times n}(\mathbb{F}) \oplus M_{n\times n}(\mathbb{F}) \oplus M_{n\times n}(\mathbb{F})$, where $M_{n\times n}(\mathbb{F})$ is the ordinary associative algebra of all n by n matrices over $\mathbb{F}$. It follows that there are four nontrivial finite-dimensional irreducible representations of J. In [4], we have studied the representations of two special Jordan triple systems (with respect to the product $xyz + zyx$): The Jordan triple system $\mathcal{J}_{\mathcal{S}}$ of all symmetric n by n ($n \geq 2$) matrices over a field $\mathbb{F}$ of characteristic zero, and the Jordan triple system $\mathcal{J}_{\mathcal{H}}$ of all Hermitian n by n ($n \geq 2$) matrices over the complex numbers $\mathbb{C}$. We proved that the universal (associative) envelope of $\mathcal{J}_{\mathcal{S}}$ is isomorphic to $M_{n\times n}(\mathbb{F}) \oplus M_{n\times n}(\mathbb{F})$, while the universal (associative) envelope of $\mathcal{J}_{\mathcal{H}}$ is isomorphic to $M_{n\times n}(\mathbb{C}) \oplus M_{n\times n}(\mathbb{C}) \oplus M_{n\times n}(\mathbb{C}) \oplus M_{n\times n}(\mathbb{C})$. We deduced that the Jordan triple system $\mathcal{J}_{\mathcal{S}}$ has two nontrivial finite-dimensional inequivalent irreducible representations, while the Jordan triple system $\mathcal{J}_{\mathcal{H}}$ has four nontrivial inequivalent finite-dimensional irreducible representations.

In the present paper, we study the universal associative envelope of the special Jordan triple system $\mathcal{J}_{\mathbb{F}}$ of all $p \times q$ ($p \neq q$; $p, q > 1$) rectangular matrices with the triple product $xy^tz + zy^tx$. This paper is organized as follows. In Section 2, we construct the universal (associative) envelope $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ of $\mathcal{J}_{\mathbb{F}}$ and derive some identities of $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$. In Section 3, we prove that $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ is isomorphic to $M_{p+q\times p+q}(\mathbb{F})$, where $M_{p+q\times p+q}(\mathbb{F})$ is the ordinary associative algebra of all $(p+q)$ by $(p+q)$ matrices over $\mathbb{F}$ (Theorem 3.1). We also deduce that $\mathcal{J}_{\mathbb{F}}$ has only one nontrivial irreducible representation and determine the explicit form of this representation (Corollary 3.3). The center of $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ is also determined (Lemma 3.4).

2. The universal associative envelope of the special Jordan triple system of rectangular matrices

Definition 2.1. Let $\mathcal{J}_{\mathbb{F}}$ be the Jordan triple system of the rectangular matrices $M_{p\times q}(\mathbb{F})$ $(p, q > 1;\ p \neq q)$ over a field $\mathbb{F}$ of characteristic 0 with the triple product

$$\{x, y, z\} = xy^tz + zy^tx,$$

where y^t is the transpose of y.

Definition 2.2. We let $\Omega_1 = \{1, \dots, p\}$, $\Omega_2 = \{1, \dots, q\}$, and $\Omega_3 = \{p+1, \dots, p+q\}$ be three finite index sets. Let $\mathfrak{B} = \{E_{i,j} \mid i \in \Omega_1;\ j \in \Omega_2\}$ be a basis of $\mathcal{J}_{\mathbb{F}}$, where $E_{i,j}$ denotes the p-by-q matrix with a single 1, in the ith row and jth column, and zeros elsewhere.

Notation 2.3. *Throughout this paper, we use the following notations:*

- $\delta_{i,j}$ *for the Kronecker delta, and* $\widehat{\delta}_{i,j} = 1 - \delta_{i,j}$.
- $\Delta_{i,L} = 1$ *if* $i \in L$, *and* 0 *otherwise.*

Let $\mathbb{X} = \{G_{i,j} \mid i \in \Omega_1;\ j \in \Omega_2\}$ be a set of symbols in bijection with $\mathfrak{B}$ and let $\Phi : \mathfrak{B} \to \mathbb{X}$ realize the bijection $(\Phi(E_{i,j}) = G_{i,j})$. Let $\mathfrak{F}$ be the free associative algebra generated by $\mathbb{X}$. We extend Φ to a map $\Phi : \mathcal{J}_{\mathbb{F}} \to \mathfrak{F}$ (by linearity). Let I be the two-sided ideal of $\mathfrak{F}$ generated by all the elements of the form:

$$G_{i,j}G_{k,\ell}G_{s,t} + G_{s,t}G_{k,\ell}G_{i,j} - \Phi(\{E_{i,j}, E_{k,\ell}, E_{s,t}\}) \quad (i, k, s \in \Omega_1,\ j, \ell, t \in \Omega_2).$$

The universal associative envelope $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ of $\mathcal{J}_{\mathbb{F}}$ is the quotient $\mathfrak{F}/I$. Let $\pi : \mathfrak{F} \to \mathcal{U}(\mathcal{J}_{\mathbb{F}})$ be the projection, then the map $\iota = \pi \circ \Phi$ maps $\mathcal{J}_{\mathbb{F}}$ to $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$.

2.1. Identities of the universal associative envelope.

In this section we get identities of $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ that we use in the proof of the main results of the next section.

Lemma 2.4. *In* $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$, *the following identities hold:*

(1) $G_{i,j}G_{i,j}G_{i,j} \equiv G_{i,j} \quad (i \in \Omega_1,\ j \in \Omega_2)$,

(2) $G_{i,j}G_{k,\ell} \equiv 0 \quad (i \neq k,\ j \neq \ell;\ i, k \in \Omega_1,\ j, \ell \in \Omega_2)$,

(3) $G_{i,j}G_{i,\ell} \equiv G_{1,j}G_{1,\ell} \quad (j \neq \ell;\ i \in \Omega_1 \setminus \{1\},\ j,\ell \in \Omega_2)$,
(4) $G_{i,j}G_{k,j} \equiv G_{i,1}G_{k,1} \quad (i \neq k;\ i,k \in \Omega_1,\ j \in \Omega_2 \setminus \{1\})$,
(5) $G_{1,j}G_{1,\ell}G_{1,s} \equiv 0 \quad (j \neq \ell \neq s;\ j,\ell,s \in \Omega_2)$,
(6) $G_{i,1}G_{k,1}G_{t,1} \equiv 0 \quad (i \neq k \neq t;\ i,k,t \in \Omega_1)$,
(7) $G_{1,1}G_{i,1}G_{i,1} \equiv G_{1,j}G_{1,j}G_{1,1} \quad (i \in \Omega_1 \setminus \{1\},\ j \in \Omega_2 \setminus \{1\})$,
(8) $G_{i,1}G_{i,1}G_{1,1} \equiv G_{1,1}G_{1,j}G_{1,j} \quad (i \in \Omega_1 \setminus \{1\},\ j \in \Omega_2 \setminus \{1\})$.

Proof. For (1): It is obvious, since $2G_{i,j}G_{i,j}G_{i,j} \equiv 2G_{i,j}$ $(i \in \Omega_1;\ j \in \Omega_2)$. For (2): Let $i,k \in \Omega_1$, $j,\ell \in \Omega_2$, $i \neq k$, and $j \neq \ell$. By (1) (of the present lemma), we have $G_{i,j}G_{i,j}G_{i,j} \equiv G_{i,j}$. Multiplying by $G_{k,\ell}$ from the right, we get

$$G_{i,j}G_{i,j}G_{i,j}G_{k,\ell} \equiv G_{i,j}G_{k,\ell}. \tag{1}$$

Using $G_{i,j}G_{i,j}G_{k,\ell} \equiv -G_{k,\ell}G_{i,j}G_{i,j}$ in (1) gives

$$-G_{i,j}G_{k,\ell}G_{i,j}G_{i,j} \equiv G_{i,j}G_{k,\ell},$$

which implies (2), since $G_{i,j}G_{k,\ell}G_{i,j} \equiv 0$. For (3): Let $i \in \Omega_1 \setminus \{1\}$, $j,\ell \in \Omega_2$, and $j \neq \ell$, we have

$$G_{i,j}G_{1,j}G_{1,j} + G_{1,j}G_{1,j}G_{i,j} - G_{i,j} \equiv 0.$$

Multiplying by $G_{i,\ell}$ from the right and observing that $G_{1,j}G_{i,\ell} \equiv 0$ (by (2) (of the present lemma)), we get

$$G_{1,j}G_{1,j}G_{i,j}G_{i,\ell} - G_{i,j}G_{i,\ell} \equiv 0. \tag{2}$$

Using $G_{1,j}G_{i,j}G_{i,\ell} \equiv -G_{i,\ell}G_{i,j}G_{1,j} + G_{1,\ell}$ in (2) gives

$$G_{1,j}\left(-G_{i,\ell}G_{i,j}G_{1,j} + G_{1,\ell}\right) - G_{i,j}G_{i,\ell} \equiv 0,$$

which implies (3), since $G_{1,j}G_{i,\ell} \equiv 0$ (by (2) (of the present lemma)). For (4): Let $i,k \in \Omega_1$, $j \in \Omega_2 \setminus \{1\}$, and $i \neq k$. We have

$$G_{i,j}G_{i,1}G_{i,1} + G_{i,1}G_{i,1}G_{i,j} - G_{i,j} \equiv 0.$$

Multiplying by $G_{k,j}$ from the right and observing that $G_{i,1}G_{k,j} \equiv 0$ (by (2) (of the present lemma)), we obtain

$$G_{i,1}G_{i,1}G_{i,j}G_{k,j} - G_{i,j}G_{k,j} \equiv 0. \tag{3}$$

Using $G_{i,1}G_{i,j}G_{k,j} \equiv -G_{k,j}G_{i,j}G_{i,1} + G_{k,1}$ in (3) gives

$$G_{i,1}(-G_{k,j}G_{i,j}G_{i,1} + G_{k,1}) - G_{i,j}G_{k,j} \equiv 0,$$

which implies (4), since $G_{i,1}G_{k,j} \equiv 0$ (by (2) (of the present lemma)). For (5): Let $i \in \Omega_1 \setminus \{1\}$, $j, \ell, s \in \Omega_2$, and $j \neq \ell \neq s$. By (3) (of the present lemma), we have

$$G_{i,j}G_{i,\ell} \equiv G_{1,j}G_{1,\ell}.$$

Multiplying by $G_{1,s}$ from the right, we get

$$G_{i,j}G_{i,\ell}G_{1,s} \equiv G_{1,j}G_{1,\ell}G_{1,s},$$

which implies (5), since $G_{i,\ell}G_{1,s} \equiv 0$ (by (2) (of the present lemma)). For (6): Let $i, k, t \in \Omega_1$, $i \neq k \neq t$, and $j \in \Omega_2 \setminus \{1\}$. By (3) (of the present lemma), we have

$$G_{i,j}G_{k,j} \equiv G_{i,1}G_{k,1}.$$

Multiplying by $G_{t,1}$ from the right gives

$$G_{i,j}G_{k,j}G_{t,1} \equiv G_{i,1}G_{k,1}G_{t,1},$$

which implies (6); since $G_{k,j}G_{t,1} \equiv 0$ (by (2) (of the present lemma)). For (7): Let $i \in \Omega_1 \setminus \{1\}$ and $j \in \Omega_2 \setminus \{1\}$. By (4) (of the present lemma), we have

$$G_{1,j}G_{i,j} \equiv G_{1,1}G_{i,1}.$$

Multiplying by $G_{i,1}$ from the right and observing that $G_{i,j}G_{i,1} \equiv G_{1,j}G_{1,1}$ (by (3) (of the present lemma)), we get (7). For (8), let $i \in \Omega_1 \setminus \{1\}$, and $j \in \Omega_2 \setminus \{1\}$, we have

$$G_{i,1}G_{i,1}G_{1,1} \equiv -G_{1,1}G_{i,1}G_{i,1} + G_{1,1}. \tag{4}$$

By (7) (of the present lemma), we have $G_{1,1}G_{i,1}G_{i,1} \equiv G_{1,j}G_{1,j}G_{1,1}$. Using this in (4) gives

$$G_{i,1}G_{i,1}G_{1,1} \equiv -G_{1,j}G_{1,j}G_{1,1} + G_{1,1} \equiv G_{1,1}G_{1,j}G_{1,j}.$$

This completes the proof. □

Remark 2.5. By (7) and (8) of Lemma 2.4, for all $i, k \in \Omega_1 \setminus \{1\}$, $i \neq k$, and $j \in \Omega_2 \setminus \{1\}$, we have

$$G_{1,1}G_{i,1}G_{i,1} \equiv G_{1,j}G_{1,j}G_{1,1} \equiv G_{1,1}G_{k,1}G_{k,1},$$

and

$$G_{i,1}G_{i,1}G_{1,1} \equiv G_{1,1}G_{1,j}G_{1,j} \equiv G_{k,1}G_{k,1}G_{1,1}.$$

That is, the products

$$G_{1,1}G_{i,1}G_{i,1} \text{ and } G_{i,1}G_{i,1}G_{1,1} \text{ (resp. } G_{1,j}G_{1,j}G_{1,1} \text{ and } G_{1,1}G_{1,j}G_{1,j})$$

do not depend on the choice of $i \in \Omega_1 \setminus \{1\}$ (resp. $j \in \Omega_2 \setminus \{1\}$).

Corollary 2.6. *In $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$, the following identities hold:*

(1) $G_{1,\ell}G_{1,1}G_{1,1}G_{1,j} \equiv -\delta_{\ell,j}G_{1,1}G_{1,1}G_{1,j}G_{1,j} + G_{1,\ell}G_{1,j} \quad (\ell, j \in \Omega_2 \setminus \{1\})$,
(2) $G_{i,1}G_{1,1}G_{1,1}G_{k,1} \equiv -\delta_{i,k}G_{1,1}G_{1,1}G_{i,1}G_{i,1} + G_{i,1}G_{k,1} \quad (i, k \in \Omega_1 \setminus \{1\})$,
(3) $G_{1,j}G_{1,\ell}G_{1,\ell}G_{1,1} \equiv -\delta_{j,1}G_{1,1}G_{1,1}G_{1,\ell}G_{1,\ell} + G_{1,j}G_{1,1} \quad (j, \ell \in \Omega_2; \ell \neq 1)$,
(4) $G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,1} \equiv G_{1,j}G_{1,j}G_{1,1} \quad (j \in \Omega_2 \setminus \{1\})$,
(5) $G_{1,j}G_{1,j}G_{1,1}G_{1,1}G_{1,\ell} \equiv G_{1,1}G_{1,1}G_{1,\ell} \quad (j, \ell \in \Omega_2 \setminus \{1\})$,
(6) $G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,1}G_{i,1} \equiv G_{1,1}G_{i,1} \quad (i \in \Omega_1 \setminus \{1\},\ j \in \Omega_2 \setminus \{1\})$,
(7) $G_{1,1}G_{1,j}G_{1,j}G_{1,\ell} \equiv G_{1,1}G_{1,\ell} \quad (j,\ \ell \in \Omega_2 \setminus \{1\})$,
(8) $G_{i,1}G_{1,1}G_{1,j}G_{1,j} \equiv G_{i,1}G_{1,1} \quad (i \in \Omega_1 \setminus \{1\},\ j \in \Omega_2 \setminus \{1\})$.

Proof. For (1): Let $\ell, j \in \Omega_2 \setminus \{1\}$, we have

$$\begin{aligned} G_{1,\ell}G_{1,1}G_{1,1}G_{1,j} &\equiv (-G_{1,1}G_{1,1}G_{1,\ell} + G_{1,\ell})\, G_{1,j} \\ &\equiv -\delta_{\ell,j}G_{1,1}G_{1,1}G_{1,j}G_{1,j} + G_{1,\ell}G_{1,j}, \end{aligned}$$

using $G_{1,\ell}G_{1,1}G_{1,1} \equiv -G_{1,1}G_{1,1}G_{1,\ell} + G_{1,\ell}$ and (5) of Lemma 2.4. The proof of (2) is similar. For (3): Let $\ell, j \in \Omega_2$ and $\ell \neq 1$, we have

$$\begin{aligned} G_{1,j}G_{1,\ell}G_{1,\ell}G_{1,1} &\equiv G_{1,j}(-G_{1,1}G_{1,\ell}G_{1,\ell} + G_{1,1}) \\ &\equiv -\delta_{j,1}G_{11}G_{1,1}G_{1,\ell}G_{1,\ell} + G_{1,j}G_{1,1}, \end{aligned}$$

using $G_{1,\ell}G_{1,\ell}G_{1,1} \equiv -G_{1,1}G_{1,\ell}G_{1,\ell} + G_{1,1}$ and (5) of Lemma 2.4. For (4): Let $j \in \Omega_2 \setminus \{1\}$ and choose any $t \in \Omega_1 \setminus \{1\}$, we have

$$G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,1} \equiv G_{1,1}G_{1,1}G_{1,1}G_{t,1}G_{t,1} \equiv G_{1,1}G_{t,1}G_{t,1} \equiv G_{1,j}G_{1,j}G_{1,1},$$

by (7) and (1) of Lemma 2.4. For (5): Let $j, \ell \in \Omega_2 \setminus \{1\}$ and choose any $t \in \Omega_1 \setminus \{1\}$, we have

$$G_{1,j}G_{1,j}G_{1,1}G_{1,1}G_{1,\ell} \equiv G_{1,1}G_{t,1}G_{t,1}G_{1,1}G_{1,\ell} \equiv G_{1,1}G_{1,1}G_{1,\ell}G_{1,\ell}G_{1,\ell} \equiv G_{1,1}G_{1,1}G_{1,\ell},$$

by (7), (8), and (1) of Lemma 2.4. For (6): Let $i \in \Omega_1 \setminus \{1\}$ and $j \in \Omega_2 \setminus \{1\}$, we have

$$G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,1}G_{i,1} \equiv G_{1,j}G_{1,j}G_{1,1}G_{i,1} \equiv G_{1,1}G_{i,1}G_{i,1}G_{i,1} \equiv G_{1,1}G_{i,1},$$

by (4) (of the present lemma) and (7), (1) of Lemma 2.4. For (7): Let $\ell,\ j \in \Omega_2 \setminus \{1\}$, we have

$$G_{1,1}G_{1,j}G_{1,j}G_{1,\ell} \equiv G_{1,1}G_{1,\ell}G_{1,\ell}G_{1,\ell} \equiv G_{1,1}G_{1,\ell},$$

by Remark 2.5 and (1) of Lemma 2.4. For (8): Let $i \in \Omega_1 \setminus \{1\}$ and $j \in \Omega_2 \setminus \{1\}$, we have

$$G_{i,1}G_{1,1}G_{1,j}G_{1,j} \equiv G_{i,1}G_{i,1}G_{i,1}G_{1,1} \equiv G_{i,1}G_{1,1},$$

by (8) and (1) of Lemma 2.4. This completes the proof. □

3. Main results

In this section we present the main results of this paper on the representations of the Jordan triple system $\mathcal{J}_{\mathbb{F}}$. Our goal is to use the specialty of the Jordan triple system $\mathcal{J}_{\mathbb{F}}$ and the identities of Section 2.1 to get the decomposition of $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ into matrix algebras.

Theorem 3.1. *With notation as above. If $\mathcal{J}_{\mathbb{F}}$ is the Jordan triple system of all $p \times q$ ($p \neq q$; $p, q > 1$) rectangular matrices over a field $\mathbb{F}$ of characteristic 0 and $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ is the universal associative envelope of $\mathcal{J}_{\mathbb{F}}$, then*

$$\mathcal{U}(\mathcal{J}_{\mathbb{F}}) \cong M_{p+q\times p+q}(\mathbb{F}),$$

where $M_{p+q\times p+q}(\mathbb{F})$ is the ordinary associative algebra of all $(p+q)\times(p+q)$ matrices over $\mathbb{F}$.

Proof. We set

$$A_{i,k} = G_{i,1}G_{k,1} \quad (i, k \in \Omega_1;\ i \neq k).$$
$$A_{1,p+1} = G_{1,j}G_{1,j}G_{1,1} \quad (\text{for any } j \in \Omega_2 \setminus \{1\}).$$
$$A_{i,k} = G_{i,1}G_{1,1}G_{1,k-p} \quad (i \in \Omega_1,\ k \in \Omega_3;\ (i,k) \neq (1, p+1)).$$
$$A_{i,k} = -A_{k,i} + G_{k,i-p} \quad (i \in \Omega_3,\ k \in \Omega_1).$$
$$A_{i,k} = G_{1,i-p}G_{1,k-p} \quad (i, k \in \Omega_3;\ i \neq k).$$
$$A_{1,1} = G_{1,1}G_{1,1}G_{1,j}G_{1,j} \quad (\text{for any } j \in \Omega_2 \setminus \{1\}).$$
$$A_{i,i} = -G_{1,1}G_{1,1}G_{t,1}G_{t,1} + G_{i,1}G_{i,1} \quad (i \in \Omega_1 \setminus \{1\}; \text{for any } t \in \Omega_1 \setminus \{1\}).$$
$$A_{i,i} = -G_{1,1}G_{1,1}G_{1,j}G_{1,j} + G_{1,i-p}G_{1,i-p} \quad (i \in \Omega_3;\ \text{for any } j \in \Omega_2 \setminus \{1\}).$$

We wish to show that the elements $A_{i,j}$ (for all $i, j \in \Omega_1 \cup \Omega_3$) satisfy the multiplication table for matrix units. We first observe that the elements $A_{1,1}$, $A_{1,p+1}$, and the first term of $A_{i,i}$ ($i \in (\Omega_1 \cup \Omega_3) \setminus \{1\}$) do not depend on the choice of $j \neq 1$ (see Remark 2.5). Let $i \in (\Omega_1 \cup \Omega_3) \setminus \{1\}$ and choose any $j \in \Omega_2 \setminus \{1\}$, we have

$$A_{1,i} = \Delta_{i,\Omega_1}G_{1,1}G_{i,1} + \delta_{i,p+1}G_{1,j}G_{1,j}G_{1,1} + \widehat{\delta}_{i,p+1}\Delta_{i,\Omega_3}G_{1,1}G_{1,1}G_{1,i-p}.$$
$$A_{i,1} = \Delta_{i,\Omega_1}G_{i,1}G_{1,1} + \Delta_{i,\Omega_3}(-A_{1,i} + G_{1,i-p}).$$

For all $i \in (\Omega_1 \cup \Omega_3) \setminus \{1\}$, we first consider the following four products: $A_{1,i}A_{1,1}$, $A_{1,1}A_{1,i}$, $A_{1,1}A_{i,1}$, and $A_{i,1}A_{1,1}$. For $A_{1,i}A_{1,1}$: We have

$$\begin{aligned} A_{1,i}A_{1,1} &= \Delta_{i,\Omega_1} G_{1,1}G_{i,1}G_{1,1}G_{1,1}G_{1,j}G_{1,j} \\ &\quad + \delta_{i,p+1}G_{1,j}G_{1,j}G_{1,1}G_{1,1}G_{1,1}G_{1,j}G_{1,j} \\ &\quad + \widehat{\delta}_{i,p+1}\Delta_{i,\Omega_3}G_{1,1}G_{1,1}G_{1,i-p}G_{1,1}G_{1,1}G_{1,j}G_{1,j} \\ &\equiv 0, \end{aligned} \tag{5}$$

since $G_{11}G_{11}G_{11} \equiv G_{11}$ (by (1) of Lemma 2.4) and the elements, $G_{1,1}G_{i,1}G_{1,1}$ $(i \neq 1)$, $G_{1,j}G_{1,1}G_{1,j}$ $(j \neq 1)$ and $G_{1,1}G_{1,i-p}G_{1,1}$ $(i \neq p+1)$ vanish (by (6), (5) of Lemma 2.4),

For $A_{1,1}A_{1,i}$: We have

$$\begin{aligned} A_{1,1}A_{1,i} &= \Delta_{i,\Omega_1}G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,1}G_{i,1} + \delta_{i,p+1}G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,j}G_{1,j}G_{1,1} \\ &\quad + \widehat{\delta}_{i,p+1}\Delta_{i,\Omega_3}G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,1}G_{1,1}G_{1,i-p} \qquad (6) \\ &\equiv \Delta_{i,\Omega_1}G_{1,1}G_{i,1} + \delta_{i,p+1}G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,1} \\ &\quad + \widehat{\delta}_{i,p+1}\Delta_{i,\Omega_3}G_{1,1}G_{1,1}G_{1,1}G_{1,1}G_{1,i-p} \\ &\equiv \Delta_{i,\Omega_1}G_{1,1}G_{i,1} + \delta_{i,p+1}G_{1,j}G_{1,j}G_{1,1} + \widehat{\delta}_{i,p+1}\Delta_{i,\Omega_3}G_{1,1}G_{1,1}G_{1,i-p} \\ &= A_{1,i}, \end{aligned}$$

by (6), (4), (5) of Corollary 2.6 and (1) of Lemma 2.4. For $A_{1,1}A_{i,1}$: We have

$$A_{1,1}A_{i,1} = \Delta_{i,\Omega_1}A_{1,1}G_{i,1}G_{1,1} + \Delta_{i,\Omega_3}(-A_{1,1}A_{1,i} + A_{1,1}G_{1,i-p}).$$

We observe that $A_{1,1}G_{i,1} = G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{i,1} \equiv 0$; since $G_{1,j}G_{i,1} \equiv 0$ (by (2) of Lemma 2.4). Using this and (6) (of the present proof), (4) and (7) of Corollary 2.6, we obtain

$$\begin{aligned} A_{1,1}A_{i,1} &= \Delta_{i,\Omega_3}(-A_{1,i} + A_{1,1}G_{1,i-p}) \\ &= \Delta_{i,\Omega_3}(-A_{1,i} + G_{1,1}G_{1,1}G_{1,j}G_{1,j}G_{1,i-p}) \\ &\equiv \Delta_{i,\Omega_3}(-A_{1,i} + \delta_{i,p+1}G_{1,j}G_{1,j}G_{1,1} + \widehat{\delta}_{i,p+1}G_{1,1}G_{1,1}G_{1,i-p}) \\ &\equiv 0. \end{aligned}$$

For $A_{i,1}A_{1,1}$: We have

$$\begin{aligned}A_{i,1}A_{1,1} &= \Delta_{i,\Omega_1}G_{i,1}G_{1,1}A_{1,1} + \Delta_{i,\Omega_3}(-A_{1,i}A_{1,1} + G_{1,i-p}A_{1,1})\\ &= \Delta_{i,\Omega_1}G_{i,1}G_{1,1}G_{1,1}G_{1,1}G_{1,j}G_{1,j} + \Delta_{i,\Omega_3}G_{1,i-p}G_{1,1}G_{1,1}G_{1,j}G_{1,j}\\ &\equiv \Delta_{i,\Omega_1}G_{i,1}G_{1,1}G_{1,j}G_{1,j} + \Delta_{i,\Omega_3}\big[\delta_{i,p+1}G_{1,1}G_{1,1}G_{1,1}G_{1,j}G_{1,j}\\ &\quad + \widehat{\delta}_{i,p+1}G_{1,i-p}G_{1,1}G_{1,1}G_{1,i-p}G_{1,i-p}\big]\\ &\equiv \Delta_{i,\Omega_1}G_{i,1}G_{1,1} + \Delta_{i,\Omega_3}\big[\delta_{i,p+1}G_{1,1}G_{1,j}G_{1,j}\\ &\quad + \widehat{\delta}_{i,p+1}(-G_{1,1}G_{1,1}G_{1,i-p} + G_{1,i-p})\big]\\ &= A_{i,1},\end{aligned}$$

by (5) (of the present proof), (1) of Lemma 2.4, Remark 2.5, and (8), (1) of Corollary 2.6. Summarizing, for all $i \in (\Omega_1 \cup \Omega_3) \setminus \{1\}$, we have

$$A_{1,i}A_{1,1} = 0 = A_{1,1}A_{i,1},\ A_{1,1}A_{1,i} = A_{1,i},\ A_{i,1}A_{1,1} = A_{i,1}. \tag{7}$$

Throughout the rest of the proof, we assume that $i, k \in (\Omega_1 \cup \Omega_3) \setminus \{1\}$. Using the products of (7), we get

$$A_{1,i}A_{1,k} = A_{1,i}A_{1,1}A_{1,k} = 0, \quad A_{i,1}A_{k,1} = A_{i,1}A_{1,1}A_{k,1} = 0. \tag{8}$$

We next consider the two products: $A_{1,i}A_{k,1}$ and $A_{i,1}A_{1,k}$.

For $A_{1,i}A_{k,1}$: Using (8) (of the present proof), we get

$$\begin{aligned}A_{1,i}A_{k,1} &= \Delta_{k,\Omega_1}A_{1,i}G_{k,1}G_{1,1} + \Delta_{k,\Omega_3}A_{1,i}G_{1,k-p}.\\ &= \Delta_{k,\Omega_1}\big(\Delta_{i,\Omega_1}G_{1,1}G_{i,1} + \delta_{i,p+1}G_{1,j}G_{1,j}G_{1,1}\\ &\quad + \widehat{\delta}_{i,p+1}\Delta_{i,\Omega_3}G_{1,1}G_{1,1}G_{1,i-p}\big)G_{k,1}G_{1,1}\\ &\quad + \Delta_{k,\Omega_3}\big(\Delta_{i,\Omega_1}G_{1,1}G_{i,1} + \delta_{i,p+1}G_{1,j}G_{1,j}G_{1,1}\\ &\quad + \widehat{\delta}_{i,p+1}\Delta_{i,\Omega_3}G_{1,1}G_{1,1}G_{1,i-p}\big)G_{1,k-p}.\end{aligned}$$

By (2), (5), and (6) of Lemma 2.4, the following products vanish: $G_{i,1}G_{k,1}G_{1,1}$ $(i \neq k)$, $G_{1,1}G_{k,1}G_{1,1}$, $G_{1,i-p}G_{k,1}$ $(i \neq p{+}1)$, $G_{i,1}G_{1,k-p}$ $(k \neq p{+}1)$, $G_{1,j}G_{1,1}G_{1,k-p}$ $(k \neq p+1)$, $G_{1,1}G_{1,i-p}G_{1,k-p}(i \neq k, p+1)$. It follows that

$$\begin{aligned}A_{1,i}A_{k,1} &= \Delta_{k,\Omega_1}\delta_{i,k}G_{1,1}G_{k,1}G_{k,1}G_{1,1} + \Delta_{k,\Omega_3}\big(\delta_{i,p+1}\delta_{k,p+1}G_{1,j}G_{1,j}G_{1,1}G_{1,1}\\ &\quad + \Delta_{i,\Omega_3}\widehat{\delta}_{i,p+1}\delta_{i,k}G_{1,1}G_{1,1}G_{1,k-p}G_{1,k-p}\big).\end{aligned}$$

We now choose any $\ell \in \Omega_1 \setminus \{1\}$ and $t \in \Omega_2 \setminus \{1\}$. By (7) and (8) of Lemma 2.4 and Remark 2.5, we get $G_{k,1}G_{k,1}G_{1,1} \equiv G_{1,1}G_{1,t}G_{1,t}$, $G_{1,j}G_{1,j}G_{1,1}G_{1,1} \equiv$

$G_{1,1}G_{\ell,1}G_{\ell,1}G_{1,1} \equiv G_{1,1}G_{1,1}G_{1,t}G_{1,t}$ $(j \neq 1)$, $G_{1,1}G_{1,k-p}G_{1,k-p} \equiv G_{1,1}G_{1,t}G_{1,t}$ $(k-p \neq 1)$. Using this discussion in the last equation implies

$$\begin{aligned} A_{1,i}A_{k,1} &\equiv \Delta_{k,\Omega_1}\delta_{i,k}G_{1,1}G_{1,1}G_{1,t}G_{1,t} + \Delta_{k,\Omega_3}\big(\delta_{i,p+1}\delta_{k,p+1}G_{1,1}G_{1,1}G_{1,t}G_{1,t} \\ &\qquad + \Delta_{i,\Omega_3}\widehat{\delta}_{i,p+1}\delta_{i,k}G_{1,1}G_{1,1}G_{1,t}G_{1,t}\big) \\ &= \delta_{ik}A_{1,1}. \end{aligned} \tag{9}$$

For the product $A_{i,1}A_{1,k}$: Using (8)(of the present proof) and (1) of Lemma 2.4, we get

$$\begin{aligned} A_{i,1}A_{1,k} &\equiv \Delta_{i,\Omega_1}G_{i,1}G_{1,1}A_{1,k} + \Delta_{i,\Omega_3}(-A_{1,i}A_{1,k} + G_{1,i-p}A_{1,k}) \\ &\equiv \Delta_{i,\Omega_1}G_{i,1}G_{1,1}\big[\Delta_{k,\Omega_1}G_{1,1}G_{k,1} + \delta_{k,p+1}G_{1,t}G_{1,t}G_{1,1} \\ &\quad + \Delta_{k,\Omega_3}\widehat{\delta}_{k,p+1}G_{1,k-p}\big] + \Delta_{i,\Omega_3}G_{1,i-p}\big[\Delta_{k,\Omega_1}G_{1,1}G_{k,1} \\ &\quad + \delta_{k,p+1}G_{1,t}G_{1,t}G_{1,1} + \widehat{\delta}_{k,p+1}\Delta_{k,\Omega_3}G_{1,1}G_{1,1}G_{1,k-p}\big]. \end{aligned}$$

Using (2), (8), (3), and (1) of Corollary 2.6 implies

$$\begin{aligned} A_{i,1}A_{1,k} &\equiv \Delta_{i,\Omega_1}\big[\Delta_{k,\Omega_1}(-\delta_{i,k}G_{1,1}G_{1,1}G_{i,1}G_{i,1} + G_{i,1}G_{k,1}) + \delta_{k,p+1}G_{i,1}G_{1,1}G_{1,1} \\ &\quad + \Delta_{k,\Omega_3}\widehat{\delta}_{k,p+1}G_{i,1}G_{1,1}G_{1,k-p}\big] + \Delta_{i,\Omega_3}\big[\Delta_{k,\Omega_1}G_{1,i-p}G_{1,1}G_{k,1} \\ &\quad + \delta_{k,p+1}(-\delta_{i,p+1}G_{1,1}G_{1,1}G_{1,t}G_{1,t} + G_{1,i-p}G_{1,1}) + \widehat{\delta}_{k,p+1}\Delta_{k,\Omega_3}\big(\delta_{i,p+1}G_{1,1}G_{1,k-p} \\ &\quad + \widehat{\delta}_{i,p+1}(-\delta_{i,k}G_{1,1}G_{1,1}G_{1,i-p}G_{1,i-p} + G_{1,i-p}G_{1,k-p})\big)\big] \\ &= A_{i,k}. \end{aligned}$$

Summarizing, for all $i, k \in (\Omega_1 \cup \Omega_3) \setminus \{1\}$, we have

$$A_{1,i}A_{1,k} = 0 = A_{i,1}A_{k,1},\ A_{1,i}A_{k,1} = \delta_{ik}A_{1,1},\ A_{i,1}A_{1,k} = A_{i,k}. \tag{10}$$

We now use the products of (7) and (10) to get all the others. For all $i, k, \ell, t \in (\Omega_1 \cup \Omega_3) \setminus \{1\}$, we have $A_{1,i}A_{i,1} = A_{1,1}$, hence $A_{1,1}A_{1,i}A_{i,1} = A_{1,1}A_{1,1}$. Thus $A_{1,1} = A_{1,1}A_{1,1}$. We also have $A_{i,k} = A_{i,1}A_{1,k}$. Hence $A_{i,k}A_{\ell,t} = A_{i,1}A_{1,k}A_{\ell,1}A_{1,t} = \delta_{k,\ell}A_{i,1}A_{1,1}A_{1,t} = \delta_{k,\ell}A_{i,1}A_{1,t} = \delta_{k,\ell}A_{i,t}$. Summarizing, for all $i, k, \ell, t \in \Omega_1 \cup \Omega_3$, we have

$$A_{i,k}A_{\ell,t} = \delta_{k,\ell}A_{i,t}.$$

Now let $\mathcal{S}$ denote the subspace of $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ generated by $A_{i,j}$ $(i, j \in \Omega_1 \cup \Omega_3)$. We have shown that $\mathcal{S}$ is a subalgebra of $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ and is isomorphic to $M_{p+q\times p+q}(\mathbb{F})$. By the definition of $A_{i,j}$, we get

$$G_{i,j} = A_{i,j+p} + A_{j+p,i} \quad (\text{for all } i \in \Omega_1,\ j \in \Omega_2).$$

Thus all $G_{i,j} \in \mathcal{S}$. Hence $\mathcal{U}(\mathcal{J}_{\mathbb{F}}) \cong M_{p+q\times p+q}(\mathbb{F})$. □

Corollary 3.2. *The universal associative envelope of the Jordan triple system $\mathcal{J}_{\mathbb{F}}$ (of Definition 2.1) is semisimple.*

Corollary 3.3. *The Jordan triple system $\mathcal{J}_{\mathbb{F}}$ (of Definition 2.1) has only one nontrivial representation (up to equivalence) defined by:*

$$\Theta : \mathcal{J}_{\mathbb{F}} \to M_{p+q\times p+q}(\mathbb{F}), \quad E_{i,j} \to \begin{pmatrix} O_{p\times p} & E_{i,j} \\ E_{j,i} & O_{q\times q} \end{pmatrix}.$$

Proof. By Theorem 3.1, the Jordan triple system $\mathcal{J}_{\mathbb{F}}$ has only one nontrivial representation. We now verify that Θ is a representation of $\mathcal{J}_{\mathbb{F}}$. We first observe that,

$$\Theta(E_{i,j}) = \mathbb{E}_{i,p+j} + \mathbb{E}_{p+j,i} \quad (\text{for all } i \in \Omega_1,\ j \in \Omega_2).$$

where $\mathbb{E}_{i,j}$ is the $(p+q)\times(p+q)$ matrices whose (i,j) entry is 1 and all the other entries are 0. For all $i,k,s \in \Omega_1$ and $j,\ell,t \in \Omega_2$, we have

$$\begin{aligned}\Theta\{E_{i,j}, E_{k,\ell}, E_{s,t}\} &= \Theta\left(E_{i,j}E_{\ell,k}E_{s,t} + E_{s,t}E_{\ell,k}E_{i,j}\right) \\ &= \Theta\left(\delta_{j,\ell}\delta_{k,s}E_{i,t} + \delta_{t,\ell}\delta_{k,i}E_{s,j}\right) \\ &= \delta_{j,\ell}\delta_{k,s}\left(\mathbb{E}_{i,p+t} + \mathbb{E}_{p+t,i}\right) + \delta_{t,\ell}\delta_{k,i}\left(\mathbb{E}_{s,p+j} + \mathbb{E}_{p+j,s}\right).\end{aligned}$$

On the other hand

$$\begin{aligned}\{\Theta(E_{i,j}), \Theta(E_{k,\ell}), \Theta(E_{s,t})\} &= \Theta(E_{i,j})(\Theta(E_{k,\ell}))^t\Theta(E_{s,t}) + \Theta(E_{s,t})(\Theta(E_{k,\ell}))^t\Theta(E_{i,j}) \\ &= \left(\mathbb{E}_{i,p+j} + \mathbb{E}_{p+j,i}\right)\left(\mathbb{E}_{k,p+\ell} + \mathbb{E}_{p+\ell,k}\right)^t\left(\mathbb{E}_{s,p+t} + \mathbb{E}_{p+t,s}\right) \\ &\quad + \left(\mathbb{E}_{s,p+t} + \mathbb{E}_{p+t,s}\right)\left(\mathbb{E}_{k,p+\ell} + \mathbb{E}_{p+\ell,k}\right)^t\left(\mathbb{E}_{i,p+j} + \mathbb{E}_{p+j,i}\right) \\ &= \left(\mathbb{E}_{i,p+j} + \mathbb{E}_{p+j,i}\right)\left(\mathbb{E}_{k,p+\ell} + \mathbb{E}_{p+\ell,k}\right)\left(\mathbb{E}_{s,p+t} + \mathbb{E}_{p+t,s}\right) \\ &\quad + \left(\mathbb{E}_{s,p+t} + \mathbb{E}_{p+t,s}\right)\left(\mathbb{E}_{k,p+\ell} + \mathbb{E}_{p+\ell,k}\right)\left(\mathbb{E}_{i,p+j} + \mathbb{E}_{p+j,i}\right) \\ &= \delta_{j,\ell}\delta_{k,s}\mathbb{E}_{i,p+t} + \delta_{i,k}\delta_{\ell,t}\mathbb{E}_{p+j,s} + \delta_{t,\ell}\delta_{i,k}\mathbb{E}_{s,p+j} + \delta_{s,k}\delta_{\ell,j}\mathbb{E}_{p+t,i}.\end{aligned}$$

Hence Θ is a representation of $\mathcal{J}_{\mathbb{F}}$. □

Lemma 3.4. *The center $\mathfrak{C}(\mathcal{U}(\mathcal{J}_{\mathbb{F}}))$ of the universal associative envelope $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ has dimension* 1 *with a basis*

$$e = (1-q)\,G_{1,1}G_{1,j}G_{1,j} + (1-p)\,G_{1,1}G_{1,1}G_{t,1}G_{t,1} + \sum_{i=2}^{p} G_{i,1}G_{i,1} + \sum_{s=1}^{q} G_{1,s}G_{1,s},$$

for any $t \in \Omega_1 \setminus \{1\}$ and $j \in \Omega_2 \setminus \{1\}$.

Proof. By Theorem 3.1, we have $\mathfrak{C}(\mathcal{U}(\mathcal{J}_{\mathbb{F}})) \cong \mathbb{F}$. It follows that $e = \sum_{i=1}^{p+q} A_{i,i}$ is the only idempotent in $\mathcal{U}(\mathcal{J}_{\mathbb{F}})$ that span the center. Using the proof of Theorem 3.1, we get

$$\begin{aligned} e &= G_{1,1}G_{1,1}G_{1,j}G_{1,j} + \sum_{i=2}^{p} (-G_{11}G_{1,1}G_{t,1}G_{t,1} + G_{i,1}G_{i,1}) \\ &\quad + \sum_{i=p+1}^{p+q} (-G_{1,1}G_{1,1}G_{1,j}G_{1,j} + G_{1,i-p}G_{1,i-p}) \\ &= G_{1,1}G_{1,1}G_{1,j}G_{1,j} - (p-1)\, G_{11}G_{1,1}G_{t,1}G_{t,1} + \sum_{i=2}^{p} G_{i,1}G_{i,1} \\ &\quad - q\, G_{1,1}G_{1,1}G_{1,j}G_{1,j} + \sum_{i=p+1}^{p+q} G_{1,i-p}G_{1,i-p}. \end{aligned}$$

This completes the proof. □

Acknowledgement. We thank the referee for helpful suggestions which improved the exposition.

References

[1] C. Chu, *Jordan triples and Riemannian symmetric spaces*, Adv. Math., 219 (2008), 2029-2057.

[2] E. Corrigan and T. Hollowood, *String construction of a commutative nonassociative algebra related to the exceptional Jordan algebra*, Phys. Lett., 203 (1988), 47-51.

[3] H. Elgendy, *On the universal envelope of a Jordan triple system of $n \times n$ matrices*, J. Algebra Appl., 21(6) (2022), 2250126 (19 pp).

[4] H. Elgendy, *Representations of special Jordan triple systems of all symmetric and hermitian n by n matrices*, Linear Multilinear Algebra, DOI: 10.1080/03081087.2021.1970097, in press.

[5] D. Fairlie and C. Manogue, *Lorentz invariance and the composite string*, Phys. Rev., 34 (1986), 1832-1834.

[6] J. Faulkner and J. Ferrar, *Exceptional Lie algebras and related algebraic and geometric structures*, Bull. London Math. Soc., 9 (1977), 1-35.

[7] R. Foot and G. Joshi, *String theories and the Jordan algebras*, Phys. Lett., 199 (1987), 203-208.

[8] P. Goddard, W. Nahm, D. Olive, H. Ruegg and A. Schwimmer, *Fermions and octonions*, Comm. Math. Phys., 112 (1987), 385-408.

[9] M. Günaydin and F. Gürsey, *Quark structure and octonions*, J. Mathematical Phys., 14 (1973), 1651-1667.

[10] M. Günaydin, G. Sierra and P. Townsend, *The geometry of* $N = 2$ *Maxwell-Einstein supergravity and Jordan algebras*, Nuclear Phys., 242 (1984), 244-268.

[11] F. Gürsey, *Super Poincaré groups and division algebras*, Modern Phys. Lett., 2 (1987), 967-976.

[12] N. Jacobson, *Lie and Jordan triple systems*, Amer. J. Math., 71 (1949), 149-170.

[13] W. Kaup and D. Zaitsev, *On symmetric Cauchy-Riemann manifolds*, Adv. Math., 149 (2000), 145-181.

[14] M. Koecher, *Imbedding of Jordan algebras into Lie algebras I*, Amer. J. Math., 89 (1967), 787-816.

[15] M. Koecher, An Elementary Approach to Bounded Symmetric Domains, Lecture Notes, Rice University, Houston, 1969.

[16] K. McCrimmon, *Jordan algebras and their applications*, Bull. Amer. Math. Soc., 84 (1978), 612-627.

[17] K. McCrimmon, A Taste of Jordan Algebras, Universitext, Springer-Verlag, 2004.

[18] K. Meyberg, Lectures on Algebras and Triple Systems, Lecture Notes, University of Virginia, Charlottesville, 1972.

[19] E. Neher, Jordan Triple Systems by the Grid Approach, Lecture Notes in Mathematics, Vol. 1280, Springer-Verlag, Berlin, 1987.

[20] S. Okubo, *Triple products and Yang-Baxter equation. II. Orthogonal and symplectic ternary systems*, J. Math. Phys., 34 (1993), 3292-3315.

[21] G. Sierra, *An application of the theories of Jordan algebras and Freudenthal triple systems to particles and strings*, Classical Quantum Gravity, 4 (1987), 227-236.

[22] H. Upmeier, *Jordan algebras and harmonic analysis on symmetric spaces*, Amer. J. Math., 108 (1986), 1-25.

Hader A. Elgendy
Department of Mathematics
Faculty of Science
Damietta University
Damietta 34517, Egypt
e-mail: haderelgendy42@hotmail.com