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ON AUTOMORPHISM-INVARIANT MULTIPLICATION MODULES OVER A NONCOMMUTATIVE RING

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Abstract. One of the important classes of modules is the class of multiplication modules over a commutative ring. This topic has been considered by many authors and numerous results have been obtained in this area. After that, Tuganbaev also considered the multiplication module over a noncommutative ring. In this paper, we continue to consider the automorphism-invariance of multiplication modules over a noncommutative ring. We prove that if R is a right duo ring and M is a multiplication, finitely generated right R-module with a generating set $\{m_1, \dots, m_n\}$ such that $r(m_i) = 0$ and $[m_iR : M] \subseteq C(R)$ the center of R, then M is projective. Moreover, if R is a right duo, left quasi-duo, CMI ring and M is a multiplication, non-singular, automorphism-invariant, finitely generated right R-module with a generating set $\{m_1, \dots, m_n\}$ such that $r(m_i) = 0$ and $[m_iR : M] \subseteq C(R)$ the center of R, then $M_R \cong R$ is injective.

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1. Introduction

Throughout this paper, all rings are associative rings with unit and all modules are right unital modules. We use $N \leq M$ ($N \lneq M$) to mean that N is a submodule (respectively, a proper submodule) of M. $E(M), C(R), J(R)$ denote the injective envelope of M, the center of the ring R and the Jacobson radical of R, respectively. A submodule N of a module M is said to be *essential* if $N \cap X \neq 0$ for every nonzero submodule X of M, denoted by $N \leq^e M$. In this case, M is called an *essential extension* of N.

A ring R is called *right duo* if every right ideal is an ideal. A right R-module M is called *multiplication* if for every submodule N of M, there exists an ideal B of R such that $N = MB$. So it is easy to see that R is right duo if and only if R_R is multiplication. Indeed, if R is right duo, then for every right ideal I of R,

I is an ideal of R, so we can write $I = RI$, i.e., R_R is multiplication. Conversely, if R_R is multiplication and I is a right ideal of R, then there exists an ideal J of R such that $I = RJ = J$. So I is an ideal of R, i.e., R is right duo. A ring R is called *right multiplication* if R_R is multiplication. Note that over a right duo (or a right multiplication) ring, every cyclic right R-module is multiplication. In [19], Tuganbaev gave the definition of concept "commutative multiplication of ideals" (briefly CMI) and obtained many results on multiplication modules over a right duo ring or a ring with CMI. A ring R is called *commutative multiplication of ideals* if $AB = BA$ for any ideals A, B of R. Two above conditions are followed from commutativity of a ring but the converses are not true, in general. So it makes sense if we consider a multiplication module over a right duo rings (or a ring with CMI).

For a subset X of a right R-module M over a ring R, we denote that $r_R(X)$ or $r(X)$ the right annihilator of X in R. A right R-module M is said to be *faithful* if $r(M) = 0$. Now let X and Y be two subsets of a right R-module M, the subset $\{r \in R | Xr \subseteq Y\}$ of R is denoted by $[Y : X]$. Recall that if $Y \leq M_R$, then $[Y : X] \leq R_R$ and if $X \leq M_R$ and $Y \leq M_R$, then $[Y : X]$ is an ideal of R. A submodule N of the module M is said to be *closed* in M if N' is an essential extension of N in M, then $N = N'$. A module M is called *square-free* if M does not have nonzero submodules of the form $X \oplus Y$ with $X \cong Y$. Recall that $Z(M) = \{m \in M | \ r(m) \leq^e R_R\}$ is called the *singular submodule* of M, and if $Z(M) = M$ (resp. $Z(M) = 0$), then M is called *singular* (resp. non-singular). A ring R is said to be right *non-singular* if R_R is non-singular. A ring is said to be *reduced* if each of its nilpotent elements is equal to zero. Left-sided for these notations are defined similarly.

For a module N, a module M is said to be *injective with respect to* N or N-*injective* if for any submodule $X \leq N$, every homomorphism $X \to M$ can be extended to a homomorphism $N \to M$. A module is said to be *injective* if it is injective with respect to each module. A module is said to be *quasi-injective* if it is injective with respect to itself. It is well known that a module M is quasi-injective if and only if $f(M) \leq M$ for any endomorphism f of the injective envelope of the module M (see [7]). A module M is said to be *automorphism-invariant* if $f(M) \leq M$ for any automorphism f of the injective envelope of M. Automorphism-invariant modules are studied in [4], [6], [17], [21], and [22].

Multiplication modules over a commutative ring were considered by many authors, for examples, see [3], [10], [12], and [13]. However, when we consider this kind of modules over a noncommutative ring, we will meet many difficulties. Although many difficulties arise, many results about the multiplication modules over a noncommutative ring were obtained, for example see [18], [19], and [20]. In [10], S. Singh and Y. Al-Shaniafi obtained many results on quasi-injective multiplication modules over a commutative ring. In this paper we continue to consider the quasi-injectivity of multiplication modules over a noncommutative ring. From this we obtain the result on automorphism-invariant multiplication modules over a noncommutative ring.

All terms such as "duo" and 'non-singular" when applied to a ring will apply all both sided. For any terms not defined here the reader is referred to [1], [2], [8], [9], [15] and [23].

2. Results

The following properties are interesting when considering a multiplication module over a noncommutative ring.

Proposition 2.1. *The following statements are equivalent for a right R-module M:*

(1) *M is a multiplication module.*
(2) *$N \leq M.[N : M]$ for every $N \leq M_R$.*
(3) *$N = M.[N : M] = Mr(M/N)$ for every $N \leq M_R$.*

Proof. See [19, Note 1.3]. □

Proposition 2.2. *Let R be a right duo ring with commutative multiplication of ideals. Then the following conditions are equivalent for a right R-module M:*

(1) *M is a multiplication module.*
(2) *For every nonempty collection of right ideals $\{B_i\}_{i\in I}$ of R, we have*

$$\bigcap_{i\in I}(MB_i) = M[\bigcap_{i\in I}(B_i + r(M))],$$

and for any submodule N of M and each right ideal C of R with $N \subsetneq MC$, there exists an ideal B of R such that $B \subsetneq C$ and $N \subsetneq MB$.

Proof. See [19, Theorem 4.3]. □

Proposition 2.3. *Let M be a non-singular automorphism-invariant right R-module. Then there exists a direct decomposition $M = X \oplus Y$ such that X is a quasi-injective non-singular module, Y is a square-free non-singular automorphism-invariant module, the modules X and Y are injective with respect to each other, any sum of closed *submodules of the module Y is an automorphism-invariant module, $Hom(X,Y) = Hom(Y,X) = 0$, and $Hom(Y_1, Y_2) = 0$ for any two submodules Y_1 and Y_2 in Y with $Y_1 \cap Y_2 = 0$.*

Proof. See [11, Theorem 3.6]. □

Next, we study the non-singularity of rings and faithful multiplication modules over a right duo ring.

Proposition 2.4. *Let R be a right duo (or right multiplication) ring. Then the following conditions hold.*

(1) *R is right non-singular if and only if R is reduced.*
(2) *Let M_R be a faithful multiplication module. Then M is non-singular if and only if R is right non-singular.*
(3) *Let M_R be a non-singular faithful multiplication module and N be a closed submodule of M. Then $[N : M]$ is a closed ideal of R.*

Proof. (1) It's clear that "reduced" $\Rightarrow$ 'right non-singular".

To prove the converse, let a be an element of R such that $a^2 = 0$. Then for aR there exists a right ideal B of R such that $B \cap aR = 0$ and $B \oplus aR$ is an essential right ideal of R. Since B is an ideal of R, $aB \leq B \cap A = 0$, and so $aB = 0$. From this, $a(B + aR) = 0$ and $B \oplus aR \leq^e R_R$, it follows that $a = 0$. Hence R is reduced.

(2) Let R be a right non-singular ring. Take $x \in Z(M)$. Then there exists an essential right ideal I of R such that $xI = 0$. Since M is multiplication, there exists an ideal A of R such that $xR = MA$. Then $0 = xI = xRI = MAI = 0$. It follows that $AI = 0$. We have that I is essential in R and obtain $A = 0$, and so $x = 0$ or $Z(M) = 0$.

To prove the converse, if $Z(M) = 0$, then by [19, Proposition 3.13], $Z(M) = MZ_r(R)$, it follows that $MZ_r(R) = 0$. But M is faithful, and so $Z_r(R) = 0$.

(3) By (2), R is right non-singular. Since $N \leq M_R$, $[N : M]$ is an ideal of R. We have $R/[N : M]$ is a cyclic right R-module, so it is multiplication. Note that since N is a closed submodule of a non-singular module M, by [15, Corollary 4.2], M/N is non-singular. Now, let $r + [N : M] \in Z(R/[N : M])$. Then there exists an essential right ideal I of R such that $(r + [N : M])I = 0$, so $rI \leq [N : M]$ and hence $MrI \leq N$. It follows that $Mr + N \leq Z(M/N) = 0$, so $Mr \leq N$ or $r \in [N : M]$. Thus $Z(R/[N : M]) = 0$ or $R/[N : M]$ is a non-singular right R-module.

Assume that $[N : M] \leq^e I$ for some ideal I of R. Then, $I/[N : M]$ is singular. It follows that

$$0 \neq Z(I/[N : M]) \leq Z(R/[N : M]) = 0.$$

It contradicts, and so $[N : M]$ is closed in R. $\square$

Corollary 2.5. *If R a right duo (or right multiplication) right non-singular ring, then $r_R(x) = l_R(x)$ for all $x \in R$.*

Proof. Let x be an element of R. If $y \in r_R(x)$, then $xy = 0$ and so $(yx)^2 = 0$. By Proposition 2.4(2), it immediately infers that $yx = 0$. This means that $y \in l_R(x)$. It is shown that $r_R(x) \subseteq l_R(x)$. It is similar to prove that $l_R(x) \subseteq r_R(x)$. $\square$

Proposition 2.6. *Let R be a right duo, CMI ring and M be a faithful, multiplication right R-module. Then for any closed ideal A of R and $N = MA$, N is a closed submodule of M and $A = [N : M]$.*

Proof. Let K be a closed closure of N in M. Then by Proposition 2.1, $K = MB$, where $B = [K : M]$. It implies that $A \leq B$. We show that A is essential in B. In fact, take b an arbitrary nonzero element in B. Then, we have $Mb \leq K$. Since M is faithful, $Mb \neq 0$ and so $MbR \leq K$ and $MbR \neq 0$. We have that K is an essentially extension of N and obtain that $MbR \cap N \neq 0$ and $MbR \cap MA \neq 0$. By Proposition 2.2, $M(bR \cap A) = MbR \cap MA \neq 0$. It follows $bR \cap A \neq 0$. Thus, A is essential in B. Since A is closed in R_R, $A = B$ and so $N = K$.

By the same above proof, we show that $A = [N : M]$. One can check that $A \leq [N : M]$. Let y be an arbitrary nonzero element in $[N : M]$. Then, $My \leq N = MA$ and so $MyR \leq MA$. We have, from Proposition 2.2, that

$$M(yR \cap A) = MyR \cap MA = MyR \neq 0.$$

It follows $yR \cap A \neq 0$. It is shown that A is essential in $[N : M]$. Since A is closed in R_R, $A = [N : M]$. $\square$

Corollary 2.7. *Let R be a ring with commutative multiplication of right ideals. If M is a faithful, multiplication right R-module, then for any closed ideal A of R and $N = MA$, N is a closed submodule of M and $A = [N : M]$.*

Proposition 2.8. *Let R be a right duo ring and M be a faithful, non-singular, multiplication right R-module. Then $E(R) \cong E(M)$.*

Proof. By Proposition 2.4(1), R is right non-singular. Then, there exists an embedding of M into $E(R)$. By Zorn's Lemma, there exists a maximal embedding of $K \leq M_R$ into $E(R)$, that is $t : K \longrightarrow E(R)$. It is easy to see that K is a closed submodule of M. Let N be a complement of K in M, then $N \cap K = 0$. Let $A = [K : M]$ and $B = [N : M]$. Then by Proposition 2.4(3), A and B are closed ideals of R. Now if $r \in A \cap B$, then $Mr \subseteq K, Mr \subseteq N$, and so

$Mr \subseteq K \cap N = 0$ and since M is faithful, $r = 0$. It means that $A \cap B = 0$. This gives $AB \subseteq A \cap B = 0 \Rightarrow AB = 0, BA = 0, K = MA, N = MB$. Hence $A \leq r(B)$. Now if $Br = 0$, then $MBr = 0 = Nr$. It follows that $r \in r(N)$ and $r(B) \leq r(N)$. Now let $r \in r(N)$, then $Nr = 0$. From this $Mr = (K + N)r \subseteq Kr \subseteq K$. So $r(N) \leq A$. Thus $A = r(B) = r(N)$. Similarly, $B = r(A) = r(K)$.

Assume that $N \neq 0$. Then $A = r(N) \neq R$. So it is easy to see that R/A is a right non-singular ring and N is a non-singular, faithful right R/A-module. Now we consider any $y \neq 0, y \in N$. Then by Zorn's Lemma, there exists a nonzero ideal of C such that $C \cap A = 0$. Let $\theta : C \to yC$ defined by $c \longmapsto yc$. Then if $y(c - c') = 0$ and $c \neq c'$ in C, then $c - c' \in r(y) \geq N = A$. It follows that $yA = 0$, and so $y \in Z(N) = 0$, a contradiction. Thus, we have $C \cong yC$.

We can consider an embedding $\mu : yC \longrightarrow E(R)$. Now if $x \in t(K) \cap \mu(yC) \leq E(R)$, then $x(A + B) \leq K \cap N = 0$. Therefore, $x(A + B) = 0$ or $x \in Z(E(R) = 0$. So $x = 0$. It follows that $t(K) \cap \mu(yC) = 0$. So we obtain a large embedding, a contradiction. Thus, $N = 0$ and then $K = M$.

Assume that $E(M) \cong E(t(M)) \neq E(R)$. Then, there exists a nonzero right ideal C of R (and hence ideal) such that $t(M) \cap C = 0$. Take $L = t(M) \cap R$. Note that $LR = L$, then L is a right ideal and hence an ideal of R. So $CL \leq C$ and $CL = LC \leq t(M)C \leq t(M)$, hence $CL = 0$. Since R is right non-singular and $L \leq^e t(M) \leq R \leq^c E(R)$, $C \leq Z_r(R) = 0$, a contradiction. Hence $E(M) \cong E(t(M)) = E(R)$. □

Proposition 2.9. *Let R be a right duo ring with commutative multiplication of ideals and M be a faithful, multiplication right R-module. Then the following conditions hold.*

(1) *There exists a smallest ideal $\tau(M)$ of R such that $M = M\tau(M)$. Moreover, $\tau(M) = R$ if and only if M is finitely generated.*
(2) *Let $\tau(M)$ be in (1) and $M = N \oplus K$ for some submodules N and K of M, $A = [N : M], B = [K : M]$.*

Then

$$A \cap \tau(M) = A\tau(M) = (A\tau(M))^2, \tau(M) = \tau(M)A \oplus \tau(M)B.$$

Moreover

$$r(\tau(M)A) \cap \tau(M) = \tau(M)B, N = N\tau(M)A$$

and

$$r(\tau(M)B) \cap \tau(M) = \tau(M)A, K = K\tau(M)B.$$

Proof. (1) We have $M = MB$ for some ideal B of R. Let τ be the set of all ideals B of R such that $M = MB$. Now we take $\tau(M)$ the intersection of all ideals in τ. By [19, Theorem 4.3],

$$M[\bigcap_{B\in\tau} B] = M(\tau(M)) = \bigcap_{B\in\tau}(MB) = M.$$

We deduce that $\tau(M)$ is the smallest ideal such that $M = M\tau(M)$.

Now if M is finitely generated, faithful, then by [19, Theorem 3.11], $M \neq MB$ for every proper ideal B of R. So $\tau(M) = R$. Conversely, if $\tau(M) = R$, then R is the smallest ideal B of R such that $M = MB$. Then $M \neq MB$ for every proper ideal B of R. Also by [19, Theorem 3.11], M is finitely generated.

(2) From (1), it infers that $M = M\tau(M)$ and $\tau(M) = \tau(M)^2$. Assume that $M = N \oplus K$ for some submodules N and K of M, $A = [N : M], B = [K : M]$. Then, $N = MA$ and $K = MB$. We have $A \cap B = 0$ and obtain $M = N \oplus K = MA \oplus MB = M\tau(M)A \oplus M\tau(M)B$. It follows that $M = M(\tau(M)A \oplus \tau(M)B)$ and $\tau(M)A \oplus \tau(M)B \leq \tau(M)$. Since $\tau(M)$ is the smallest ideal of R such that $M = M.\tau(M)$, $\tau(M) = \tau(M)A \oplus \tau(M)B$. From this, it immediately infers that $A \cap \tau(M) = \tau(M)A$ and $B \cap \tau(M) = \tau(M)B$. We have $AB = BA \leq A \cap B = 0$ and so

$$(\tau(M)A)^2 \leq \tau(M)A = [\tau(M)A \oplus \tau(M)B]A = \tau(M)A^2 = \tau(M)^2A^2 = (\tau(M)A)^2.$$

It follows that $A \cap \tau(M) = A\tau(M) = (A\tau(M))^2$. Next, we show that $r(\tau(M)A) \cap \tau(M) = \tau(M)B$. In fact, let $x \in \tau(M)B = B \cap \tau(M)$. Then, $(\tau(M)A)x \subseteq (\tau(M)A) \cap (\tau(M)B) \subseteq A \cap B = 0$ and so $x \in r(\tau(M)A) \cap \tau(M)$. Thus, $\tau(M)B$ is contained in $r(\tau(M)A)$. To prove the converse, take $x \in r(\tau(M)A) \cap \tau(M)$, and so $Mx = M\tau(M)x = M(\tau(M)A \oplus \tau(M)B)x = M\tau(M)Bx \subseteq K$, since $\tau(M)Ax = 0$. It follows that $x \in B \cap \tau(M) = \tau(M)B$.

Moreover, we have

$$\begin{aligned} N = MA &= (M\tau(M)A \oplus M\tau(M)B)A \\ &= M\tau(M)A^2 = MA\tau(M)A \\ &= N\tau(M)A. \end{aligned}$$

Similarly, we have $r(\tau(M)B) \cap \tau(M) = \tau(M)A, K = K\tau(M)B$. $\square$

Theorem 2.10. *Let R be a right duo ring and M be a multiplication, finitely generated right R-module with a generating set $\{m_1, \ldots, m_n\}$ such that $r(m_i) = 0$ and $[m_iR : M] \subseteq C(R)$ for every $i = 1, 2, \ldots, n$. Then, M is projective.*

Proof. By [19, Theorem 3.11],

$$R = \sum_{i=1}^{n} [m_i R : M].$$

Thus, there exist elements $r_i \in C(R)$ $(1 \le i \le n)$ such that $Mr_i \subseteq m_i R$ and

$$1 = r_1 + r_2 + \cdots + r_n.$$

We show that

$$R = \sum_{i=1}^{n} r_i^2 R.$$

In fact, assume that $B = \sum\limits_{i=1}^{n} r_i^2 R$ and $B \neq R$. Then, there exists a maximal ideal P of A containing B. So for every element $r_i^2 \in P$, $r_i \in P$, since R/P is a division ring. Thus

$$P \ni \sum_{i=1}^{n} r_i = 1 \notin P.$$

It contradicts. We deduce that $R = B$.

From this, there exist $s_i \in R$ $(1 \le i \le n)$ such that

$$1 = \sum_{i=1}^{n} r_i^2 s_i.$$

Now for each $1 \le i \le n$, we define $\theta_i : M \longrightarrow R$ as follows: for each $m \in R$, $\theta_i(m) = r_m.r_i.s_i$ where $r_m \in R$ is any element such that satisfying the condition $mr_i = m_i r_m$.

Assume that $m_i r_m = m_i r'_m$ with $r_m, r'_m \in R$. From this $r_i(r_m - r'_m) = 0$. Then $r_m = r'_m$. Hence θ_i is well defined.

Now we show that θ_i is a homomorphism. Indeed, for all $m, m' \in M, \theta_i(m+m') = r_{m+m'} r_i s_i$ such that $(m + m')r_i = m_i r_{m+m'}$ and $\theta_i(m) = r_m r_i s_i, \theta_i(m') = r_{m'} r_i s_i$, such that $mr_i = m_i r_m, m' r_i = m_i r_{m'}$. Then, $m_i(r_{m+m'} - r_m - r_{m'}) = 0$. It follows that $r_{m+m'} = r_m + r_{m'}$. Moreover, for all $a \in R, \theta_i(ma) = r_{ma} r_i s_i$ such that $mar_i = m_i r_{ma}$. Since $mar_i = mr_i a$, $m_i(r_{ma} - r_m a) = 0$. Hence $r_{ma} = r_m a$. Now, $Mr_1 s_i \subseteq m_1 R s_i \subseteq m_1 R$, and so $r_1 s_i \in C(R)$. Similarly $r_n s_i \in C(R)$. From this, $s_i \in C(R)$. One can check that $\theta_i(ma) = \theta_i(m)a$ for all $a \in R$.

It is shown that θ_i is an R-homomorphism for each $1 \le i \le n$. Now, for each $m \in M$, we can write

$$\begin{aligned} m = m.1 \quad &= m(r_1^2 s_1) + \cdots + m s_n^2 s_n \\ &= m r_1 r_1 s_1 + \cdots + m r_n r_n s_n \\ &= m_1 r_{1m} r_1 s_1 + \cdots + m_n r_{nm} r_n s_n \\ &= m_1 \theta_1(m) + \cdots + m_n \theta_n(m). \end{aligned}$$

By the Dual Basis Lemma, it infers that M is projective. □

From this result, we can obtain the following general case.

Corollary 2.11. *Let R be a right duo ring and M be a multiplication, finitely generated right R-module with a generating set $\{m_1, \ldots, m_n\}$ such that $r(m_i) = eR$ for every $i = 1, 2, \ldots, n$ and for some central idempotent $e \in R$ with $[m_i(1-e)R : M_{(1-e)R}] \subseteq C((1-e)R)$. Then, M is projective.*

Proof. Note that $r(M) \leq r(m_i) = eR$, and so M is a finitely generated multiplication right $(R/eR \cong)$ $(1-e)R$-module such that $r(m_i) = 0_{R/eR}$ and $[m_i(1-e)R : M_{(1-e)R}] \subseteq C((1-e)R)]$ for every $i = 1, 2, \ldots, n$. By Theorem 2.10, M is a projective $(1-e)R$-module. Since a right R-module and homomorphism are also a right $(1-e)R$-module and homomorphism respectively, M is a projective right R-module. □

P. Smith ([12, Theorem 11]) proved the following result for a multiplication module over a commutative ring and A. A. Tuganbaev reproved it in [19, Theorem 7.6].

Corollary 2.12. *Let R be a commutative ring with identity and M be a multiplication, finitely generated R-module such that $r(M) = eR$ for some idempotent e in R. Then, M is projective.*

Proof. See [12, Theorem 11] and [19, Theorem 7.6]. □

Let R be a right duo ring and P be a maximal ideal of R. Then it is easy to prove that $R \setminus P$ is multiplicatively closed and satisfies the following condition

$$(S1) : \forall s \in R \setminus P \text{ and } r \in R, \text{ there exist } t \in R \setminus P \text{ and } u \in R \text{ such that } su = rt.$$

Moreover, if R satisfies ACC on right annihilators, then by [15, Proposition 1.5], $R \setminus P$ is a right denominator set. In this case, the ring $R(R \setminus P)^{-1}$ is called the *right localization with respect to* P and we write R_P and M_P instead of $R(R \setminus P)^{-1}$ and $M(R \setminus P)^{-1} = M \otimes_R R_P$, respectively. A ring R is called *right localizable* if for each maximal right ideal P of R, the right localization R_P exists. A ring R is said to be left *quasi-duo* if each of its maximal left ideals is an ideal of R. Now we give another condition for $R \setminus P$ to be a right denominator set.

Lemma 2.13. *Let R be a right duo right non-singular ring and P be a maximal ideal of R. Then, $R \setminus P$ is a right denominator set, i.e., the right localization R_P exists.*

Proof. We show that $R\backslash P$ satisfies the condition (S2): If $x \in R\backslash P$ and $a \in R$ with $xa = 0$, then there exists $y \in R \setminus P$ such that $ay = 0$. Indeed, we take $y = x$. □

Corollary 2.14. [19, Theorem 4.18] *Let R be a right duo ring with commutative multiplication of ideals. Then, for every maximal right ideal P of R, the right localization R_P exists and R_P is a right duo ring with commutative multiplication of ideals.*

Proof. We show that R satisfies the condition: $l(x) = r(x)$ for all $x \in R$. Indeed, we take $a \in R, a \in l(x)$. Then $ax = 0$ and $RaxR = 0$. Since R is a right duo ring, $RaRxR = 0$ and $RaRRxR = 0$. We have that R has the commutative multiplication of ideals and obtain $RxRRaR = 0$, and so $xa = 0$ or $a \in r(x)$.

Conversely, let $b \in r(x)$. Then $xbR = 0$ and so $0 = xRbR = RxRRbR$, since R is a right duo ring. And hence $RbRRxR = 0$. It follows that $bx = 0$ or $b \in l(x)$. □

Recall that a ring R is called *right QF*-3^+ (see [16]) if the injective envelope $E = E(R)$ of R is a projective right R-module.

Proposition 2.15. *Let R be a right duo right non-singular ring. If R is a right QF-3^+, then E_P is a free right R_P-module.*

Proof. Let P be a maximal ideal of R and $\theta : E \to E_P$ be the canonical map. By Lemma 2.13, the right localization R_P exists. We have that E is projective and obtain $E \oplus A = R^{(X)}$ with some A_R and index set X. It is well-known $E_P = E \otimes_R R_P$, and so

$$\begin{aligned}(E \oplus A) \otimes_R R_P &= (E \otimes_R R_P) \oplus (A \otimes_R R_P)\\ &= R^{(X)} \otimes_R R_P \cong R_P^{(X)}\end{aligned}$$

Hence E_P is a projective right R_P-module.

Let $F = \{x \in E|\ [EP : x] \nsubseteq P\}$. With assumption $\theta(1) \in E_P P$ and by [19, Lemma 3.17], it infers that $[EP : 1] \nsubseteq P$. It means that $1 \in F$. Similarly, by [19, Lemma 3.17], $\theta(x) \in E_P P$ if and only if $[EP : x] \nsubseteq P$. It follows that $F = \{x \in E|\theta(x) \in E_P P\}$. Because θ is an R-homomorphism, we can prove easily that F is a submodule of E.

Now we will prove that F is quasi-injective. This is equivalent to F being invariant under all endomorphisms of injective envelope $E(F)$. Since $E(F)$ is a direct summand of E, we show that F is invariant under all endomorphisms of E. Let $\psi : E \longrightarrow E$ be an endomorphism of E. There exists an R_P-homomorphism

$\sigma : E_P \longrightarrow E$ such that $\sigma\theta = \psi$, i.e., the following diagram is commutative:

$$\begin{tikzcd} E \arrow[r, "\theta"] \arrow[dr, dotted, "\psi"'] & E_P \arrow[d, "\sigma"] \\ & E \end{tikzcd}$$

Now, let t be an element in F. Then $t \in E$ and there exists $r \notin P$ such that $tr \in EP$. Moreover, $\theta(t) \in E_P P$. Hence there exist $p \in P, e_t \in E_p$ such that $\theta(t) = e_t p$. So $\psi(t) = (\sigma\theta)(tr) = \sigma(\theta(t))r = (\sigma\theta)(e_t p)r = (\sigma\theta)(e_t)pr \in EP$. It follows that $\psi(t) \in L$.

Since F is invariant under any homomorphism of E, F is quasi-injective. Now since $1 \in F$, there exists $r \in EP$ such that $r \notin P$. Let $e \in E$. We have $r \in (EP) \cap R$ and obtain $er \in E[(EP) \cap R] \leq EP$, and so $e \in F$. It follows that $E = F$.

Note that $E_P \neq E_P P$. So there exists $e \in E$ such that $\theta(e) \notin E_P P$. We have $E = L$, $e \in L$ and obtain that $[EP : e] \nsubseteq P$. Then there is $v \notin P$ with $ev \in EP$. Hence $\theta(e) \in EP$, a contradiction. It follows that $\theta(1) \notin E_P P$. Since R_P is a local ring and E_P is a nonzero projective R_P-module, so it is free and then

$$E_P = \bigoplus_{i \in I} A_i, \quad A_i \cong R_P. \qquad \square$$

S. Singh and Y. Al-Shaniafi (see [10, Theorem 1.10]) proved that if R is a commutative, QF-3^+ ring with identity, then R is self-injective. We will extend this result to the noncommutative case as follows.

Lemma 2.16. *Let R be a right duo right non-singular, right QF-3^+, left quasi-duo ring. Then, R is right self-injective.*

Proof. Now we show that E/R is a flat right R-module. By [15, Exercise 39, p. 48] we need to show that for every maximal left ideal P of R, $EP \neq E$. Note that P is an ideal and since $\theta(1) \notin E_P P$, $R \cap EP \leq P$. Assume that $EP = E$. Then $x \in R \Rightarrow x \in E \Rightarrow x \in EP \Rightarrow x \in P$. So $R = P$, a contradiction. Since E is projective and by [9, Lemma 7.30], E is also finitely generated, so for some $n \in \mathbb{N}$, we obtain that $R^n \to E/R \to 0$ is exact. From [15, Corollary 11.4, p.38], it infers that E/R is projective. We deduce that $E = R$, and so R is right self-injective. $\square$

From Lemma 2.16 and [24, Theorem 2.7], we have the following result.

Theorem 2.17. *Let R be a right non-singular, right QF-3^+ ring. Then R is right self-injective if and only if R is right automorphism-invariant.*

Proof. Assume that R is a right non-singular, right QF-3^+, right automorphism-invariant ring. Then, R has a ring decomposition $R = S \oplus T$, where S is a right self-injective and T_T is square-free by [14, Theorem 4.12]. It follows, from the [5, Theorem 15], that T is a right and left quasi-duo ring. Note that T is also a right non-singular, right QF-3^+, right automorphism-invariant ring. Thus, T is a von Neumann regular ring by Proposition 1 in [4]. Applying Theorem 2.7 in [24] we have that T is a right and left duo ring. From Lemma 2.16, we deduce that T is a right self-injective ring. Thus, R is a right self-injective ring. $\square$

Corollary 2.18. *The following conditions are equivalent for a ring R:*

(1) *R is a right automorphism-invariant right non-singular, right QF-3^+ ring.*
(2) *R is a right automorphism-invariant regular, right QF-3^+ ring.*
(3) *R is a right self-injective regular ring.*

Lemma 2.19. *Every idempotent element of a right duo ring is central.*

Proof. Let e be an idempotent element of a right duo ring R. We have that $1 - e$ is in $r(e)$ and obtain that $R(1 - e) \subseteq r(e)$, since R is a right duo ring. It follows that $eR(1 - e) = 0$. It is similar to see that $(1 - e)Re = 0$. Thus, e is central. $\square$

S. Singh and Y. Al-Shaniafi (see [10, Theorem 1.11]) proved that if R is a commutative ring with identity and M is a finitely generated, faithful, quasi-injective multiplication right R-module, then $M \cong R$ (and M is injective). We will extend this result to the noncommutative case as follows.

Theorem 2.20. *Let R be a right duo, left quasi-duo, CMI ring and M be a multiplication, quasi-injective, finitely generated right R-module with a generating set $\{m_1, \ldots, m_n\}$ such that $r(m_i) = 0$ and $[m_iR : M] \subseteq C(R)$. Then $M_R \cong R$ is injective.*

Proof. For some $n \geq 1, R$ is embedded in M^n. We have that M is quasi-injective and obtain that M^n is injective and so $M^n = E(R_R) \oplus L$ for some injective right R-module L. By Theorem 2.10, M is projective and then so is M^n. Then $E(R_R)$ is projective. From Lemma 2.16, we infer that $R = E(R)$. Since L is injective, by [8, Theorem 1.21] we can apply the exchange property to the injective module L, so we obtain that

$$L \oplus R = L \oplus \bigoplus_{i=1}^{n} B_i,$$

where each B_i is a direct summand of M. And then $R \cong \overset{n}{\underset{i=1}{\oplus}} B_i$.

So there exists a direct summand of R is embedded in M. By Zorn's Lemma, there exists a maximal embedding $\alpha : A \to M$, where $A = eR$ is a direct summand of R_R for some idempotent e of R. We obtain

$$M = \alpha(A) \oplus N,$$

for some submodule N of M. Suppose that $e \neq 1$. Now if $m(1-e) \in \alpha(eR)$, then $m(1-e) = \alpha(er)$ for some $r \in R$. Then, we have

$$m(1-e)(1-e) = \alpha(er)(1-e) = \alpha(er(1-e) = \alpha(0) = 0.$$

So $M(1-e) \cap \alpha(A) = 0$.

Now take any $m \in M$. Then, $m = \alpha(er)+n$ for some $r \in R, n \in N$. From this we have $m(1-e) = \alpha(er)(1-e)+n(1-e)$ and by Lemma 2.19, $m(1-e) = n(1-e)$. We write $m = \alpha(er)+n(1-e)+ne \Rightarrow m-m(1-e) = \alpha(er')$ for some $r' \in R$. And then $\alpha(er') - \alpha(er) = ne$. It follows that $ne = 0$. Hence $m = \alpha(er) + m(1-e)$ and then $M = \alpha(A) \oplus M(1-e)$ and $M(1-e)$ is finitely generated by $\{m_i(1-e) | i = 1, \ldots, n\}$ since

$$M(1-e) = M(1-e)^2R = \sum_{i=1}^{n} m_iR(1-e)R = \sum_{i=1}^{n} m_i(1-e)R = \sum_{i=1}^{n} m_i(1-e)(1-e)R.$$

Moreover, $M(1-e)$ is a quasi-injective, multiplication module over the ring $(1-e)R$. We also have

$$r_{(1-e)R}(m_i(1-e)) = \{(1-e)r | m_i(1-e)(1-e)r = 0\} = 0,$$

since $r(m_i) = 0$. Let $(1-e)r \in [m_i(1-e)(1-e)R : M(1-e)]$ for some $r \in R$. Then, $m(1-e)r \in m_i(1-e)R \leq m_iR$ for every $m \in M$. It means that $(1-e)r \in [m_i : M] \subseteq C(R)$, and so $(1-e)r \in C(R)$. Of course, $(1-e)r \in C((1-e)R)$. It follows that $[m_i(1-e)(1-e)R : M(1-e)] \subseteq C((1-e)R)$. So a nonzero direct summand of $(1-e)R$ embeds in $M(1-e)$. This contradicts the maximality of α.

Hence $e = 1$. We deduce that $M = K \oplus N$, where $R \overset{\varphi}{\cong} K$. From Proposition 2.9(2), it infers that $R = A \oplus B, N = NA, K = KB$, where $A = [N : M], B = [K : M]$. Therefore, $K = KB = \varphi(R)B = \varphi(RB) = \varphi(BR) = \varphi(B)$. Inasmuch as $\varphi(R) = \varphi(A) \oplus \varphi(B)$ we have $K = \varphi(R) = \varphi(A) \oplus K$. It follows $\varphi(A) = 0$ so that $A = 0$. From this, we have $N = 0$. It is shown that $M = K$ and so $M \cong R$. □

Now we will give a condition for an automorphism-invariant module to be injective. In this case it is isomorphic to the ring R.

Theorem 2.21. *Let R be a right duo, left quasi-duo, CMI ring and M be a multiplication, non-singular automorphism-invariant, finitely generated right R-module with a generating set $\{m_1, \ldots, m_n\}$ such that $r(m_i) = 0$ and $[m_iR : M] \subseteq C(R)$. Then $M_R \cong R$ is injective.*

Proof. By Proposition 2.3, there exists a direct decomposition $M = X \oplus Y$ such that X is a quasi-injective non-singular module, Y is a square-free non-singular automorphism-invariant module, the modules X and Y are injective with respect to each other, any sum of closed submodules of the module Y is an automorphism-invariant module, $Hom(X, Y) = Hom(Y, X) = 0$. By [19, Note 1.7], X is a multiplication module satisfying Theorem 2.20. It follows that $X_R \cong R$. We have $0 = Hom(X, Y) \cong Hom(R, Y) \cong Y$, and so $Y = 0$. Thus $M_R \cong R$ is injective. □

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