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THE ONE-DIMENSIONAL NONREDUCED LINE SCHEME OF TWO FAMILIES OF QUADRATIC QUANTUM $\mathbb{P}^3$S

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Abstract. The classification of quantum $\mathbb{P}^2$s was completed by M. Artin et al. decades ago, but the classification of quadratic algebras that are viewed as quantum $\mathbb{P}^3$s is still an open problem. Based on work of M. Van den Bergh, it is believed that a "generic" quadratic quantum $\mathbb{P}^3$ should have a finite point scheme and a one-dimensional line scheme. Two families of quadratic quantum $\mathbb{P}^3$s with these geometric properties are presented herein, where each family member has a line scheme that is either a union of lines or is a union of a line, a conic and a curve. Moreover, we prove that, under certain conditions, if A is a quadratic quantum $\mathbb{P}^3$ that contains a subalgebra B that is a quadratic quantum $\mathbb{P}^2$, then the point scheme of B embeds in the line scheme of A.

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Introduction

An AS-regular algebra of global dimension $n+1$ is viewed by many researchers as being a noncommutative (or quantized) analogue of the polynomial ring on $n+1$ variables. For this reason, such an algebra is often called a quantum $\mathbb{P}^n$. Although the classification of quantum $\mathbb{P}^2$s was completed many years ago (cf. [1,2,3,20,21,22]), the classification of quantum $\mathbb{P}^3$s remains an open problem. Given the large role played by the geometric techniques developed by M. Artin, J. Tate and M. Van den Bergh in their seminal papers [2,3] in the classification of quantum $\mathbb{P}^2$s,

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it is reasonable to expect that similar geometric techniques will play an analogous role in the classification of quantum $\mathbb{P}^3$s. Hence, identification of any geometric properties satisfied by quantum $\mathbb{P}^3$s, or by "generic" quadratic quantum $\mathbb{P}^3$s, is desirable.

Based on unpublished notes of M. Van den Bergh, and as explained in [25, Section 1D], a "generic" quadratic quantum $\mathbb{P}^3$ has a one-dimensional line scheme, and has defining relations whose zero locus consists of twenty points (counted with multiplicity). In particular, such an algebra has at most twenty nonisomorphic point modules. Additionally, in [7], it is proved that if the line scheme of a quadratic quantum $\mathbb{P}^3$ has dimension one, then, when viewed as a subscheme of $\mathbb{P}^5$ via the Plücker embedding, the line scheme has degree twenty. This raises the question (posed by S. P. Smith to the authors) of whether or not there could exist a quadratic quantum $\mathbb{P}^3$ that has a finite point scheme while simultaneously having a line scheme consisting of a union of lines.

Addressing Smith's question is one of our main objectives in this article, and we answer the question in the affirmative, in Section 3, while computing, in Theorems 3.1 and 3.5, the line scheme of algebras that belong to two families of quadratic algebras whose members are candidates for being "generic" quadratic quantum $\mathbb{P}^3$s. Indeed, via the Plücker embedding, we exhibit the line scheme of the algebras as a subscheme of $\mathbb{P}^5$, showing that, for the algebras in one family, the line scheme is the union of ℓ lines, where $\ell \in \{3, 4\}$, and that, for the algebras in the other family, the line scheme is the union of a line, a conic and a curve. In so doing, we accomplish one of our other main objectives which is to add to the few examples in the literature (cf. [5,6,9,19,24]) of quadratic algebras that are candidates for being "generic" quadratic quantum $\mathbb{P}^3$s.

Moreover, the algebras in the two families we study are Ore extensions of certain quadratic quantum $\mathbb{P}^2$s, and we noticed that the point scheme of the quadratic quantum $\mathbb{P}^2$ in each case is embedded in the line scheme of the quadratic quantum $\mathbb{P}^3$. This observation led to Theorem 1.4 proving that this behavior is not a coincidence.

The paper is outlined as follows. Section 1 lays out some notation to be used throughout the article, and presents the two families of algebras to be studied herein. The algebras first appeared in [23], so some results from [23] are reviewed in Section 1. In the second half of the section, we prove Theorem 1.4, showing that, under certain conditions, if a quadratic quantum $\mathbb{P}^3$ contains a quadratic quantum $\mathbb{P}^2$ as a subalgebra, then the point scheme of the quadratic quantum $\mathbb{P}^2$ embeds in the line scheme of the quadratic quantum $\mathbb{P}^3$.

In Section 2, we compute the point scheme, in Proposition 2.1, of the algebras presented in Section 1, and, in Section 3, we compute their line schemes in Theorems 3.1 and 3.5. We describe which lines in $\mathbb{P}^3$ are parametrized by the line schemes in Corollaries 3.3 and 3.7.

This article is based on the Ph.D. dissertations, [13,15], of the first and third authors and on Chapter 3 of the Ph.D. dissertation, [14], of the second author. Hence, some details have been omitted that can be found in [13,14,15].

1. Two families of quadratic quantum $\mathbb{P}^3$s

In this section, we introduce some notation to be used throughout the article, and present two families of quadratic AS-regular algebras that are candidates for being generic quadratic quantum $\mathbb{P}^3$s. In Section 1.2, in Theorem 1.4, we prove that, under certain conditions, if $\mathfrak{b}$ is a quadratic quantum $\mathbb{P}^2$ that is a subalgebra of an algebra, $\mathfrak{a}$, that is a quadratic quantum $\mathbb{P}^3$, then the point scheme of $\mathfrak{b}$ embeds in the line scheme of $\mathfrak{a}$.

Throughout the article, $\Bbbk$ denotes a field. After this section, we will impose the additional assumptions that $\Bbbk$ be algebraically closed and satisfy $\mathrm{char}(\Bbbk) = 0$. The dual of any finite-dimensional $\Bbbk$-vector space W will be denoted by W^*. For any nonzero ring $\mathcal{R}$, we write $\mathcal{R}^\times$ to denote the subset of all units in $\mathcal{R}$, whereas for $i \in \mathbb{Z}$ and any $\mathbb{Z}$-graded $\Bbbk$-algebra $\mathcal{R}$, we write $\mathcal{R}_i$ for the span of the homogeneous degree-i elements in $\mathcal{R}$. The scheme-theoretic zero locus in $\mathbb{P}^n$ of homogeneous polynomials $f_1, \ldots, f_r$ will be denoted by $\mathcal{V}(f_1, \ldots, f_r)$.

1.1. Two families of algebras. In this subsection, we present the two families of AS-regular algebras of global dimension four that were introduced in [23]. In particular, let A denote the $\Bbbk$-algebra on four generators, x_1, x_2, x_3, x_4, subject to the defining relations

$$\begin{array}{rclcrcl} x_2x_1 &=& -x_1x_2, & \quad & x_4x_1 &=& -x_1x_4 + x_2^2 + bx_3^2, \\ x_3x_1 &=& x_1x_3, & & x_4x_2 &=& -x_2x_4 + x_1^2 + dx_3^2, \\ x_3x_2 &=& x_2x_3, & & x_4x_3 &=& x_3x_4 + x_1x_2, \end{array} \tag{1.1}$$

where $b \in \Bbbk^\times$ or $d \in \Bbbk^\times$. In fact, A is a member of the 4-parameter family of algebras given in [23, Proposition 2.1], and we have taken two of the parameters (a and c) in that family to be zero (and retained the name of the other two parameters for any readers who wish to compare A with the algebras in [23, Proposition 2.1]). By [23, Proposition 2.1], A is a noetherian, AS-regular domain of global dimension four, is an infinite module over its center, has a finite point scheme consisting of at most five closed points, and has a line scheme of dimension one. Moreover, in the proof of [23, Proposition 2.1], it was established that A is an Ore extension, $A'[x_4\,;\,\sigma,\,\delta]$, of the subalgebra A' that is generated by x_1, x_2, x_3, where $\sigma \in \operatorname{Aut}(A')$. In particular, A is a free A'-module, so it follows that the defining relations of A' are given by the three relations in (1.1) that do not use x_4, and so A' is a quadratic quantum $\mathbb{P}^2$, with point scheme given by the "triangle" $\mathcal{V}(x_1x_2x_3) \subset \mathbb{P}^2$.

For the second family, let B denote the $\Bbbk$-algebra on four generators, x_1, x_2, x_3, x_4, subject to the defining relations

$$\begin{aligned} x_2x_1 &= -x_1x_2 + 2x_3^2, & x_4x_1 &= -x_1x_4 + x_2^2 + \lambda x_1x_3, \\ x_3x_1 &= x_1x_3, & x_4x_2 &= -x_2x_4 + x_1^2 + (\lambda-2)x_2x_3, \\ x_3x_2 &= x_2x_3, & x_4x_3 &= x_3x_4 + x_1x_2 - x_3^2, \end{aligned} \tag{1.2}$$

where $\lambda \in \Bbbk$. Analogous to the situation for A, the algebra B is a member of the 5-parameter family of algebras given in [23, Proposition 2.2], and we have taken four of the parameters to be zero. By [23, Proposition 2.2], B is a noetherian, AS-regular domain of global dimension four, is an infinite module over its center, has a finite point scheme consisting of exactly seven closed points, and has a line scheme of dimension one. Moreover, as mentioned in [23], B is an Ore extension, $B'[x_4\,;\,\sigma,\,\delta]$, of the subalgebra B' that is generated by x_1, x_2, x_3, where $\sigma \in \operatorname{Aut}(B')$. In particular, B is a free B'-module, so it follows that the defining relations of B' are given by the three relations in (1.2) that do not use x_4, and so B' is a quadratic quantum $\mathbb{P}^2$, with point scheme given by the union of a line and a conic, namely $\mathcal{V}(x_3) \cup \mathcal{V}(x_1x_2 - x_3^2) \subset \mathbb{P}^2$.

Before proceeding, we make the observation in our next result that each of the algebras A and B encode some symmetry.

Lemma 1.1. *Writing A as $A(b,\,d)$ and B as $B(\lambda)$, we have that $A(b,\,d) \cong A(d,\,b)$ for all $b,\,d \in \Bbbk$, and $B(\lambda) \cong B(2-\lambda)$ for all $\lambda \in \Bbbk$.*

Proof. Let $V = \sum_{1\le i\le 4} \Bbbk x_i$, so $A_1 = V = B_1$. The linear map $\phi : V \to V$ given by

$$x_1 \mapsto x_2, \qquad x_2 \mapsto x_1, \qquad x_3 \mapsto -x_3, \qquad x_4 \mapsto x_4,$$

induces an isomorphism from $A(b,\,d)$ to $A(d,\,b)$, for all $b,\,d \in \Bbbk$, and, similarly, induces an isomorphism from $B(\lambda)$ to $B(2-\lambda)$, for all $\lambda \in \Bbbk$. □

Owing to Lemma 1.1, we henceforth take $d \neq 0$ in all discussions regarding the algebra A, and $\lambda \neq 0$ in all discussions regarding B.

The algebras A and B were presented in [23] since each algebra is an example of a quadratic AS-regular algebra of global dimension four that is not a finite module over its center, yet the algebra has a finite point scheme with a finite-order automorphism acting on the point scheme. This was discussed in detail in [10], in which the fat-point modules over A and B were computed and found to exhibit an automorphism of infinite order.

1.2. Point schemes embedded in line schemes. Before concluding this section, we prove in Corollary 1.5 that the point scheme of A' (respectively, B') is contained in the line scheme of A (respectively, B), and that this containment is a special case of a more general result, namely Theorem 1.4. Our results on this issue build on results in [13, Section 4.4]. We remind the reader that the point scheme (respectively, line scheme) is the scheme that represents the functor of point modules (respectively, line modules); cf. [2,18]. We refer to the reduced variety of the point scheme (respectively, line scheme) as the point variety (respectively, line variety).

Lemma 1.2. *Suppose that $\mathfrak{a}$ is a $\mathbb{Z}$-graded $\Bbbk$-algebra that contains a $\mathbb{Z}$-graded subalgebra $\mathfrak{b}$ such that $\mathfrak{b}_1 \subset \mathfrak{a}_1$, where $\dim(\mathfrak{b}_1) = 3$ and $\dim(\mathfrak{a}_1) = 4$. If $u, v \in \mathfrak{b}_1$ satisfy $\dim(u\mathfrak{b}_1 + v\mathfrak{b}_1) \leq 5$, then $\dim(u\mathfrak{a}_1 + v\mathfrak{a}_1) \leq 7$.*

Proof. With the given hypotheses, we may write $\mathfrak{a}_1 = \mathfrak{b}_1 \oplus \Bbbk w$ for some $w \in \mathfrak{a}_1 \setminus \mathfrak{b}_1$. It follows that $u\mathfrak{a}_1 + v\mathfrak{a}_1 = u\mathfrak{b}_1 + v\mathfrak{b}_1 + \Bbbk uw + \Bbbk vw$. Thus,

$$\dim(u\mathfrak{a}_1 + v\mathfrak{a}_1) \leq \dim(u\mathfrak{b}_1 + v\mathfrak{b}_1) + \dim(\Bbbk uw + \Bbbk vw) \leq 5 + 2,$$

which yields the result. □

Proposition 1.3. *Suppose $\mathfrak{a}$ is a quadratic Auslander-regular algebra of global dimension four that satisfies the Cohen-Macaulay property with Hilbert series $H_{\mathfrak{a}}(t) = 1/(1-t)^4$. If $\mathfrak{a}$ contains a quadratic AS-regular subalgebra $\mathfrak{b}$ of global dimension three, where $\mathfrak{b}_1 \subset \mathfrak{a}_1$, then the point variety of $\mathfrak{b}$ embeds in the line variety of $\mathfrak{a}$.*

Proof. Let M denote a right point module over $\mathfrak{b}$. Since $\mathfrak{b}$ is a quadratic AS-regular algebra of global dimension three, we have $M \cong \frac{\mathfrak{b}}{u\mathfrak{b}+v\mathfrak{b}}$, where $u, v \in \mathfrak{b}_1$ are linearly independent and $\dim(u\mathfrak{b}_1 + v\mathfrak{b}_1) = 5$. The Hilbert series of $\mathfrak{a}$ implies that $\dim(\mathfrak{a}_1) = 4$, so Lemma 1.2 implies that $\dim(u\mathfrak{a}_1 + v\mathfrak{a}_1) \leq 7$. Since $\mathfrak{a}$ is connected and Auslander regular, it follows from [11, Theorem 4.8] that $\mathfrak{a}$ is a domain. Thus, $u\mathfrak{a}_1 \cap v\mathfrak{a}_1 \neq \{0\}$, so [12, Proposition 2.8] implies that $\frac{\mathfrak{a}}{u\mathfrak{a}+v\mathfrak{a}}$ is a line module over $\mathfrak{a}$.

In other words, the point $\mathcal{V}(u, v) \subset \mathbb{P}^2$ corresponds to a point module over $\mathfrak{b}$, while the line $\mathcal{V}(u, v) \subset \mathbb{P}^3$ corresponds to a line module over $\mathfrak{a}$, which completes the proof. □

In Proposition 1.3, if $\mathfrak{p}$ belongs to the point scheme of $\mathfrak{b}$, then the image of $\mathfrak{p}$ in the line scheme of $\mathfrak{a}$ could conceivably have multiplicity lower than that of $\mathfrak{p}$. The following result shows that this situation cannot arise.

Theorem 1.4. *Under the hypotheses of Proposition 1.3, we have that the point scheme of $\mathfrak{b}$ embeds in the line scheme of $\mathfrak{a}$.*

Proof. Our method of proof builds on the proof of Proposition 1.3 by using the notion of a family of point (respectively, line) modules, $M = \oplus_{i\geq 0} M_i$, parametrized by $S = \mathrm{Spec}(R)$, where R is a commutative $\mathbb{k}$-algebra, as discussed in [2, Section 3]. In other words, in our setting, M is a graded module over the algebra $R \otimes_{\mathbb{k}} \mathfrak{b}$ (respectively, $R \otimes_{\mathbb{k}} \mathfrak{a}$). In fact, since M is either a family of point modules or line modules, we have $M_0 = R$, so M is compatible with descent, which implies that we may assume that R is a commutative local $\mathbb{k}$-algebra, and that M_i is a free R-module of rank one (respectively, rank $i + 1$), for all i. Thus, we consider the module M, from the proof of Proposition 1.3, as determining a point module $\frac{R\otimes_{\mathbb{k}}\mathfrak{b}}{u(R\otimes_{\mathbb{k}}\mathfrak{b})+v(R\otimes_{\mathbb{k}}\mathfrak{b})}$ over $R \otimes_{\mathbb{k}} \mathfrak{b}$, and, using the proof of Proposition 1.3, we produce a line module $\frac{R\otimes_{\mathbb{k}}\mathfrak{a}}{u(R\otimes_{\mathbb{k}}\mathfrak{a})+v(R\otimes_{\mathbb{k}}\mathfrak{a})}$ over $R \otimes_{\mathbb{k}} \mathfrak{a}$, thereby showing that a family of point modules over $\mathfrak{b}$ is determining a family of line modules over $\mathfrak{a}$. The result follows. □

Corollary 1.5. *With A, A', B, B' as defined in Section 1.1, the point scheme of A' (respectively, B') embeds in the line scheme of A (respectively, B).*

Proof. Since A is an Ore extension of A', A' is a subalgebra of A, and similarly for B and B'. The result follows by applying Theorem 1.4. □

The embedding maps that are intrinsic to Corollary 1.5 are described in greater detail in Remarks 3.4 and 3.6.

2. The point schemes of A and B

Henceforth, we impose the additional assumption that $\mathbb{k}$ be algebraically closed and satisfy $\mathrm{char}(\mathbb{k}) = 0$. Although a count was provided in [23] for the total number of distinct points in the point scheme of A, and in the point scheme of B, the actual point schemes were not computed in [23]. Thus, in this section, we determine the point scheme of the algebra A in Proposition 2.1 and the point scheme of the algebra B in Proposition 2.2.

It will be convenient to write $V = A_1$ (respectively, $V = B_1$) and identify $\mathbb{P}^3$ with $\mathbb{P}(V^*)$.

Proposition 2.1. *Suppose $d \neq 0$, and let δ_{ij} denote the Kronecker-delta symbol, where $i, j \in \Bbbk$. The point scheme, $P_A \subset \mathbb{P}^3$, of A consists of*

(*a*) $e_4 = (0, 0, 0, 1) = \mathcal{V}(x_1, x_2, x_3)$ *with multiplicity* $16 + 2\delta_{0b}$, *and*

(*b*) *two distinct points of the form* $(2d, 0, \alpha_1, -b)$, *where* $\alpha_1^2 = -4d$ *and each point has multiplicity one, and*

(*c*) $2(1 - \delta_{0b})$ *distinct points of the form* $(0, 2b, \alpha_2, -d)$, *where* $\alpha_2^2 = -4b$ *and each point has multiplicity one.*

In particular, all the points lie on the union $\mathcal{V}(x_1x_2x_3) \subset \mathbb{P}^3$ of the first three coordinate planes.

Proof. Following the method in [2], we write the defining relations (1.1) of A as Mx, where M is the 6×4 matrix

$$\begin{bmatrix} x_2 & x_1 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 \\ 0 & x_3 & -x_2 & 0 \\ x_4 & -x_2 & -bx_3 & x_1 \\ -x_1 & x_4 & -dx_3 & x_2 \\ 0 & -x_1 & x_4 & -x_3 \end{bmatrix},$$

and x is the column vector given by the transpose of $[x_1, \ldots, x_4]$. As explained in [2], the point scheme of A can be computed as a subscheme of $\mathbb{P}^3$ by finding the zero locus of the 4×4 minors of M. Three of the 4×4 minors are:

$$bx_2x_3^3 + x_1^2x_2^2 + x_2^3x_3, \qquad -dx_1x_3^3 + x_1^2x_2^2 - x_1^3x_3, \qquad -bx_2x_3^3 + dx_1x_3^3 + x_1^3x_3 - x_2^3x_3,$$

which sum to $4x_1^2x_2^2$. We first address finding the closed points of the point scheme, after which we compute the multiplicity of the points.

Considering where $x_1 = 0 = x_2$, we find that, since $d \neq 0$, there is only one solution, namely the point $e_4 = \mathcal{V}(x_1, x_2, x_3)$. On the other hand, considering where $x_1 = 0 \neq x_2$, we find that if $b = 0$, then there is no solution, whereas if $b \neq 0$, then we have $x_2^2 + bx_3^2 = 0 = dx_2 + 2bx_4$. This latter case, where $b \neq 0$, yields two distinct solutions, namely $(0, 2b, \pm 2\sqrt{-b}, -d)$. Turning to the case where $x_1 \neq 0 = x_2$, we find that, since $d \neq 0$, we may use the isomorphism from Lemma 1.1 to reveal exactly two distinct points, namely $(2d,\ 0,\ \pm 2\sqrt{-d},\ -b)$.

In order to find the multiplicity of any point in P_A, we need to consider the coordinate ring $R = \Bbbk[x_1, \ldots, x_4]/J$ of P_A, where J is the ideal generated by the 4×4 minors of the matrix M. In particular, when computing the multiplicity at $\mathfrak{p} \in P_A$, we consider the localization $R_{\mathfrak{p}}$ of R at $\mathfrak{p}$.

If $b \neq 0$, then x_2 and x_3 are both nonzero at the point $\mathfrak{p} = (0,\ 2b,\ \alpha_2,\ -d)$, where $\alpha_2^2 = -4b$. Moreover, $x_1 x_2 x_3^2 \in J$, so the image of x_1 in $R_{\mathfrak{p}}$ is zero. By inverting x_2, the remaining elements of J imply that $2X_4 - dX_3^2$ and $bX_3^2 + 1$ are zero in $R_{\mathfrak{p}}$, where X_i denotes the image of x_i in $R_{\mathfrak{p}}$, for $i \in \{3,\ 4\}$. It follows that $\dim(R_{\mathfrak{p}}) \leq 2$. However, since $bX_3^2 + 1$ vanishes at two distinct points in P_A, we have that $\dim(R_{\mathfrak{p}}) = 1$, so $\mathfrak{p}$ has multiplicity one. By using Lemma 1.1, we deduce that the multiplicity of $(2d,\ 0,\ \alpha_1,\ -b)$, where $\alpha_1^2 = -4d$, is also one.

Since P_A is finite, an unpublished result of M. Van den Bergh (cf. [25, Section 1D]) implies that the sum of the multiplicities of all the points in P_A is twenty, and hence the multiplicity of e_4 is as claimed in the statement, which concludes the proof. (Alternatively, the reader is referred to [13, Pages 20-21] and [14, Page 28] for details concerning explicit computation of the multiplicity of e_4. However, in [13], the last exponent in the first polynomial on the last line of Page 20 should be 3, not 2.) □

Proposition 2.2. *The point scheme, $P_B \subset \mathbb{P}^3$, of B consists of*

(*a*) $e_4 = (0,\ 0,\ 0,\ 1) = \mathcal{V}(x_1,\ x_2,\ x_3)$ *with multiplicity* 14, *and*

(*b*) *six distinct points of the form* $(2\alpha^2(\alpha^3 + 2),\ 2\alpha,\ 2,\ \alpha^3 + \lambda)$, *where* $\alpha^6 + 2\alpha^3 - 1 = 0$ *and each point has multiplicity one.*

In particular, all seven points lie on the singular quadric $\mathcal{V}(x_1 x_2 - x_3^2) \subset \mathbb{P}^3$.

Proof. Following the proof of Proposition 2.1, the defining relations (1.1) of B may be written as Mx, where M is the 6×4 matrix

$$\begin{bmatrix} x_2 & x_1 & -2x_3 & 0 \\ x_3 & 0 & -x_1 & 0 \\ 0 & x_3 & -x_2 & 0 \\ x_4 & -x_2 & -\lambda x_1 & x_1 \\ -x_1 & x_4 & (2-\lambda)x_2 & x_2 \\ 0 & -x_1 & x_3 + x_4 & -x_3 \end{bmatrix},$$

and x is the column vector given by the transpose of $[x_1, \ldots, x_4]$. One of the 4×4 minors of M is $(x_1x_2 - x_3^2)x_3^2$, while another three 4×4 minors of M belong to $x_1^3x_2 + x_3\Bbbk[x_1, \ldots, x_4]$, $-x_1^4 + (\Bbbk x_2 + \Bbbk x_3)\Bbbk[x_1, \ldots, x_4]$ and $-x_2^4 + (\Bbbk x_1 + \Bbbk x_3)\Bbbk[x_1, \ldots, x_4]$. It follows that if $x_3 = 0$, then $x_1 = 0 = x_2$, yielding the point $e_4 = \mathcal{V}(x_1, x_2, x_3)$ as a solution. On the other hand, inverting x_3, the ideal generated by the images of the 4×4 minors of M is generated by

$$X_1 - X_2^5 - 2X_2^2, \qquad 2X_4 - X_2^3 - \lambda \qquad \text{and} \qquad X_2^6 + 2X_2^3 - 1,$$

where X_i is the image of x_i, for all $i \neq 3$, in the ring obtained by inverting x_3. Comparing the latter polynomial with its derivative (given that $\operatorname{char}(\Bbbk) = 0$), we find that there are six distinct solutions for X_2. Thus, if $x_3 \neq 0$, then there are six distinct points of the form

$$(2\alpha^2(\alpha^3 + 2),\ 2\alpha,\ 2,\ \alpha^3 + \lambda),$$

where $\alpha^6 + 2\alpha^3 - 1 = 0$. Moreover, our discussion implies that the coordinate ring of $P_B \setminus \{e_4\}$ is isomorphic to

$$\frac{\Bbbk[X]}{\langle X^6 + 2X^3 - 1\rangle},$$

which has dimension six, so each of the six distinct points has multiplicity one. Appealing to the unpublished result of M. Van den Bergh that was mentioned in the proof of Proposition 2.1 completes the proof. □

3. The line schemes of A and B

A method for computing the line scheme of a quadratic AS-regular algebra of global dimension four on four generators with six relations is given in [19]. For the algebra A, the method is described in [13] and, for the algebra B, the method is described in [15]. In this section, Theorems 3.1 and 3.5 identify the line schemes of A and B when viewed via the Plücker embedding. In particular, the line scheme of A is the union of ℓ lines, where $\ell \in \{3, 4\}$, and the line scheme of B is the union of a line, a conic and a degree-8 curve. Corollaries 3.3 and 3.7 describe the lines in $\mathbb{P}^3$ that are parametrized by the line schemes of A and B.

We continue to assume that $\Bbbk$ is algebraically closed and satisfies $\operatorname{char}(\Bbbk) = 0$.

The method in [19] for computing the line scheme of a quadratic AS-regular algebra of global dimension four on four generators with six relations entails the following. One finds the Koszul dual, D, of the quadratic algebra, and writes the defining relations of D using a 10×4 matrix $\mathcal{M}$, and then forms a 10×8 matrix by juxtaposing $\mathcal{M}$ evaluated at an arbitrary element of V with $\mathcal{M}$ evaluated at a potentially different arbitrary element of V. The method then requires one to compute the 45 8×8 minors of the 10×8 matrix. After substituting Plücker coordinates M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34} into the 45 minors, one seeks the zero locus of the ideal generated by the 45 polynomials so obtained together with the quadratic Plücker polynomial,

$$p = M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23}.$$

For the algebra A (respectively, B), we denote the ideal generated by these 46 polynomials as J_A (respectively, J_B).

Using the above Plücker coordinates, in accordance with the recipe in [19], has the advantage of allowing us to carry information from the line scheme back to $\mathbb{P}(V^*)$. In particular, if $\mathfrak{p} = (p_1, \ldots, p_4)$ and $\mathfrak{q} = (q_1, \ldots, q_4)$ are distinct points in $\mathbb{P}(V^*)$, then M_{ij} (where $1 \le i < j \le 4$) can be taken to be the 2×2 minor using columns i and j from the 2×4 rank-2 matrix

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{bmatrix}.$$

We view this 2×4 rank-2 matrix as representing the line in $\mathbb{P}(V^*)$ that passes through $\mathfrak{p}$ and $\mathfrak{q}$, and it corresponds to a point on the Grassmannian, $\mathcal{V}(p)$, of all lines in $\mathbb{P}^3$. This description of lines in $\mathbb{P}^3$ via Plücker coordinates and the 2×4 matrices is called the Plücker embedding. For example, viewing the line scheme of the polynomial ring, $\Bbbk[x_1, \ldots, x_4]$, in $\mathbb{P}^5$ via the Plücker embedding yields its line scheme to be $\mathcal{V}(p)$.

It was proved, in [7], that if the line scheme of a quadratic quantum $\mathbb{P}^3$ has dimension one, then, when viewed as a subscheme of $\mathbb{P}^5$ via the Plücker embedding, the line scheme has degree twenty. As mentioned in the Introduction, this raises the question (posed by S. P. Smith to the authors) of whether or not there could exist a quadratic quantum $\mathbb{P}^3$ that has a finite point scheme while simultaneously having a line scheme consisting of a union of lines, when viewed via the Plücker embedding in $\mathbb{P}^5$. The next result addresses this question in the affirmative by showing that the algebra A, with the restrictions imposed on the parameters in Section 1.1, has a line scheme that is a union of lines. It is still an open question whether or not a quadratic quantum $\mathbb{P}^3$ could have a line scheme that is a union of 20 distinct lines. It is also an open question as to what the line scheme of a generic quadratic quantum $\mathbb{P}^3$, or of classes of generic quantum $\mathbb{P}^3$, is likely to be, although a few examples are provided in [5,6,9,19,24].

Theorem 3.1. *If $d \neq 0$, then the line scheme, $\mathcal{L}_A$, of A has dimension one. In this case, viewing $\mathcal{L}_A$ as a subscheme of $\mathbb{P}^5$ via the Plücker embedding, the reduced variety of $\mathcal{L}_A$ is a union of lines. More precisely, writing*

$$L_1 = \mathcal{V}(M_{12},\ M_{13},\ M_{23},\ M_{14}) \subset \mathbb{P}^5,$$

$$L_2 = \mathcal{V}(M_{12},\ M_{13},\ M_{23},\ M_{24}) \subset \mathbb{P}^5,$$

$$L_3 = \mathcal{V}(M_{12},\ M_{13},\ M_{23},\ M_{34}) \subset \mathbb{P}^5,$$

$$L_4 = \mathcal{V}(M_{12},\ dM_{13} - bM_{23},\ dM_{14} - bM_{24}, (b^3 + d^3)M_{23} - 2bd^2M_{34}) \subset \mathbb{P}^5,$$

we have

(*a*) $\mathcal{L}_A = L_1 \cup L_2 \cup L_3$, *if $b = 0$, where L_1, L_2 and L_3 have multiplicities* 8, 6 *and* 6 *respectively, and*

(*b*) $\mathcal{L}_A = L_1 \cup L_2 \cup L_3 \cup L_4$, *if $b \neq 0$, where L_i has multiplicity* 6 *for $i \leq 3$ and L_4 has multiplicity* 2.

Moreover, $L_i \cap L_j$ is nonempty whenever $i, j \leq 3$ or $i, j \geq 3$, whereas, if $b \neq 0$, then $L_i \cap L_4$ is empty for $i \leq 2$.

Proof. By [7, Proposition 3.2], if $\dim(\mathcal{L}_A) = 1$, then every irreducible component of $\mathcal{L}_A$ has dimension one; in particular, there are no embedded components. We first compute the closed points of $\mathcal{L}_A$, after which we discuss the multiplicities of the points. For any omitted details, the reader is referred to [13,14].

For arbitrary b and d, the 45 polynomials obtained from the 10×8 matrix discussed above are provided in [13, Section 6.2.2] and, for $b = 0$, they are provided in [14, Section 6.2]. In seeking the reduced variety of $\mathcal{L}_A$, we note that, since one of the polynomials is M_{12}^4, we may assume that $M_{12} = 0$. With the assumption that $d \neq 0$, a Gröbner basis computed with Wolfram's Mathematica, [16], on the generators of $J_A + \langle M_{12} \rangle$ yields that either $dM_{13} = bM_{23}$ or that $M_{13} = 0 = M_{23}$. In the latter case, we find that $M_{14}M_{24}M_{34} = 0$, which proves that $L_i \subset \mathcal{L}_A$ for all $i \leq 3$, and no other solutions exist in this case. In the case where $dM_{13} = bM_{23}$, we may assume that $M_{23} \neq 0$, so another Gröbner basis computation reveals that $dM_{14} = bM_{24}$ and $(b^3 + d^3)M_{23} = 2bd^2M_{34}$, which gives us precisely L_4. It follows that the first two assertions and part of the third assertion in the statement are proved.

By Lemma 1.1, in order to compute the multiplicity of the points that lie on only one component of $\mathcal{L}_A$, it suffices to consider such points on L_2 and L_3 only.

Let $\mathfrak{p} \in L_2 \setminus L_i$ for all $i \neq 2$. In homogeneous coordinates, $\mathfrak{p}$ may be written as

$$\mathfrak{p} = (0,\, 0,\, 1,\, 0,\, 0,\, \gamma),$$

for some $\gamma \in \Bbbk^\times$. We wish to intersect L_2 at $\mathfrak{p}$ with a hyperplane $\mathcal{V}(f)$, where

$$f = M_{34} - \gamma M_{14} + \beta_1 M_{12} + \beta_2 M_{13} + \beta_3 M_{23} + \beta_4 M_{24},$$

for some $\beta_1, \ldots, \beta_4 \in \Bbbk$. For this purpose, we first localize by inverting M_{14} and substitute for M_{23} using the image of p in the localized ring. Our method at this stage entailed using Mathematica to compute different Gröbner bases in order to find a dozen or so elements in J_A that are of the form gh, for some polynomials g and h, where $g|_{\mathfrak{p}} \neq 0$. Each time, the factors given by h were appended to the list of generators. Once this process stopped yielding such elements gh, we then used the image of f in the localized ring in order to substitute for M_{34}, while removing our earlier substitution for M_{23} from the computation (but the image of p in the localized ring is an element of the image of J_A in the localized ring, so that information is not lost). After computing another Gröbner basis, it was found to have a zero locus that contained only $\mathfrak{p}$. The ideal, $\hat{J}_A$, generated by this last Gröbner basis was generated by three polynomials in two variables, and the quotient ring was isomorphic to $\Bbbk[y,\, z]_{\langle y,\, z\rangle}$ modulo the ideal generated by

$$y^5, \qquad 4\gamma^2 z^2 - (d^2\gamma^4 + 1)y^4,$$

$$2\gamma^2 y^3 + (\beta_4 d + 2\beta_2\gamma + \beta_4)y^4 - 4\gamma^3 yz,$$

so the quotient ring has dimension at most six. A computation using Singular, [8], verified the dimension to be six, and so did a computation with the Affine package in Maxima, [17]. In fact, the latter revealed that the only denominator used in the application of Bergman's Diamond Lemma was $4\gamma^{-2}$, which is nonzero. It follows that the multiplicity of any point of L_2 that lies on only L_2 is six.

The procedure is very similar for the computation of the multiplicity of any point of L_3 that lies on only L_3. In this case, we take the point

$$\mathfrak{p} = (0,\, 0,\, \gamma,\, 0,\, 1,\, 0),$$

for some $\gamma \in \Bbbk^\times$. For the hyperplane $\mathcal{V}(f)$ this time, we use

$$f = M_{14} - \gamma M_{24} + \beta_1 M_{12} + \beta_2 M_{13} + \beta_3 M_{23} + \beta_4 M_{34},$$

for some $\beta_1, \ldots, \beta_4 \in \Bbbk$. As in the case of L_2, this time the ideal $\hat{J}_A$ was again generated by three polynomials in two variables, and the quotient ring was isomorphic to $\Bbbk[y,\, z]_{\langle y,\, z\rangle}$ modulo the ideal generated by

$$y^5, \qquad 2z^2 - (b^2 - 2bd\gamma + d^2\gamma^2)y^4,$$

$$4\gamma yz + 2\gamma(b - d\gamma)y^3 + d(2\gamma\beta_3 + 2\gamma^2\beta_2 + \gamma^3\beta_4 + \beta_4)y^4,$$

so again the quotient ring has dimension at most six. A computation using Singular, [8], verified the dimension to be six, and so did a computation with the Affine package in Maxima, [17]. In fact, the latter revealed that the only denominator used in the application of Bergman's Diamond Lemma was 4γ, which is nonzero. It follows that the multiplicity of any point of L_3 that lies on only L_3 is six.

If $b \neq 0$, then Lemma 1.1 implies that the multiplicity of L_1 is six, in which case the multiplicity of L_4 is $20 - 18 = 2$, by [7, Proposition 3.2] (see Remark 3.2(b)). On the other hand, if $b = 0$, then the multiplicity of L_1 is $20 - 12 = 8$. (This latter case highlights that, if $b = 0$, then L_4 degenerates into L_1 and adds its multiplicity from the case where $b \neq 0$ to the multiplicity of L_1.)

The proof of the last assertion in the statement is left to the reader. □

Remark 3.2.

(a) If $b = -d$, then the defining polynomials of L_4 are symmetric in b and d. If $b \neq -d$, then the last polynomial defining L_4 can be replaced by

$$(b^2 - bd + d^2)(M_{13} + M_{23}) - 2bdM_{34}$$

(since

$$(b^3 + d^3)M_{13} - 2b^2dM_{34}$$

belongs to the ideal that determines the line L_4), which is again symmetric in b and d.

(b) If one wishes to compute explicitly the multiplicity of L_4 in Theorem 3.1(b), one should likely change coordinates first as follows in order to help choose the hyperplane $\mathcal{V}(f)$:

$$\begin{aligned} &M_{12} \mapsto X_1, && M_{13} \mapsto (X_2 + bX_5)d^{-1}, \\ &M_{23} \mapsto X_5, && M_{14} \mapsto (X_3 + bX_6)d^{-1}, \\ &M_{24} \mapsto X_6, && M_{34} \mapsto \big((b^3 + d^3)X_5 - X_4\big)b^{-1}d^{-2}/2. \end{aligned}$$

Consequently, one may choose $f = X_6 - sX_5 - \sum_{1\leq i\leq 4} \beta_i X_i$, where $s,\, \beta_1,\, \ldots, \beta_4 \in \Bbbk$. In this case, one can reduce the number of variables to one, say y, and compute the ideal $\hat{J}_A$ (cf. the proof of Theorem 3.1 for notation) to be $\langle y^2\rangle$, so the quotient ring has dimension two, as expected.

(c) As mentioned in Section 1.1, the algebras A belong to a 4-parameter family of algebras introduced in [23, Proposition 2.1]. The line scheme of the larger family is computed in [13, Theorem 3.3.1].

Corollary 3.3. *Let $L_1, \ldots, L_4$ be as in Theorem 3.1 concerning the line scheme of A. For $i \leq 3$, the lines in $\mathbb{P}^3$ that are parametrized by L_i are those that lie on the plane $\mathcal{V}(x_i)$ and pass through the point $\mathcal{V}(x_1,\, x_2,\, x_3) = e_4$. If $b,\, d \in \Bbbk^\times$, then the lines in $\mathbb{P}^3$ that are parametrized by L_4 are those that lie on the plane $\mathcal{V}(dx_1 - bx_2)$ and pass through the point*

$$\mathcal{V}(x_3,\ dx_1 - bx_2,\ b^2x_1 + d^2x_2 + 2bdx_4).$$

Proof. The proof uses the description of the lines using Plücker coordinates and the 2×4 matrices discussed at the start of this section. Details can be found in [13, Section 4.2]. □

Remark 3.4. In view of our last result, we can see how Corollary 1.5 embeds the point scheme $\mathcal{V}(x_1x_2x_3)$ of A' into the line scheme of A. More precisely, for $i \leq 3$, the line $\mathcal{V}(x_i)$ in $\mathbb{P}^2$ embeds in the line scheme of A as the line L_i from Theorem 3.1. Moreover, the plane that houses the point scheme of A' can be identified with the plane $\mathcal{V}(M_{12},\ M_{13},\ M_{23}) \subset \mathbb{P}^5$.

Theorem 3.5. *The line scheme, $\mathcal{L}_B$, of B has dimension one. If $\lambda \neq 0$, then, viewing $\mathcal{L}_B$ as a subscheme of $\mathbb{P}^5$ via the Plücker embedding, the reduced variety of $\mathcal{L}_B$ is the union of a line L, a conic C_1 and a degree-8 curve C_2, where*

$$L = \mathcal{V}(M_{12},\ M_{13},\ M_{23},\ M_{34}) \subset \mathbb{P}^5,$$

$$C_1 = V(M_{12},\ M_{13},\ M_{23},\ M_{14}M_{24} - M_{34}^2) \subset \mathbb{P}^5,$$

and the affine open subset of C_2 where the first coordinate is nonzero is given by

$$\begin{aligned} \mathcal{V}\big(M_{12} - 1,\ 2M_{13}M_{23} + 1,\ M_{14} - \tfrac{1}{2}(M_{23}^2 - 2M_{13}^4 + (\lambda - 2)M_{13}),& \\ M_{24} - \tfrac{1}{2}(2M_{23}^4 + \lambda M_{23} - M_{13}^2),\ M_{34} + M_{23}^3 + M_{13}^3 + \tfrac{1}{2}\big) \subset \mathbb{A}^5.& \end{aligned} \tag{3.1}$$

As subschemes of $\mathcal{L}_B$, L has multiplicity four, C_1 has multiplicity four and C_2 has multiplicity one. Moreover, $L \cap C_1 = L \cap C_2 = C_1 \cap C_2 = L \cap \mathcal{V}(M_{14}M_{24})$.

Proof. As in the proof of Theorem 3.1, if $\dim(\mathcal{L}_B) = 1$, then there are no embedded components. We first compute the closed points of $\mathcal{L}_B$, after which we discuss the multiplicities of the points. For any omitted details, the reader is referred to [15], where the case $\lambda = 2$ is discussed.

In seeking the reduced variety of $\mathcal{L}_B$, we note that, for points where M_{12} is zero, we also have

$$M_{13} = 0 = M_{23} \qquad \text{and} \qquad M_{34}(M_{14}M_{24} - M_{34}^2) = 0,$$

which shows that $L \cup C_1 \subset \mathcal{L}_B \cap \mathcal{V}(M_{12})$. Moreover, there are no other solutions in this case. On the other hand, if $M_{12} \neq 0$, then inverting M_{12} and computing a Gröbner basis (with standard lexicographical ordering) shows that the remaining points of $\mathcal{L}_B$ yield C_2. It follows that the first two assertions in the statement are proved.

Let $\mathfrak{p} \in L \setminus (C_1 \cup C_2)$. In homogeneous coordinates, $\mathfrak{p}$ may be written as

$$\mathfrak{p} = (0,\, 0,\, 1,\, 0,\, \gamma,\, 0),$$

for some $\gamma \in \Bbbk^\times$. We wish to intersect L at $\mathfrak{p}$ with a hyperplane $\mathcal{V}(f)$, where

$$f = M_{24} - \gamma M_{14} + \beta_1 M_{12} + \beta_2 M_{13} + \beta_3 M_{23} + \beta_4 M_{34},$$

for some $\beta_1, \ldots, \beta_4 \in \Bbbk$. Using the same method as that employed in the proof of Theorem 3.1, and after computing a Gröbner basis that has zero locus containing only $\mathfrak{p}$, we find that the ideal, $\hat{J}_B$, generated by this last Gröbner basis is generated by three polynomials in two variables, and the quotient ring is isomorphic to $\Bbbk[y,\, z]_{\langle y,\, z\rangle}$ modulo the ideal generated by

$$y^3, \qquad (\gamma^3+1)y^2 - 2\gamma^2 yz, \qquad 4\gamma^4 z^2 - (\gamma^6 + 4\gamma^3 + 1)y^2,$$

so the quotient ring has dimension at most four. A computation using Singular, [8], verified the dimension to be four, and so did a computation with the Affine package in Maxima, [17]. In fact, the latter revealed that the only denominators used in the application of Bergman's Diamond Lemma were elements of $\Bbbk^\times\gamma^4$, which are nonzero. It follows that the multiplicity of any point of L that lies on only L is four.

A similar procedure may be used to compute the multiplicity of a point $\mathfrak{p} \in C_1 \setminus (L \cup C_2)$, where, in homogeneous coordinates, we may write

$$\mathfrak{p} = (0,\, 0,\, 1,\, 0,\, \gamma^2,\, \gamma),$$

for some $\gamma \in \Bbbk^\times$. In this case, we may take the hyperplane $\mathcal{V}(f)$ to be given by

$$f = M_{24} - \gamma^2 M_{14} + \beta_1(M_{34} - \gamma M_{14}) + \beta_2 M_{12} + \beta_3 M_{13} + \beta_4 M_{23},$$

for some $\beta_1, \ldots, \beta_4 \in \Bbbk$. Assuming that the values of $\beta_1, \ldots, \beta_4$ are generic, the ideal $\hat{J}_B$ is generated, in this case, by six polynomials in two variables, where the quotient ring is isomorphic to $\Bbbk[y,\,z]_{\langle y,\,z\rangle}$ modulo the ideal generated by

$$y^3,\quad y^2z,\quad yz^2,\quad z^3,\quad \alpha_1y^2+\alpha_2yz,\quad \alpha_3y^2+\alpha_4z^2,$$

where $\alpha_1,\ldots,\alpha_4$ are scalars that depend on γ, λ, $\beta_1,\ldots,\beta_4$. Hence, the quotient ring has dimension at most four. A computation using Singular, [8], verified the dimension to be four, and so did a computation with the Affine package in Maxima, [17]. In fact, the latter revealed that the only denominators used in the application of Bergman's Diamond Lemma were nonzero scalar multiples of powers of γ and of $2\gamma+\beta_1$, and thus are nonzero for a generic hyperplane $\mathcal{V}(f)$. It follows that the multiplicity of any point of C_1 that lies on only C_1 is four.

An analogous argument for the curve C_2 entails using a point of the form

$$\mathfrak{p}=\big(1,\;\;\gamma,\;\;\tfrac{1}{8\gamma^2}(1+4(\lambda-2)\gamma^3-8\gamma^6),\;\;-\tfrac{1}{2\gamma},\;\;\tfrac{1}{16\gamma^4}(1-4\lambda\gamma^3-8\gamma^6),\;\;\tfrac{1}{8\gamma^3}(1-4\gamma^3-8\gamma^6)\big),$$

where $\gamma\in\Bbbk^\times$, and a hyperplane $\mathcal{V}(f)$, with

$$f=\gamma M_{12}-M_{13}+\sum_{1\le i\le 4}\beta_iN_i,$$

where

$$N_1=M_{12}+2\gamma M_{23},$$

$$N_2=8\gamma^3M_{24}+(1-4\lambda\gamma^3-8\gamma^6)M_{23},$$

$$N_3=4\gamma^2M_{34}+(1-4\gamma^3-8\gamma^6)M_{23},$$

$$N_4=M_{14}-\gamma M_{34}+(\lambda-1)\gamma^2M_{23},$$

and $\beta_1,\ldots,\beta_4\in\Bbbk$. In this case, by inverting M_{12}, one may reduce the number of variables to two by using the polynomials in (3.1). Intersecting with $\mathcal{V}(f)$ and taking $\beta_4\neq 0$, one may solve for the image of M_{13}, leaving only one variable, namely, the image of M_{23}. In this manner, one obtains an ideal in a ring isomorphic to $\Bbbk[y]_{\langle 1+2\gamma y\rangle}$ that is generated by a polynomial of the form $(1+2\gamma y)g$, where $g\in\Bbbk[y]_{\langle 1+2\gamma y\rangle}$. In this setting, $\mathfrak{p}$ corresponds to the solution of $1+2\gamma y=0$. If the values of $\beta_1,\ldots,\beta_4$ are generic, then $g|_{\mathfrak{p}}\neq 0$ and $\deg(g)=7$. It follows that the multiplicity of $\mathcal{L}_B$ at $\mathfrak{p}$ is one, and that a generic hyperplane intersects C_2 at eight points (counted with multiplicity), so $\deg(C_2)=8$.

It is straightforward to see that $L\cap C_1=L\cap\mathcal{V}(M_{14}M_{24})$. In order to consider where C_2 meets L or C_1, we first compute, using degree reverse lexicographical ordering (with the aid of algebra software, such as Mathematica, [16]), a Gröbner basis for the image of J_B from inverting M_{12}, after which we homogenize the elements of the Gröbner basis. The ideal generated by these homogenized elements and the element M_{12} is generated by M_{23}^4, $M_{13}^3 + M_{23}^3$, M_{34}^4, $2M_{14}M_{24} - M_{34}^2$ and a dozen other polynomials that vanish on the closed points in the zero locus of the given four polynomials. The last assertion in the statement now follows. □

We observe that computing the degree of $\mathcal{L}_B$ from the multiplicities and degrees of the components of $\mathcal{L}_B$ confirms a total degree of $4 + 2(4) + 8 = 20$, as expected from [7, Proposition 3.2].

Remark 3.6. In light of Theorem 3.5, it is clear how Corollary 1.5 embeds the point scheme $\mathcal{V}(x_3) \cup \mathcal{V}(x_1x_2 - x_3^2)$ of B' into the line scheme of B. The line $\mathcal{V}(x_3)$ in $\mathbb{P}^2$ corresponds to the line L in Theorem 3.5, and the conic $\mathcal{V}(x_1x_2 - x_3^2)$ in $\mathbb{P}^2$ corresponds to the conic C_1 in Theorem 3.5. Also, the plane that houses the point scheme of B' can be identified with the plane $\mathcal{V}(M_{12},\ M_{13},\ M_{23}) \subset \mathbb{P}^5$.

Corollary 3.7.

(*a*) *The lines in* $\mathbb{P}^3$ *that are parametrized by* L *are the lines that lie on the plane* $\mathcal{V}(x_3) \subset \mathbb{P}^3$ *and pass through the point* $e_4 = \mathcal{V}(x_1,\ x_2,\ x_3) \in \mathbb{P}^3$.

(*b*) *The lines in* $\mathbb{P}^3$ *that are parametrized by* C_1 *are those that lie on the singular quadric* $\mathcal{V}(x_1x_2 - x_3^2) \subset \mathbb{P}^3$ *and pass through the singular point,* e_4*, of the quadric.*

(*c*) *For every* $\lambda \in \Bbbk^\times$*, the lines in* $\mathbb{P}^3$ *that are parametrized by* $C_2 \setminus (L \cup C_1)$ *are all the lines*

$$\mathcal{V}(x_3 - a_3x_1 - \tfrac{1}{2a_3}x_2,\quad x_4 - a_4x_1 - b_4x_2)$$

where $a_3 \in \Bbbk^\times$, $\ a_4 = a_3(\frac{1}{8a_3^3} - a_3^3 + \frac{\lambda}{2})$ *and* $b_4 = \frac{1}{2a_3}(a_3^3 - \frac{1}{8a_3^3} + \frac{\lambda}{2} - 1)$.

Proof. For (c), the defining polynomials of C_2 imply that a line parametrized by $C_2 \setminus (L \cup C_1)$ may be depicted using a rank-2 2×4 matrix

$$\begin{bmatrix} 1 & 0 & a_3 & a_4 \\ 0 & 1 & b_3 & b_4 \end{bmatrix},$$

where the entries are scalars that satisfy the equations

$$\begin{aligned} b_4 &= \tfrac{1}{2}(a_3^2 - 2b_3^4 + (\lambda - 2)b_3), & 0 &= 2a_3b_3 - 1, \\ a_4 &= -\tfrac{1}{2}(2a_3^4 - \lambda a_3 - b_3^2), & 0 &= a_3b_4 - a_4b_3 - a_3^3 + b_3^3 + \tfrac{1}{2}. \end{aligned}$$

However, the last equation follows from the previous three equations, and substitution of $b_3 = \frac{1}{2a_3}$ into a_4 and b_4 leads to the result for (c). The proof of (a) and (b) is similar and is omitted. □

Since the point schemes of A and B are finite, it is difficult to see if the point scheme in each case is informing us of anything useful about elements in each algebra. It is therefore reasonable to investigate whether or not the line scheme might be informing us of such data.

In particular, regarding the algebra A, the line L_i is contained in the plane $\mathcal{V}(M_{12}, M_{13}, M_{23})$, for all $i \leq 3$, and every component of $\mathcal{L}_A$ is contained in the hypersurface $\mathcal{V}(M_{12})$. Within the algebra A, the elements x_1^2 and x_2^2 are central elements, so possibly the line scheme is noting this behavior. Moreover, the images of $dx_1 - bx_2$ and of $(dx_1 - bx_2)x_3$ are normal elements in the factor algebra $A/\langle x_1^2, x_2^2\rangle$, and the images of x_3^2 and of x_1x_2 are normal elements in the factor algebra $A/\langle x_1^2, x_2^2, (dx_1 - bx_2)x_3\rangle$. Turning to the algebra B, both the line L and the conic C_1 are contained in the plane $\mathcal{V}(M_{12}, M_{13}, M_{23})$, and, in B, the elements x_1^2 and x_2^2 are central elements. Although some of this geometric behavior is a consequence of Theorem 1.4 and of A (respectively, B) being an Ore extension of A' (respectively, of B'), this relationship between the line scheme and elements in A and B is reminiscent of observations noted in [5,24] regarding certain quadratic quantum $\mathbb{P}^3$s that are regular graded skew Clifford algebras ([4]), and perhaps suggests, more generally, that the line scheme might encode algebraic properties of certain elements in a quadratic quantum $\mathbb{P}^3$.

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References

[1] M. Artin and W. Schelter, *Graded algebras of global dimension* 3, Adv. in Math., 66(2) (1987), 171-216.

[2] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, in The Grothendieck Festschrift 1, Eds. P. Cartier et al., Progr. Math., 86 (1990), 33-85.

[3] M. Artin, J. Tate and M. Van den Bergh, *Modules over regular algebras of dimension* 3, Invent. Math., 106(2) (1991), 335-388.

[4] T. Cassidy and M. Vancliff, *Generalizations of graded Clifford algebras and of complete intersections*, J. Lond. Math. Soc. (2), 81(1) (2010), 91-112. (Corrigendum: 90(2) (2014), 631-636.)

[5] R. G. Chandler and M. Vancliff, *The one-dimensional line scheme of a certain family of quantum* $\mathbb{P}^3$*s*, J. Algebra, 439 (2015), 316-333.

[6] A. Chirvasitu and S. P. Smith, *Exotic elliptic algebras of dimension* 4, with an appendix by D. Tomlin, Adv. Math., 309 (2017), 558-623.

[7] A. Chirvasitu, S. P. Smith and M. Vancliff, *A geometric invariant of* 6*-dimensional subspaces of* 4×4 *matrices*, Proc. Amer. Math. Soc., 148(3) (2020), 915-928.

[8] W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann, *Singular* 4-4-0 — *A Computer Algebra System for Polynomial Computations*, 2024: https://www.singular.uni-kl.de.

[9] P. Goetz, E. Kirkman, W. F. Moore and K. B. Vashaw, *Some Artin-Schelter regular algebras from dual reflection groups and their geometry*, (2024), arXiv:2410.08959v1 [math.RA].

[10] P. Goetz and B. Shelton, *Representation theory of two families of quantum projective* 3*-spaces*, J. Algebra, 295(1) (2006), 141-156.

[11] T. Levasseur, *Some properties of non-commutative regular graded rings*, Glasgow Math. J., 34(3) (1992), 277-300.

[12] T. Levasseur and S. P. Smith, *Modules over the* 4*-dimensional Sklyanin algebra*, Bull. Soc. Math. France, 121(1) (1993), 35-90.

[13] I. C. Lim, Some Quadratic Quantum $\mathbb{P}^3$s with a Linear One-Dimensional Line Scheme, Ph.D. Dissertation, Univ. of Texas at Arlington, 2021: https://mavmatrix.uta.edu/math_dissertations/174/.

[14] J. E. Lozano, Point Modules and Line Modules of Certain Quadratic Quantum Projective Spaces, Ph.D. Dissertation, Univ. of Texas at Arlington, 2024: https://mavmatrix.uta.edu/math_dissertations/2/.

[15] A. Mastriania IV, Some Quadratic Regular Algebras on Four Generators with a 1-Dimensional Nonreduced Line Scheme, Ph.D. Dissertation, Univ. of Texas at Arlington, 2019: https://mavmatrix.uta.edu/math_dissertations/214/.

[16] Mathematica, Version 12.0, Wolfram Research Inc., Champaign, IL, 2019.

[17] Maxima, Version 5.47.0 — A Computer Algebra System, 2023: https://maxima.sourceforge.io/.

[18] B. Shelton and M. Vancliff, *Schemes of line modules I*, J. London Math. Soc. (2), 65(3) (2002), 575-590.

[19] B. Shelton and M. Vancliff, *Schemes of line modules II*, Comm. Algebra, 30(5) (2002), 2535-2552.

[20] D. R. Stephenson, *Artin-Schelter regular algebras of global dimension three*, J. Algebra, 183(1) (1996), 55-73.

[21] D. R. Stephenson, *Algebras associated to elliptic curves*, Trans. Amer. Math. Soc., 349(6) (1997), 2317-2340.

[22] D. R. Stephenson, *Quantum planes of weight* $(1, 1, n)$, J. Algebra, 225(1) (2000), 70-92.

[23] D. R. Stephenson and M. Vancliff, *Some finite quantum* $\mathbb{P}^3$*s that are infinite modules over their centers*, J. Algebra, 297(1) (2006), 208-215.

[24] D. Tomlin and M. Vancliff, *The one-dimensional line scheme of a family of quadratic quantum* $\mathbb{P}^3$*s*, J. Algebra, 502 (2018), 588-609.

[25] M. Vancliff, *The interplay of algebra and geometry in the setting of regular algebras*, in Commutative Algebra and Noncommutative Algebraic Geometry, Eds. D. Eisenbud et al., Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, New York, 67 (2015), 371-390.

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