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ON MODULES WITH CHAIN CONDITION ON NON-SMALL SUBMODULES

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Abstract. In 1979, Fleury studied a class of modules with finite spanning dimension and dually a class of modules with ascending chain condition on non-small submodules was studied by Lomp and Ozcan in 2011. In the present work, we explore and investigate some new characterizations and properties of these classes of modules.

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1. Introduction

We recall a classic notion "Goldie dimension" from [5], a module M is said to have *finite Goldie dimension* if for any strictly ascending sequence of submodules $U_1 \subset U_2 \subset \cdots$, there exists i such that $U_j \leq_e U_{j+1}$ for every $j \geq i$ or equivalently M does not contain a direct sum of infinite number of nonzero submodules of M.

In 1979, P. Fleury [3] introduced a class of "modules with finite spanning dimension" in order to dualize the concept of Goldie dimension. According to [3], a module M has *finite spanning dimension* if for every strictly decreasing sequence of submodules $U_1 \supset U_2 \supset \cdots$ there exists i such that $U_j \leq_s M$ for every $j \geq i$. This class is actually a class of modules with descending chain condition on non-small submodules.

In 2011, Lomp and Ozcan [9] discussed some properties of a class of modules with ascending chain condition on non-small submodules. In this paper, we continue the study of this class of modules and call it as ns-Noetherian modules. A module is

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called *ns-Noetherian* if, it satisfies ascending chain condition on non-small submodules; a ring R is called *right (respectively, left) ns-Noetherian* if, R_R (respectively, ${}_RR$) is an ns-Noetherian module; and a ring R is called an *ns-Noetherian ring* if it is left and right ns-Noetherian. Similarly, the notion of ns-Artinian modules is equivalent to the modules with finite spanning dimension.

The main aim of this work is to investigate some new properties of ns-Noetherian modules neither analogous to Noetherian modules nor to the modules with finite spanning dimensions. Furthermore, we provide some examples to support our view. We also explore some new properties of the class of modules with finite spanning dimension. We generalize [9, Theorem 4.2] by giving an alternate method of proof. As a consequence of Proposition 4.10 and Proposition 4.11, we find that the condition of quasi-projective is superfluous in [11, Proposition 3.5].

Throughout this paper, all rings are associative rings with identity and all modules are right unital modules unless otherwise stated. Recall from [13], a submodule N of a module M is called *essential* (in M) and denoted by $N \leq_e M$ if $N \cap K \neq 0$ for every nonzero submodule K of M, otherwise N is called *non-essential*. We denote a non-essential submodule N in M by $N \leq_{ne} M$. Dually, a submodule N of a module M is called *small* (in M) and denoted by $N \leq_s M$ if $N + K \neq M$ for any proper submodule K of M, otherwise N is called *non-small*. We denote a non-small submodule N in M by $N \leq_{ns} M$. We refer readers to [13] for all undefined terminologies and notions.

2. ns-Noetherian modules

We begin by observing the following facts:

Lemma 2.1. *Let M be a module.*

(1) *M is ns-Noetherian if and only if for every nonempty family of non-small submodules of M has a maximal element.*
(2) *M is Noetherian if and only if it is ns-Noetherian and satisfies ACC on small submodules.*

Let M be a module over a commutative ring R. Then, the set consisting of all ordered pairs (r, m) where $r \in R, m \in M$ form a commutative ring with respect to the addition and multiplication defined by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$ and $(r_1, m_1)(r_2, m_2) = (r_1r_2, m_1r_2 + r_1m_2)$, respectively, for all $r_1, r_2 \in R, m_1, m_2 \in M$. This ring is known as *trivial extension* of M by R and usually denoted by $R(+)M$ [1].

Lemma 2.2. *Let M be a divisible module over a commutative integral domain R and K be an ideal of $R(+)M$. Then,*

(1) *K is non-small if and only if $K = I(+)M$ for some non-small ideal I of R.*
(2) *K is small if and only if $K = I(+)M$ for some small ideal I of R or $K = 0(+)N$ for some submodule N of M.*

Proof. Let K be a proper ideal of $S = R(+)M$. Then, by [1, Corollary 3.4], either $K = I(+)M$ for some proper ideal I of R or $K = 0(+)N$ for some submodule N of M. First assume that $K = 0(+)N$ for some submodule N of M. Then, K is always a small ideal as $K + (0(+)N') \neq S$ for any submodule N' of M and $K + (I'(+)M) = S$ if and only if $I' = R$. Next assume that $K = I(+)M$ for some proper ideal I of R. Then, $K + (0(+)N') \neq S$ for any submodule N' of M and $K + (I'(+)M) = S$ if and only if $I + I' = R$. It follows that $K = I(+)M$ is small if and only if I is small in R. Thus, the result follows. □

Recall from [13], a module M is called a *hollow module* if its every proper submodule is small. By applying Lemma 2.2, in Proposition 2.3(1), we generalize [9, Example 4.4] analogous to [12, Corollary 4.2].

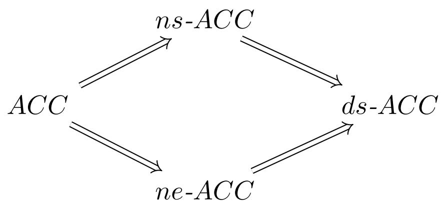
Proposition 2.3. *Let M be a divisible module over a commutative integral domain R. Then,*

(1) *$R(+)M$ is ns-Noetherian (has finite spanning dimension) if and only if R is so.*
(2) *$R(+)M$ is hollow if and only if R is hollow.*

Remarks 2.4. (1) Obviously, Noetherian modules and hollow modules are ns-Noetherian modules. But, an ns-Noetherian module need not be Noetherian or hollow. Also, a right ns-Noetherian ring need not be right Noetherian. For example, since $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module and $\mathbb{Z}$ is an ns-Noetherian commutative domain, the ring $R = \mathbb{Z}(+)\mathbb{Q}$ is an ns-Noetherian ring but not hollow by Proposition 2.3. Since $\mathbb{Q}$ is not a Noetherian $\mathbb{Z}$-module, the ring R is not Noetherian as it has ideals of the form $0(+)N$, where N's are submodules of $\mathbb{Q}$.
(2) We observe that some properties of modules with finite spanning dimension do not hold in case of ns-Noetherian modules. For example, $\mathbb{Z}$ as a $\mathbb{Z}$-module is ns-Noetherian but it does not satisfy [3, Lemma 2.2, Lemma 2.3, Theorem 3.1 and Theorem 5.1].

Since every nonzero proper direct summand of a module is non-essential and non-small, *ACC* on either non-small submodules or non-essential submodules implies

ACC on direct summands. Throughout this paper, we use the following notations for ACC on all submodules, non-small submodules, non-essential submodules and direct summands: *ACC*, *ns-ACC* (ns-Noetherian), *ne-ACC* and *ds-ACC*, respectively. We have the following diagram:



Now, we show that all the implications are strict; and there is no implication between *ns-ACC* and *ne-ACC*.

Examples 2.5. (1) An ns-Noetherian module need not be Noetherian. Also, a module satisfying *ACC* on non-essential submodules need not be Noetherian. For example, $\mathbb{Z}_{p^\infty}$ is an ns-Noetherian $\mathbb{Z}$-module and satisfies *ACC* on non-essential submodules as a $\mathbb{Z}$-module but not Noetherian.
(2) A module satisfying *ACC* on direct summands need not satisfy *ACC* on non-small and non-essential submodules. For example, let R be the formal triangular matrix ring $\begin{bmatrix} \mathbb{Z} & \mathbb{Z}_{p^\infty} \\ 0 & \mathbb{Z} \end{bmatrix}$, then R_R satisfies *ACC* on direct summands but does not satisfy *ACC* on non-small and non-essential submodules.
(3) An ns-Noetherian module need not satisfy *ACC* on non-essential submodules. For example, assume that V is an infinite dimensional vector space over a field F and R is the commutative ring $F(+)V$, then R_R is ns-Noetherian by Proposition 2.3 but does not satisfy *ACC* on non-essential submodules [12, Corollary 4.2].
(4) A module satisfying *ACC* on non-essential submodules need not be ns-Noetherian. For example, $\mathbb{Q}$ satisfies *ACC* on non-essential $\mathbb{Z}$-submodules as it is uniform. However, by Theorem 3.9(2), which will be shown further that it is not ns-Noetherian because $\mathbb{Q}$ as a $\mathbb{Z}$-module is neither hollow nor has any maximal submodule.

We observe that the Morita-equivalent ring of a right ns-Noetherian ring is not analogous to Noetherian rings. For example, let R be a ring which is right ns-Noetherian but not right Noetherian (see Remarks 2.4(2.4) for such example). Then, the matrix ring $S = M_2(R)$ is not right ns-Noetherian.

3. Characterizations and properties

Recall from [7], a submodule N of a module M is called *co-closed* in M (denoted by $N \leq_{cc} M$) if, $N/K \leq_s M/K \implies K = N$ for every submodule K of M with $K \subseteq N$. Note that every nonzero co-closed submodule is non-small.

Proposition 3.1. *Let N be a submodule of an ns-Noetherian module M. Then*

(1) *M/N is ns-Noetherian.*
(2) *N is ns-Noetherian whenever N is co-closed.*

Proof. (1) It follows from the fact that $L/N \leq_{ns} M/N \implies L \leq_{ns} M$.

(2) Since N is co-closed, $L \leq_{ns} N \implies L \leq_{ns} M$ [7, Lemma 1.1]. Hence the result follows. □

Recall from [7], a submodule K is called a *supplement* of a submodule N of M if, K is minimal with the property $K + N = M$. Since every supplement submodule is co-closed (see [7, Lemma 1.1]), the second statement of Proposition 3.1 holds for supplement submodules also.

Proposition 3.2. *Let N be a small and Noetherian submodule of a module M. Then M is ns-Noetherian if and only if M/N is ns-Noetherian.*

Proof. Suppose that M/N is ns-Noetherian and let $\{L_i\}_{i=1}^{\infty}$ be an ascending chain of non-small submodules of M. Since $L_i \leq_{ns} M$, $L_i + N \leq_{ns} M$. Therefore, $(L_i + N)/N \leq_{ns} M/N$ because N is small. Thus $\{(L_i + N)/N\}_{i=1}^{\infty}$ is an ascending chain of non-small submodules of M/N. It follows that there exists $m \in \mathbb{N}$ such that $L_i + N = L_j + N, \forall i, j \geq m$. Also, since N is Noetherian and $\{L_i \cap N\}_{i=1}^{\infty}$ is an ascending chain of submodules of N, there exists $n \in \mathbb{N}$ such that $L_i \cap N = L_j \cap N, \forall i, j \geq n$. From above two equations, it follows that there exists $k \in \mathbb{N}$ such that $L_i = L_j, \forall i, j \geq k$. Thus, M is ns-Noetherian. The converse follows from Proposition 3.1(1). □

Recall from [7], let $B \leq A \leq M$. Then B is called an *s-closure* of A in M if, $B \leq_{cc} M$ and $A/B \leq_s M/B$. Note that s-closure of every non-small submodule (if exists) is nonzero. In the following, we provide an alternate proof for [9, Theorem 4.2].

Theorem 3.3. *Consider the following statements for a module M:*

(1) *M is ns-Noetherian.*
(2) *For every non-small submodule N of M, M/N is Noetherian.*
(3) *Every decomposable factor module of M is Noetherian.*

(4) *For every nonzero co-closed submodule C of M, M/C is Noetherian.*

Then, (1) $\iff$ (2) $\iff$ (3) $\implies$ (4). *If every non-small submodule has an s-closure, then* (4) $\implies$ (1).

Proof. (1) $\implies$ (2) Let N be a non-small submodule of M. Let $N_1/N \leq N_2/N \leq N_3/N \leq \cdots$ be an ascending chain of submodules of M/N. Since N is a non-small submodule of M, each N_i is non-small in M. So, there exists $k \in \mathbb{N}$ such that $N_i = N_k$, for all $i \geq k$. Thus, M/N is Noetherian.

(2) $\implies$ (3) Let M/N be a decomposable factor module. Then we have two submodules $N \subsetneq K_1, K_2, \subsetneq M$ such that $M/N = K_1/N \oplus K_2/N$. Clearly, K_1/N and K_2/N are non-small submodules of M/N. So, K_1 and K_2 are non-small submodules of M. Therefore, by (2), M/K_1 and M/K_2 are Noetherian. Since

$$K_1/N \cong (M/N)/(K_1/N) \cong M/K_1 \text{ and } K_2/N \cong (M/N)/(K_2/N) \cong M/K_2,$$

it follows that K_1/N and K_2/N are Noetherian. Hence, $M/N = K_1/N \oplus K_2/N$ is Noetherian.

(3) $\implies$ (1) Let $M_1 \leq M_2 \leq M_3 \leq \cdots$ be an ascending chain of non-small submodules of M. If $M_1 = M$, we are done. Suppose that $M_1 \subsetneq M$. Since $M_1 \leq_{ns} M$, there exists a nonzero proper submodule M_1' such that $M = M_1 + M_1'$. Let $K = M_1 \cap M_1'$. Then K is a proper submodule of M_1 and M_1' such that $M/K = M_1/K \oplus M_1'/K$. This implies that M/K is decomposable and $M_1/K \leq M_2/K \leq M_3/K \leq \cdots$ is an ascending chain of submodules of M/K. Hence, by (3), the given chain terminates.

(2) $\implies$ (4) It follows from the fact that every nonzero co-closed submodule of M is always non-small.

(4) $\implies$ (1) Suppose that every non-small submodule has an s-closure. Let $N_1 \leq N_2 \leq N_3 \leq \cdots$ be an ascending chain of non-small submodules of M. Since N_1 is non-small, it has an s-closure N (say). It follows that N is a nonzero co-closed submodule of M such that $N \leq N_1$. Hence, by (4), the ascending chain $N_1/N \leq N_2/N \leq N_3/N \leq \cdots$ of submodules of M/N terminates and so the ascending chain $N_1 \leq N_2 \leq N_3 \leq \cdots$ terminates. Thus, M is ns-Noetherian. $\square$

By [7, Lemma 1.7], if M is an *amply supplemented* module, then every submodule of M has an s-closure in M but its converse need not be true. Therefore, as a consequence of Theorem 3.3 we have the following:

Corollary 3.4. [9, Theorem 4.2] *The following conditions are equivalent for a module M:*

(a) *M satisfies ACC on non-small submodules;*
(b) *M/N is Noetherian for every non-small submodule N of M;*
(c) *every decomposable factor module of M is Noetherian.*
If M is amply supplemented then $(a-c)$ is also equivalent to:
(d) *M/N is Noetherian for every nonzero co-closed submodule N of M.*

Corollary 3.5. *A module M is Noetherian if and only if M is ns-Noetherian and it has a non-small Noetherian submodule.*

Proof. Suppose that M is ns-Noetherian and it has a non-small Noetherian submodule, say, N. Then M/N is Noetherian by Theorem 3.3. Hence M is Noetherian. The converse is clear. □

The analogue of Hilbert's Basis Theorem does not hold for ns-Noetherian rings. For example, let R be a commutative non-Noetherian local integral domain (such rings exist, e.g.- [6, Example 2.1]). Also, since R is local, R_R is hollow and so ns-Noetherian. Since R is an integral domain, $J(R[x]) = 0$ by [8, 5.10 Amitsur's Theorem]. If possible, suppose that $R[x]$ is ns-Noetherian. Then $R[x]$ is Noetherian as $J(R[x]) = 0$. Hence, R is Noetherian which is a contradiction. Thus, $R[x]$ is not ns-Noetherian. However, we have the following result.

Proposition 3.6. *Let R be a ring. Then the polynomial ring $R[x]$ is an ns-Noetherian right R-module if and only if $R = 0$.*

Proof. Suppose that $R \neq 0$ and let $N_0 = Rx^2 + Rx^4 + Rx^6 + \cdots$ Then N_0 is a non-small R-submodule of $R[x]$ because there exists a proper R-submodule $K_0 = R + Rx + Rx^3 + Rx^5 + \cdots$ of $R[x]$ such that $N_0 + K_0 = R[x]$. Now, if we define

$$\begin{array}{lcl} N_1 & = & N_0 + Rx \\ N_2 & = & N_0 + Rx + Rx^3 \\ N_3 & = & N_0 + Rx + Rx^3 + Rx^5 \\ \cdots & .. & \cdots\cdots \\ N_n & = & N_0 + Rx + Rx^3 + Rx^5 + \cdots + Rx^{2n-1} \end{array}$$

then we have a non-terminating ascending chain $N_0 \subsetneq N_1 \subsetneq N_3 \subsetneq \cdots$ of non-small R-submodules of $R[x]$. Hence $R[x]$ is not an ns-Noetherian R-module. The converse is clear. □

Recall from [5], let S be a multiplicative set in a ring R and M be an R-module. The submodule $t_S(M) = \{m \in M : ms = 0 \ for \ some \ s \in S\}$ is called an *S-torsion* submodule of M. The module M is called *S-torsion (respectively, S-torsion free) module* if $t_S(M) = M$ (respectively, $t_S(M) = 0$). Note that for any homomorphism $f : M \to N$, $f(t_S(M)) \subseteq t_S(N)$. Recall from [4], a module is called *generalized Hopfian (gH)* if any of its surjective endomorphisms has a small kernel. It follows from [4, Proposition 1.15] that every module with finite spanning dimension is gH. It raises a natural question: Is an ns-Noetherian module gH? By following [4, Corollary 1.18], the answer is positive. In the following, we generalize [5, Proposition 4.23] and its proof is analogous to that. As a consequence, we get [4, Corollary 1.18].

Theorem 3.7. *Let S be a multiplicative set in a ring R and M be an S-torsion free, ns-Noetherian R-module. If f is an endomorphism of M such that $M/f(M)$ is S-torsion, then $ker(f) \leq_s M$. In particular, every ns-Noetherian module is gH.*

Proof. If possible, suppose that $ker(f) \leq_{ns} M$. Then we have an ascending chain $ker(f) \leq ker(f^2) \leq \cdots$ of non-small submodules of M. Since M is ns-Noetherian, there exists $n \in \mathbb{N}$ such that $ker(f^n) = ker(f^{n+1})$. Since $M/f(M)$ is S-torsion and the map $\bar{f^i} : M/f(M) \to f^i(M)/f^{i+1}(M)$ given by $\bar{f^i}(m + f(M)) = f^i(m) + f^{i+1}(M), \forall m + f(M) \in M/f(M)$ is an epimorphism for each i, $f^i(M)/f^{i+1}(M)$ is S-torsion for each i. It follows that $M/f^n(M)$ is S-torsion.

Now let $m \in ker(f)$. Then $m + f^n(M) \in M/f^n(M)$. Since $M/f^n(M)$ is S-torsion, $\exists s \in S$ such that $(m + f^n(M))s = f^n(M)$ which implies that $ms = f^n(m')$ for some $m' \in M$. Since $m \in ker(f)$, $f^{n+1}(m') = f(ms) = f(m)s = 0$ which implies that $m' \in ker(f^{n+1}) = ker(f^n)$. Thus, $ms = f^n(m') = 0$ which implies that $m \in t_S(M) = 0$. Hence $ker(f) = 0$ which is a contradiction as we have assumed that $ker(f) \leq_{ns} M$. Therefore, $ker(f) \leq_s M$.

Let M' be an ns-Noetherian module and $f \in End(M')$ be a surjective endomorphism. If we take $S = \{1\}$, then M' is S-torsion free, ns-Noetherian module and $M'/f(M') = 0$ is S-torsion. Hence $ker(f) \leq_s M'$ by above argument. □

Corollary 3.8. [4, Corollary 1.18] *Every module satisfying ACC on non-small submodules is generalized Hopfian.*

It is well known that every Noetherian module have a maximal submodule. But, an ns-Noetherian module need not has a maximal submodule. For example, $\mathbb{Z}_{p^\infty}$ is an ns-Noetherian $\mathbb{Z}$-module while it has no maximal submodule. However, in the following we observe that every non hollow ns-Noetherian module has a maximal submodule.

An ns-Noetherian module need not have a finite uniform dimension (Krull dimension). For example, let V be an infinite dimensional vector space over a field

F. Then V is a divisible F-module and F is an ns-Noetherian commutative domain. Hence the ring $R = F(+)V$ is ns-Noetherian by Proposition 2.3. Since V is an infinite dimensional vector space, it has a submodule $W := \bigoplus_{i=1}^{\infty} W_i$, where each W_i is a nonzero submodule of V. It follows that R_R has a submodule $I := \bigoplus_{i=1}^{\infty}\{0(+)W_i\}$. Therefore, R_R does not have finite uniform dimension (Krull dimension).

Theorem 3.9. *Let M be an ns-Noetherian module. Then*

(1) *M is Noetherian or indecomposable.*
(2) *M is hollow or M has a non-small maximal submodule.*
(3) *M is Dedekind finite.*
(4) *If every infinitely generated submodule of M is co-closed, then M has finite uniform dimension.*

Proof. (1) Suppose that M is not indecomposable. Then, there exist two nonzero proper submodules S_1, S_2 of M such that $M = S_1 \oplus S_2$. Clearly S_1, S_2 are non-small submodules of M. Hence $M/S_1 \cong S_2$ and $M/S_2 \cong S_1$ are Noetherian modules by Theorem 3.3. This implies that $M = S_1 \oplus S_2$ is a Noetherian module being a finite direct sum of Noetherian modules.

(2) If M is zero, nothing to prove. Suppose that M is nonzero, not hollow and let $\{N_i\}_{i\in I}$ be the family of all proper submodules of M. We can rewrite this family as $\{N_i\}_{i\in I} = \{S_j\}_{j\in J} \cup \{L_k\}_{k\in K}$, where each S_j is a small submodule and L_k is a non-small submodule of M. Since M is not hollow, M has at least one proper non-small submodule. It follows that both families $\{S_j\}_{j\in J}$ and $\{L_k\}_{k\in K}$ are nonempty and $L_k \nsubseteq S_j, \forall j, k$. Since M is ns-Noetherian, then the family $\{L_k\}_{k\in K}$ has a maximal element that will be a maximal element of the family $\{N_i\}_{i\in I}$.

(3) If M is Noetherian, we are done. Suppose that M is not Noetherian and let $f, g \in End_R(M)$ such that $fg = I$. Since $(I - gf)^2 = I - gf$ and $(I - gf)g = 0$, we have $M = g(M) \oplus (I - gf)(M)$. By Theorem 3.9(1), $g(M) = 0$ or $(I - gf)(M) = 0$. Since g is monic, $g(M) \neq 0$. Therefore, $(I - gf)(M) = 0$ and $g(M) = M$. Hence g is an isomorphism and so $gf = I$.

(4) If possible, suppose that M does not have finite uniform dimension. Then, M has a submodule of the form $K =: \bigoplus_{i=1}^{\infty} K_i$, where each K_i is a nonzero submodule of M. Since each K_i is non-small in K, we have a strictly increasing chain $K_1 \subsetneq K_1 \oplus K_2 \subsetneq K_1 \oplus K_2 \oplus K_3 \subsetneq \cdots$ of non-small submodules of K. If $K = M$, we get a contradiction as M is ns-Noetherian. If $K \neq M$, then K is a co-closed submodule

by hypothesis and so K is ns-Noetherian by Proposition 3.1(2). Hence, we again get a contradiction. Thus, M has finite uniform dimension. □

Recall from [3], a module M is called s^3-*free* if, it contains no nonzero J-semisimple supplement.

Theorem 3.10. *Let M be a module with finite spanning dimension. Then*

(1) *M is Artinian or indecomposable and s^3-free.*

(2) *M is Dedekind finite.*

Proof. (1) Since M has finite spanning dimension, by [3, Theorem 5.6], we have $M = N \oplus P$ where N is a maximal J-semisimple supplement submodule and P is s^3-free. Suppose that M is not Artinian. Then, by the similar argument as in the proof of Theorem 3.9(1), M is indecomposable. Hence, it follows that $M = N$ or $M = P$. If possible, suppose that $M = N$. Then M is a J-semisimple module with finite spanning dimension and so M is Artinian by [3, Corollary 5.2] which is a contradiction. Thus, $M = P$ is s^3-free.

(2) It is similar to that of Theorem 3.9(3). □

The direct sum of two ns-Noetherian modules need not be ns-Noetherian. For example, $\mathbb{Z}_{p^\infty}$ is an ns-Noetherian module. But, $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ is not an ns-Noetherian module as we have an strictly ascending chain $\{\mathbb{Z}_{p^\infty} \oplus (\frac{1}{p^k}\mathbb{Z})/\mathbb{Z}\}_{k=1}^{\infty}$ of non-small submodules. As a consequence of Theorem 3.9(1) and Theorem 3.10(1), we have the following conclusions.

Theorem 3.11. (1) *Let R and S be two nonzero rings and M be an (R, S)-bimodule. Then, the formal triangular matrix ring $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is right (respectively, left) Noetherian if and only if it is right (respectively, left) ns-Noetherian.*

(2) *The direct sum of two ns-Noetherian modules is ns-Noetherian if and only if both modules are Noetherian.*

(3) *The direct sum of two modules having finite spanning dimension has finite spanning dimension if and only if both modules are Artinian.*

(4) *If M is an ns-Noetherian module or a module having finite spanning dimension, then M is a direct sum of finitely many indecomposable modules.*

4. Properties over certain rings

Let I be an ideal of a commutative ring R. Recall from [14], I is called *irreducible* if, there do not exist ideals $I_1 \supsetneq I$ and $I_2 \supsetneq I$ such that $I = I_1 \cap I_2$. I is called *primary* if, for any $a, b \in R$, $ab \in I \implies a \in I$ or $\exists n \in \mathbb{N}$ such that $b^n \in I$. I is called *decomposable* if it is finite intersection of primary ideals.

Proposition 4.1. *Let I be a non-small ideal of a commutative ns-Noetherian ring R. Then*

(1) *I contains a finite product of prime ideals.*
(2) *I is the intersection of finitely many irreducible ideals.*
(3) *I is primary whenever I is irreducible.*
(4) *I is decomposable.*

Proof. (1) Let μ be a family of all non-small ideals of R which does not contain any finite product of prime ideals of R. If possible, suppose that $\mu \neq \phi$. Then, μ has a maximal element, say, A. Clearly, A is not a prime ideal and so there exist two ideals J, K of R such that $JK \subseteq A$ but $J, K \nsubseteq A$. This implies that $A+I, A+J$ are non-small ideals of R such that $A \subsetneq A+I, A+J$. Hence, by the maximality of A, $A+I$ and $A+J$ contain product of prime ideals. But then A contains a product of prime ideals as $(A+I)(A+J) \subseteq A+IJ \subseteq A$, a contradiction. Thus $\mu = \phi$. This completes the proof.

(2) Let μ be the family of all non-small ideals of R which are not intersection of finitely many irreducible ideals. Now, rest of the proof is on the same line to (1).

(3) Let $a, b \in R$ such that $ab \in I$ and $a \notin I$. Then we show that $b^n \in I$ for some positive integer n. Consider the quotient ring $\bar{R} = R/I$. Then $\bar{a}\bar{b} = \bar{0}$, $\bar{a} \neq \bar{0}$ and there is an ascending chain $ann(\bar{b}) \subseteq ann(\bar{b}^2) \subseteq \cdots$ ideals of R/I. By Theorem 3.3, there exists $n \in \mathbb{N}$ such that $ann(\bar{b}^n) = ann(\bar{b}^{n+1})$. It follows that $< \bar{b}^n > \cap < \bar{a} >= \bar{0}$. Since I is irreducible, $\bar{0}$ is irreducible and so $< \bar{b}^n >= \bar{0}$ as $< \bar{a} >\neq \bar{0}$. Thus $b^n \in I$.

(4) It follows from (2) and (3). □

In Example 2.5(3), we show that a right ns-Noetherian ring need not satisfy ACC on non-essential right ideals. Similarly, a ring having finite right spanning dimension need not satisfy DCC on non-essential right ideals. For example, let V be an infinite dimensional vector space over a field F. Then V is a divisible F-module over a commutative domain F having finite spanning dimension. Hence the commutative ring $R = F(+)V$ has finite spanning dimension by Proposition 2.3. Since V is infinite dimensional, V is not Artinian and so R is not Artinian as it has ideals of the form $0(+)W$, where W are subspaces of V. Since V is decomposable and not Artinian, it follows from Theorem 3.10 and [12, Corollary 4.2] that R does not satisfy DCC on non-essential ideals. In the following, we provide some sufficient conditions to hold it.

Proposition 4.2. *Let R be a semiprime ring. If R is right ns-Noetherian ring (with finite right spanning dimension), then R is either right uniform or right Noetherian (Artinian). In particular, R satisfies ACC (DCC) on non-essential right ideals.*

Proof. Suppose that R is not right Noetherian. Then we show that $soc(R_R) = 0$. Let S be a simple right ideal of R. Since R is not right ns-Noetherian, by Theorem 3.3, S must be a small right ideal and so $S \subseteq J(R)$. It follows that $soc(R_R) \subseteq J(R)$ and so $(soc(R_R))^2 = 0$. Since R is semiprime, $soc(R_R) = 0$. Hence R is right uniform.

In another case, suppose that R is not right Artinian. Then R_R is s^3-free by Theorem 3.10(2). Hence, by [3, Proposition 5.5], $soc(R_R) \subseteq J(R)$ and so $(soc(R_R))^2 = 0$. Since R is semiprime, $soc(R_R) = 0$. Hence R is right uniform. □

Corollary 4.3. *Let R be a semiprime ring. If R is right ns-Noetherian or R has finite right spanning dimension, then R is a direct sum of finitely many uniform right ideals.*

Proposition 4.4. *Let R be a right ns-Noetherian ring. Then for any $c \in R$, c is nilpotent or $r(c) \leq_s R_R$ or $r(c^n) \leq_{ne} R_R$, $c^n R \leq_{ne} R_R$ and $r(c^n) \cap c^n R = 0$ for some $n \in \mathbb{N}$.*

Proof. Suppose that neither c is nilpotent nor $r(c) \leq_s R_R$. Then $r(c)$ is a non-small right ideal of R and so we have an ascending chain $r(c) \subseteq r(c^2) \subseteq r(c^3) \subseteq \cdots$ of non-small right ideals of R. Since R is right ns-Noetherian, there exists $n \in \mathbb{N}$ such that $r(c^n) = r(c^{n+1})$. It follows that $r(c^n) \cap c^n R = 0$. Since $r(c)$ is non-small and c is not a nilpotent element of R, $r(c^n) \neq 0$ and $c^n \neq 0$. Hence $r(c^n) \leq_{ne} R_R$ and $c^n R \leq_{ne} R_R$. □

Recall from [12], a ring R has *many essential right ideals* provided, for every $a \in R$, $aR \leq_e R_R$ or $r(a) \leq_e R_R$. The following result is analogous to [12, Lemma 2.11].

Corollary 4.5. *Let R be a right ns-Noetherian ring. If R has many essential right ideals, then for any $c \in R$, either c is nilpotent or $r(c) \leq_s R_R$.*

Proof. If possible, suppose that neither c is nilpotent nor $r(c) \leq_s R_R$. Then, by Proposition 4.4, $r(c^n) \leq_{ne} R_R$ and $c^n R \leq_{ne} R_R$ for some $n \in \mathbb{N}$ which is a contradiction to the fact that R has many essential right ideals. So our assumption is wrong. Thus, either c is nilpotent or $r(c) \leq_s R_R$. □

The following result is a dualization to [12, Lemma 2.2].

Proposition 4.6. *Let R be a ring which is right ns-Noetherian (with finite right spanning dimension) but not right Noetherian (Artinian). Then for any $c \in R$, either $r(c) \leq_s R_R$ or $cR \leq_s R_R$.*

Proof. Suppose that R is a ring which is right ns-Noetherian but not right Noetherian. Let $c \in R$ and suppose that $r(c)$ is a non-small right ideal. Then $R/r(c) \cong cR$ is Noetherian right R-module by Theorem 3.3. This implies that R/cR is not right Noetherian R-module by the hypothesis that R is not right Noetherian. Therefore $cR \leq_s R_R$ by Theorem 3.3. The proof for the other case is similar. □

Recall from [2, 18.26], a left self-injective ring having ACC (or DCC) on essential left or right ideals is quasi-Frobenius. However, a self-injective ns-Noetherian ring need not be quasi-Frobenius. For example, let p be a prime. If $\mathbb{Z}_{(p)}$ denotes the ring of p-adic integers which is a commutative principal ideal domain with unique irreducible element, then the Prüfer p-group $\mathbb{Z}_{p^\infty}$ is a divisible $\mathbb{Z}_{(p)}$-module and $\mathbb{Z}_{(p)}$ is a hollow $\mathbb{Z}_{(p)}$-module. Hence, the ring $R = \mathbb{Z}_{(p)}(+)\mathbb{Z}_{p^\infty}$ is an ns-Noetherian ring by Proposition 2.3. But, R is not quasi-Frobenius by [2, Example 18.18]. Here, we give a sufficient condition under which it holds. First, we discuss the following lemma.

Lemma 4.7. *Let R be a ring such that $r(Z_l(R)) \leq_{ns} R_R$. If R is right ns-Noetherian, then R satisfies ACC on right annihilator ideals and cR is a Noetherian R-module for every $c \in Z_l(R)$, where $Z_l(R)$ denotes the set of all left zero divisors in R.*

Proof. Let S be a nonempty subset of R. If S has a left regular element, i.e., there exists $s \in S$ such that $sx = 0 \implies x = 0$ for any $x \in R$, then $r(S) = 0$. If S has no left regular element, i.e., every element of S is a left zero divisor, then $S \subseteq Z_l(R)$ and so $r(Z_l(R)) \subseteq r(S)$. Thus, for any $S \subseteq R$, either $r(S) = 0$ or $r(S) \leq_{ns} R_R$ as $r(Z_l(R)) \leq_{ns} R_R$. Now, it follows that R satisfies ACC on left annihilator ideals as R is right ns-Noetherian. Next, let $c \in Z_l(R)$. Then $r(c) \leq_{ns} R_R$ as $r(Z_l(R)) \subseteq r(c)$ and $r(Z_l(R)) \leq_{ns} R_R$. Therefore, by Theorem 3.3, $cR \cong R/r(c)$ is a Noetherian R-module. □

With the help of above lemma, we conclude that results analogous to [12, Lemma 2.10, Lemma 2.11 and Theorem 2.12] are true for any ring R with the property that $r(Z_l(R)) \leq_{ns} R_R$.

Proposition 4.8. *Let R be a self-injective ring such that $r(Z_l(R)) \leq_{ns} R_R$. If R is right ns-Noetherian, then R is quasi-Frobenius.*

Proof. It follows from Lemma 4.7 and [2, Proposition 18.9]. □

Recall from [10], consider the following three conditions for a module M: (D_1) For every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 \leq_s M$. (D_2) If $N \leq M$, such that M/N is isomorphic to a direct summand of M, then N is a direct summand of M. (D_3) If N and K are direct summands of M such that $M = N + K$, then $N \cap K$ is a direct summand of M.

An R-module M is called *discrete* if it has (D_1) and (D_2); M is called *quasi-discrete* if it has (D_1) and (D_3).

Proposition 4.9. *Let M be a quasi-discrete module. If M is either ns-Noetherian or M has finite spanning dimension, then M is discrete if and only if it is Hopfian.*

Proof. Suppose that M is discrete. Since M is ns-Noetherian, every surjective endomorphism of M has small kernel by Corollary 3.8. Therefore, every surjective endomorphism of M is an isomorphism by [10, Lemma 5.1]. Thus, M is Hopfian. The converse is clear by [10, Lemma 5.1]. □

Proposition 4.10. *Let M be a module satisfying D_1-condition. Then, M is ns-Noetherian (with finite spanning dimension) if and only if it is either hollow or Noetherian (Artinian).*

Proof. Suppose M is ns-Noetherian. Then M is either Noetherian or indecomposable by Theorem 3.9(1). If M is not Noetherian, then M is indecomposable and so M is a hollow module by [10, Corollary 4.9]. The converse is clear. □

Proposition 4.11. *Every module with finite spanning dimension is discrete if and only if it is quasi-projective.*

Proof. It follows from [10, Proposition 4.39] and [3, Lemma 2.2]. □

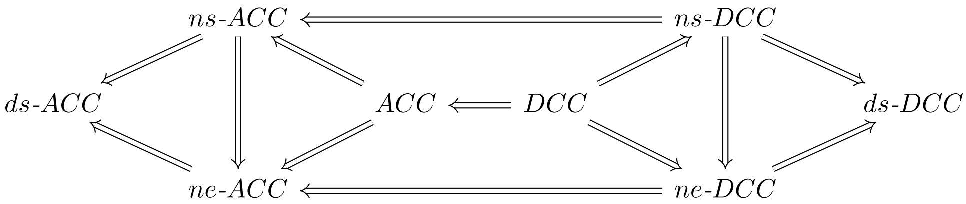
Remarks 4.12. (1) It follows by Proposition 4.10 and Proposition 4.11 that the condition of quasi-projective is superfluous in [11, Proposition 3.5] that a quasi-projective module has finite spanning dimension if and only if it is either hollow or Artinian.

(2) Every ring with right finite spanning dimension is right ns-Noetherian.

(3) In case of a ring R, R_R has finite spanning dimension if and only if it is either

hollow or Artinian. However, this fact is not true for right ns-Noetherian rings (see Remarks 2.4 for such example).

(4) Let R be a semiprime ring. Then, by Proposition 4.2, Remarks 4.12(2), [12, Theorem 2.1] and [12, Theorem 2.9], we have the following implications for the ring R:



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References

[1] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra, 1(1) (2009), 3-56.

[2] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Extending Modules, Pitman Research Notes in Mathematics Series, Vol. 313, Longman Scientific and Technical, Harlow, 1994.

[3] P. Fleury, *A note on dualizing Goldie dimension*, Canad. Math. Bull., 17(4) (1974), 511-517.

[4] A. Ghorbani and A. Haghany, *Generalized Hopfian modules*, J. Algebra, 255(2) (2002), 324-341.

[5] K. R. Goodearl and R. B. Warfield, An Introduction to Noncommutative Noetherian Rings, London Mathematical Society Student Texts, 2nd edn., Cambridge University Press, Cambridge, 2004.

[6] W. Heinzer, C. Rotthaus and S. Wiegand, *Examples of non-Noetherian domains inside power series rings*, J. Commut. Algebra, 6(1) (2014), 53-93.

[7] D. Keskin, *On lifting modules*, Comm. Algebra, 28(7) (2000), 3427-3440.

[8] T. Y. Lam, A First Course in Noncommutative Rings, Graduate Texts in Mathematics, Vol. 131, 2nd edn., Springer-Verlag, New York, 2001.

[9] C. Lomp and A. C. Ozcan, *Fleury's spanning dimension and chain conditions on non-essential elements in modular lattices*, Colloq. Math., 124(1) (2011), 133-144.

[10] S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, London Mathematical Society, Lecture Note Series, Vol. 147, Cambridge University Press, Cambridge, 1990.

[11] K. M. Rangaswamy, *Modules with finite spanning dimension*, Canad. Math. Bull., 20(2) (1977), 255-262.

[12] P. F. Smith and M. R. Vedadi, *Modules with chain conditions on non-essential submodules*, Comm. Algebra, 32(5) (2004), 1881-1894.

[13] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

[14] O. Zariski and P. Samuel, Commutative Algebra, Vol. 1, Graduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg-Berlin, 1975.

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