NP P RINGS, REDUCED RINGS AND SNF RINGS

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Received: 5 September 2007; Revised: 21 January 2008
Communicated by W. Keith Nicholson

Abstract. A ring $R$ is called left NP P if for any nilpotent element $a$ of $R$, $l(a) = Re, e^2 = e \in R$. A right $R$-module $M$ is called $Nflat$ if for each $a \in N(R)$, the $Z$-module map $1_M \otimes i : M \otimes R a \rightarrow M \otimes R$ is monic, where $i : Ra \hookrightarrow R$ is the inclusion map. A ring $R$ is called right SNF if every simple right $R$-module is $Nflat$. In this paper, we first show that a ring $R$ is left NP P iff every sum of two injective submodules of a left $R$-module is nil--injective. And some properties of left NP P rings are given, for example, if $R$ is left NP P, so is $eRe$ for any $e^2 = e \in R$ satisfying $ReR = R$. Next, we study some properties of reduced rings. A ring $R$ is reduced if and only if $R$ is $ZC$ and right SNF if and only if $R$ is left and right NP P and $R$ has no subrings which is isomorphic to the upper triangular matrix $UT_2(Z)$ or $UT(Z_p)^2$ for some prime $p$. Finally, we give some characterizations of $n$--regular rings, for example, a ring $R$ is $n$--regular if and only if every right $R$--module is $Nflat$.

Mathematics Subject Classification (2000): 16E50, 16D30

Keywords: nil--injective modules, NP P rings, reduced rings, NC2 rings, $n$--regular rings, SNF rings, $Nflat$--modules.

1. Introduction

Throughout $R$ denotes an associative ring with identity and all modules are unitary. For a subset $X$ of $R$, the left (right) annihilator of $X$ in $R$ is denoted by $l(X)$ ($r(X)$). If $X = \{a\}$, we usually abbreviate it to $l(a)$ ($r(a)$). We write $J(R)$, $Z_l(R)(Z_r(R))$, $N(R)$, $Z(R)$ for the Jacobson radical, the left (right) singular ideal, the set of nilpotent elements, the set of central elements of $R$, respectively.

A left $R$--module $M$ is called nil--injective [8] if every left $R$--homomorphism from a principal left ideal $Ra$ with $a \in N(R)$ to $M$ extends to one from $R\bar{R}$ to $M$. The ring $R$ is called left nil--injective if $R\bar{R}$ is nil--injective. Note that left principally injective rings are nil--injective, but the converse is not true by [8, Example 2.2]. A ring $R$ is called left NP P if for any $a \in N(R)$, $l(a) = Re, e^2 =
$e \in R$. Clearly, left $pp$ ring (that is: for each $a \in R, l(a) = Re, e^2 = e \in R$) is left $NPP$, but the converse is not true by [8, Example 2.8]. A ring $R$ is called left $C_2$ if $Ra$ projective implies $Ra = Re, e^2 = e \in R$ for all $a \in N(R)$. Clearly, left $C_2$ ring [7] is left $NC_2$ and by [8, Corollary 2.7], left nil--injective ring is left $NC_2$. But the converse are all not true by [8, Example 2.21 and Example 2.5]. A ring $R$ is called left $NC_2$ if $Ra$ projective implies $Ra = Re, e^2 = e \in R$ for all $a \in N(R)$. Clearly, von Neumann regular rings are $n$--regular, but the converse is not true by [8, Remark 2.19]. A ring $R$ is called $n$--regular if $aRa$ for all $a \in N(R)$. Clearly, von Neumann regular rings are $n$--regular, but the converse is not true by [8, Example 2.8]. A ring $R$ is called reduced if $N(R) = 0$, or equivalently, $a^2 = 0$ implies $a = 0$ in $R$ for all $a \in R$. Clearly, a reduced ring is left nil--injective, left $NP P$ and left $NC_2$. In this paper, we first give some characterizations of left $NP P$ rings and study some properties of left $NP P$ rings. Next, we consider some conditions for a ring $R$ being reduced. Finally, we introduce right $Nflat$ modules and right $SNF$ rings, giving some characterizations of $n$--regular rings and reduced rings in terms of them.

2. Left $NP P$ rings

**Theorem 2.1.** The following conditions are equivalent for a ring $R$.

(1) $R$ is left $NP P$.

(2) Every factor module of an injective left $R$--module is nil--injective.

(3) Every sum of two injective submodules of a left $R$--module is nil--injective.

(4) Every sum of two isomorphic injective submodules of a left $R$--module is nil--injective.

**Proof.** (3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (1) are trivial. (1) $\Rightarrow$ (2) follows from [8, Theorem 2.10(1)].

(2) $\Rightarrow$ (3) Let $N_1$ and $N_2$ be two injective submodules of a left $R$--module $M$. Since $N_1 \oplus N_2$ is injective and there is an epimorphism $N_1 \oplus N_2 \twoheadrightarrow N_1 + N_2$, $N_1 + N_2$ is nil--injective.

(4) $\Rightarrow$ (2) Let $M$ be an injective left $R$--module and $N$ a submodule. Let $U = M \oplus M, V = \{ (n, n) \mid n \in N \}, U/V, M_1 = \{ m, 0 \} \in U \mid m \in M \},$ and $M_2 = \{ (0, m) \in U \mid m \in M \}$. Then $U = M_1 + M_2$ and $M_i \cong M (i = 1, 2)$, so $U$ is nil--injective by (4). Since $M_1$ is injective, $M_1$ is a summand of $U$ and $U/M_1$ is isomorphic to a summand of $U$. Hence $U/M_1$ is nil--injective. Now there is a canonical isomorphism $M/N \cong U/M_1$, via $m + N \mapsto (0, m) + M_1$ and so $M/N$ is nil--injective.

We denote by $M_n(R)$ the ring of $n$ by $n$ matrices over $R$. Since Morita equivalence preserves summands, epimorphisms, and monomorphisms, it must preserve projective modules. Hence we have the following theorem.
The following conditions are equivalent for a ring $R$.

(1) If $r \in N(R)$ and $I$ be the principal left ideal of $M_2(R)$ generated by the diagonal matrix $\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$. Then $I$ is a projective left $M_2(R)$-module. By [4, Theorem 3.2], there is a Morita equivalence between $M_2(R)$-modules and $R$-modules via $M \rightarrow eM$, where $M$ is a left $M_2(R)$-module and $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Now $eI \cong Rr$ as $R$-modules, so $Rr$ is a projective $R$-module. Hence $R$ is left NPP.

(2) $R$ is left NPP if for each $a \in N(R)$, $Rr$ is flat. Clearly, left NPP ring is left NPF. We have the following theorem.

Theorem 2.3. The following conditions are equivalent for a ring $R$.

1. $R$ is left NPP.
2. $R$ is left NPF and for each $a \in N(R)$, $l(a)$ is finitely generated as a left $R$-module.
3. For each non-empty subset $X$ of $R$, for each $a \in r(X) \cap N(R)$, there exists $a \in r(X)$ such that $a = ba$.

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Let $\phi \neq X \subseteq R$ and $a \in r(X) \cap N(R)$. Then $Ra$ is finitely presented flat left $R$-module by (2), so $Ra$ is projective as left $R$-module. Hence $l(a) = Re$, $e_1 = e \in R$. Since $(1 - e)R = r(e) = r(Re) = r(l(a)) \subseteq rl(X) = r(X)$ and $a \in rl(a)$, $a = (1 - e)a$. Set $b = 1 - e \in r(X)$ Then $a = ba$.

(3) $\Rightarrow$ (1) Let $a \in N(R)$. Then $a \in r(l(a)) \cap N(R)$, so $a = fa$ for some $f \in r(l(a))$ by (3). Since $1 - f \in l(a) \subseteq l(f)$, $f = f^2$ and $R(1 - f) \subseteq l(a)$. Now let $x \in l(a)$. Then $xf = 0$, so $x = x(1 - f) \in R(1 - f)$. Hence $l(a) = R(1 - f)$, which implies $R$ is left NPP.

It is well known that for any left ideal $K$ of a ring $R$, $R/K$ is a flat left $R$-module if and only if for any $x \in K$, there exists $y \in K$ such that $x = xy$. Hence we have the following theorem.

Theorem 2.4. Let $e^2 = e \in R$ and $S = eRe$. Then

1. If $R$ is NPF, so is $S$.
2. Let $ReR = R$ and $x \in N(S)$. If $l_R(x)$ is finite generated as a left $R$-module, so is $l_S(x)$ as a left $S$-module.
3. Let $ReR = R$. If $R$ is left NPP, so is $S$. 

Let \( \text{NP} P \in \mathbb{F} \) for all \( \mathbb{F} \). Hence there exists \( z \in l_R(x) \) such that \( y = yz \). Thus \( y = eye = yeze \). Since \( \text{NP} P \in \mathbb{F} \), \( \text{NP} \) and \( \text{NP} \) are left \( \mathbb{F} \)-module and so \( \mathbb{F} \) is left \( \text{NP} P \).

(2) Let \( l_R(x) = \sum_{i=1}^{n} Ra_i \) where \( a_i \in R \). Since \( R = ReR, 1 = \sum_{j=1}^{m} u_j e v_j \) where \( u_j, v_j \in R \). Let \( z \in l_S(x) \), then \( z \in l_R(x) \). Set \( z = \sum_{i=1}^{n} c_i a_i \). Then \( z = \sum c_i u_j e v_j a_i e \). So, clearly, as a left \( S \)-module, \( l_S(x) \) is generated by \( e v_j a_i e, i = 1, 2, \cdots, m; j = 1, 2, \cdots, n \).

(3) follows from (1), (2) and Theorem 2.3.

By definition, we have the following theorem.

**Theorem 2.5.** Let \( R = \prod_{i \in I} R_i \) be the direct product of rings \( \{ R_i | i \in I \} \). Then

(1) \( R \) is left \( \text{NP} P \) if and only if \( R_i \) is left \( \text{NP} P \) for all \( i \in I \).

(2) \( R \) is left \( NPP \) if and only if \( R_i \) is left \( NPP \) for all \( i \in I \).

**Theorem 2.6.** (1) Left \( \text{NP} P \) rings have no nonzero central nilpotent elements.

(2) Left \( NPP \) rings have no nonzero central nilpotent elements.

(3) If \( R \) is left \( \text{NP} P \), then \( Z(R) \) is reduced.

(4) If \( R \) is left \( NPP \), then \( Z(R) \) is reduced.

**Proof.** (1) Let \( R \) be left \( \text{NP} P \) and \( x \in Z(R) \) with \( x^n = 0 \) and \( x^{n-1} \neq 0 \). Since \( R/l(x) \cong Rx \) is flat and \( x^{n-1} l(x), x^{n-1} = x^{n-1} y \) for some \( y \in l(x) \). Since \( yx = 0 \) and \( x \in Z(R), xy = 0 \). Hence \( x^{n-1} = 0 \), which is a contradiction. So left \( \text{NP} P \) rings have no nonzero central nilpotent elements.

(2), (3) and (4) follow from (1).

[8, Theorem 2.9] shows that \( R \) is reduced if and only if \( R \) is abelian left \( NPP \), where a ring \( R \) is **abelian** if every idempotent of \( R \) is central. [8, Theorem 2.24] shows that \( R \) is \( n \)-regular if and only if \( R \) is left \( NPP \) left \( NC2 \). A ring \( R \) is called **NI** if \( N(R) \) forms an ideal of \( R \). A ring \( R \) is called **2-primal** if \( N(R) = P(R) \), where \( P(R) \) is the prime radical of \( R \). A ring \( R \) is called **ZC** if \( ab = 0 \) implies that \( ba = 0 \) for all \( a, b \in R \). Clearly, (1) \( ZC \) rings are abelian, **NI** and **2-primal**; (2) abelian rings are \( NC2 \); (3) \( 2-primal \) rings are **NI**.

**Theorem 2.7.** The following conditions are equivalent for ring \( R \).

(1) \( R \) is reduced.

(2) \( R \) is \( n \)-regular and abelian.

(3) \( R \) is \( n \)-regular and \( N(R) \) forms a left ideal of \( R \).
(4) $R$ is $n$–regular and $N(R)$ forms a right ideal of $R$.
(5) $R$ is $n$–regular and $NI$.
(6) $R$ is $n$–regular and 2–primal.
(7) $R$ is left NPF and ZC.
(8) $R$ is left nil–injective left nonsingular and $NI$.

**Proof.** (1) $\iff$ (2) and (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) or (5) $\Rightarrow$ (3) are trivial.

We will prove (3) $\Rightarrow$ (4). The (4) $\Rightarrow$ (1) is similar. Let $a \in R$ with $a^2 = 0$. Then $a = aba$ for some $b \in R$ because $R$ is $n$–regular. Let $e = ba$. Then $e^2 = e$ and $a = ae$. Since $N(R)$ is a left ideal of $R$ and $a \in N(R)$, $e = ba \in N(R)$. So $e = 0$ and then $a = ae = 0$. Hence $R$ is reduced.

(1) $\Rightarrow$ (7) follows from [8, Theorem 2.9] and Theorem 2.3.

(7) $\Rightarrow$ (1) Let $x \in R$ with $x^2 = 0$. Then $R/\langle x \rangle$ is flat left $R$–module by (7). So $x = xy$ for some $y \in \langle x \rangle$ because $x \in \langle x \rangle$. Since $R$ is ZC, $xy = 0$ because $yx = 0$. Thus $x = xy = 0$.

(1) $\iff$ (8) follows from [8, Theorem 2.9].

Now we consider the $n \times n$ upper triangular matrix ring $UTR_n$ over a ring $R$.

**Theorem 2.8.** Let $R$ be a ring and $n \geq 2$. Then

(1) If $UTR_n$ is left NPF, so is $R$.
(2) If $UTR_n$ is left NPP, so is $R$.

**Proof.** (1) Let $a \in N(R)$. Then $A = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in N(UTR_n)$.

Since $UTR_n$ is left NPF, $UTR_n/\langle l \rangle_{UTR_n}(A)$ is flat left $UTR_n$–module. For any $b \in l_\langle a \rangle$, $B = \begin{pmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \in l_{UTR_n}(A)$. So there exists $C = \begin{pmatrix} c_1 & c_{12} & c_{13} & \cdots & c_{1n} \\ 0 & c_2 & c_{23} & \cdots & c_{2n} \\ 0 & 0 & c_3 & \cdots & c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_n \end{pmatrix} \in l_{UTR_n}(A)$ such that $B = BC$. Clearly, $c_1 \in l_\langle a \rangle$ and $b = bc_1$. This shows that $R$ is left NPF.
Based on the above preceding result, we consider a kind of subring of $n \times n$ upper triangular matrix rings. For a ring $R$, we consider the ring

$$SUT R_n = \left\{ \begin{pmatrix} b & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b & a_{23} & \cdots & b_{2n} \\ 0 & 0 & b & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} | b, b_{ij} \in R \right\}. $$

Then by a similar proof proceeding of Theorem 2.8, we have the following:

**Theorem 2.9.** Let $R$ be a ring and $n \geq 2$. Then

1. If $SUT R_n$ is left NP F, so is $R$.
2. If $SUT R_n$ is left NP P, so is $R$.

Let $R$ be a ring and $M$ a bimodule over $R$. The trivial extension of $R$ and $M$ is $R \ltimes M = \{(a, x) | a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$.

In fact, $R \ltimes M$ is isomorphic to the subring $\left\{ \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} : a \in R, x \in M \right\}$ of the formal $2 \times 2$ matrix ring $\left( \begin{array}{cc} R & M \\ 0 & R \end{array} \right)$, and $R \ltimes R \cong R[x]/(x^2)$. If $\sigma : R \rightarrow R$ is a ring endomorphism, let $R[x; \sigma]$ denote the ring of skew polynomials over $R$; that is all formal polynomials in $x$ with coefficients from $R$ with multiplication defined by $xr = \sigma(r)x$. Note that if $R(\sigma)$ is the $(R, R)$–bimodule defined by $R \sigma = R R(\sigma)$ and $m \circ r = m \sigma(r)$, for all $m \in R(\sigma)$ and $r \in R$, then $R[x; \sigma]/(x^2) \cong R \ltimes R(\sigma)$. Similar to the proof proceeding of Theorem 2.8, we have the following theorem.

**Theorem 2.10.** (1) If one of the following rings is left NP F, so is $R$.

1. $R \ltimes M$. 2. $R \ltimes R$. 3. $R \ltimes R(\sigma)$. 4. $R[x]/(x^2)$.

(2) If one of the following rings is left NP P, so is $R$.

1. $R \ltimes M$. 2. $R \ltimes R$. 3. $R \ltimes R(\sigma)$. 4. $R[x]/(x^2)$.

It is well known that there exists a reduced ring $R$ which is not left pp. We claim that neither $UTR_2$ nor $SUT R_2$ is left NPP. In fact, since $R$ is not left pp, there exists $a \in R$ such that $l_R(a)$ is not a direct summand of $R R$. Then $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in N(UTR_2)$. If $UTR_2$ is left NPP, then $l_{UTR_2}(A) = UTR_2 E$. 

(2) It is similar to (1). \qed
where $E^2 = E = \begin{pmatrix} e_1 & x \\ 0 & e_2 \end{pmatrix} \in UTR_2$. By computing, we have $e_1^2 = e_1 \in R$ and $l_R(a) = Re_1$, which is a contradiction. Hence $UTR_2$ is not left NPP. Similarly, we can show that $SUTR_2$ is not left NPP. Hence there exists a left NPP ring $R$ such that neither $UTR_2$ nor $SUTR_2$ is left NPP.

3. Reduced rings

In this section, we will prove that a NPP ring $R$ is reduced if and only if $R$ contains no subrings which are isomorphic to the matrix rings $UT\mathbb{Z}_2$ or $UT(\mathbb{Z}_p)_2$, where $\mathbb{Z}$ denotes the integer ring and $p$ is a prime number. For a ring $R$, let $E(R)$ denotes the set of all idempotents of $R$.

We begin with the following theorem.

**Theorem 3.1.** Let $R$ be NPP. Then the following conditions are equivalent.

1. $R$ is reduced.
2. $ef = fe$ for all $e, f \in E(R)$.
3. $E(R)$ is a subsemigroup of the semigroup $(R, \cdot)$.
4. $ef = 0$ if and only if $fe = 0$ for all $e, f \in E(R)$.
5. $N(R) \cap Re = N(R) \cap eR$ for all $e \in E(R)$.
6. $R$ is NI ring and $eN(R) = N(R)e$ for all $e \in E(R)$.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (6) $\Rightarrow$ (5) are trivial.

(3) $\Rightarrow$ (4) Let $e, f \in E(R)$ and $ef = 0$. By (3), $fe \in E(R)$. So $fe = fefe = f(ef)e = 0$.

(4) $\Rightarrow$ (5) Let $x \in N(R) \cap Re$. Then $x(1-e) = 0$, so $1-e \in r(x) = (1-f)R$ for some $f^2 = f \in R$ because $R$ is NPP and $x \in N(R)$. Hence $f(1-e) = 0$, by (4), $(1-e)f = 0$. Clearly, $1-e + (1-e)x \in E(R)$ and $f(1-e + (1-e)x) = 0$. By (4), $(1-e + (1-e)x)f = 0$. Thus $(1-e)xf = 0$. Since $r(x) = (1-f)R$, $(1-e)x(1-f) = 0$. So $(1-e)x = 0$. Hence $x = ex \in N(R) \cap eR$. This shows that $N(R) \cap Re \subseteq N(R) \cap eR$. Similarly, we can show $N(R) \cap eR \subseteq N(R) \cap Re$.

(5) $\Rightarrow$ (1) Let $x \in R$ with $x^2 = 0$. Since $R$ is NPP, $l(x) = Re, e^2 = e \in R$. So $x \in Re \cap N(R)$ and then $x \in eR$ by (5). Hence $x = ex$ and so $x = 0$ because $l(x) = Re$. Thus $R$ is reduced. □

We first denote by $o(r)$ the additive order of $r \in R$, that is, the smallest positive integer $n$ such that $nr = 0$. If $r$ is of infinite order, then we simply write $o(r) = \infty$.

The following theorem is a generalization of [3, Lemma 3.1]. For convenience, we give its brief proof.
Theorem 3.2. Let $R$ be NPP such that $ef = 0$ and $fe \neq 0$ for some $e, f \in E(R)$. Then, $o(e) = o(f) = o(fe)$. And if $o(e) < \infty$, then there exist $u, v \in E(R)$ and a prime $p$ such that $o(u) = o(v) = o(uv) = p$ with $uv = 0$ but $vu \neq 0$.

Proof. Since $R$ is NPP, by Theorem 3.1, $R$ is not reduced. Since $ef = 0$, $fe \in N(R)$. So $l(fe) = R(1 - g)$ and $r(fe) = (1 - h)R$ for some $g, h \in E(R)$. These lead to $l(fe) = l(g)$ and $r(fe) = r(h)$. Thus $g = fg$ because $1 - f \in l(fe) = l(g)$, so $gf \in E(R)$ and $l(g) = l(gf)$. Hence $fe = gff e = gfe$ because $1 - gf \in l(gf) = l(g) = l(fe)$. Similarly, there exists $h \in E(R)$ such that $h = he, eh \in E(R), r(ef) = r(fe)$ and $fe = feh$. Hence, $fe = gfeh = (gf)(eh)$ and $(ef)(gf) = efgh = 0$. Clearly, $o(gf) = o(ef) = o(fe)$. So if $o(gf) = \infty$, there is nothing to prove. If $o(gf) = pk$, where $p$ is a prime number. Then $o(kfe) = p$. By using similar arguments as above, we have $u, v \in E(R)$ such that $o(u) = o(v) = o(kfe)$ with $uv = 0$ but $vu \neq 0$. □

The following theorem also is a generalization of [3, Theorem 3.2].

Theorem 3.3. Let $R$ be NPP. Then $R$ is reduced if and only if $R$ has no subrings which are isomorphic to $UT\mathbb{Z}_2$ or $UT(\mathbb{Z}_p)_2$, where $p$ is a prime.

Proof. Since $UT\mathbb{Z}_2$ and $UT(\mathbb{Z}_p)_2$ both contain some non-commuting idempotents, by Theorem 3.1, the necessity part is clear.

To prove the sufficiency part, we suppose that $R$ is not reduced. Then by Theorems 3.1 and 3.2, there exist $e, f \in E(R)$ such that $ef = 0$, $fe \neq 0$ and $o(e) = o(f) = o(fe)$; and $o(e) = o(f) = o(fe) = p$ if $o(e) < \infty$, where $p$ is a prime. Consider the subring of $R$ generated by $e$ and $f$. Clearly, $0, e, f, fe$ forms a subsemigroup of $R$ under ring multiplication and so $S = \{af + bfe + ce \mid a, b, c \in \mathbb{Z}\}$ forms a subring of $R$.

Now let $\theta : UT\mathbb{Z}_2 \rightarrow S$ defined by $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto af + (b - c)fe + ce$. Then $\theta$ is a surjective homomorphism.

If $o(e) = o(f) = o(fe) = \infty$, then $\theta$ is an isomorphism.

If $o(e) = o(f) = o(fe) = p$, then $ker\theta = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid p|a, p|b, p|c \}$. Since $UT\mathbb{Z}_2/ker\theta \cong UT(\mathbb{Z}_p)_2$, $S \cong UT(\mathbb{Z}_p)_2$. This is a contradiction and therefore our proof is completed. □

A ring $R$ is called left GC2 [9] if for $a \in R$ and $R Ra \cong_R R$, $Ra = Re$ for some $e^2 = e \in R$. A right GC2 ring is defined similarly. A ring $R$ is called strongly regular if $a \in a^2 R$ for all $a \in R$. Since strongly regular rings are left and right C2
NPP rings, reduced rings and SNF rings

Theorem 3.4. The following conditions are equivalent for a ring $R$.

(1) $R$ is strongly regular.
(2) $R$ is abelian, left pp and left GC2.
(3) $R$ is abelian, left pp and right GC2.
(4) $R$ is von Neumann regular and $N(R)$ forms a left ideal of $R$.
(5) $R$ is von Neumann regular and $NI$.
(6) $R$ is von Neumann regular and 2-primal.

Proof. (1) ⇒ (2) and (1) ⇒ (6) ⇒ (5) ⇒ (4) are trivial.

(2) ⇒ (1) Let $a \in R$. Since $R$ is left pp, $l(a) = Re, e^2 = e \in R$. Set $b = a + e$.
Then $l(b) \subseteq l(a) \cap l(e) = 0$ because $R$ is abelian. Clearly, $(1 - e)b = (1 - e)a = a$. Since $gRb \cong_R R$ and $R$ is left GC2, $Rb = Rg, g^2 = g \in R$. Hence $b = bg = gb$, so $g = 1$ because $R$ is abelian and $l(b) = 0$. So $Ra = R(1 - e)b = (1 - e)Rb = (1 - e)Rg = (1 - e)R1 = (1 - e)R = R(1 - e)$, this implies $R$ is von Neumann regular and so $R$ is strongly regular because $R$ is abelian.

Similarly, we can show (3) ⇒ (1).

(4) ⇒ (1) By (4), $R$ is n-regular. Since $N(R)$ is a left ideal of $R$, $R$ is reduced by Theorem 2.7. So $R$ is strongly regular.

Recall that an additive subgroup $L$ of a ring $R$ is said to be a quasi-ideal if $xrx \in L$ and $xrx \in L$ whence $x \in L$ and $r \in R$. Obviously, every ideal of $R$ is a quasi-ideal. But there exists an example of a (four-dimensional) Banach algebra $A$ whose quasi-ideal $Y$ is not an ideal, since $A = A * Y$ is the exterior (Grassmann) algebra on a two dimensional real vector space $Y$ [5]. A ring $R$ is called left MC2 if $l(k) = Re, e^2 = e \in R$ whence $Rk$ is a projective minimal left ideal of $R$. By [8, Theorem 2.22], left NC2 rings are left MC2. But the converse is not true by [8, Remark 2.23]. A left $R$-module $M$ is called Wnil-injective [8] if for any $0 \neq a \in N(R)$ (if there exists), there exists a positive integer $n$ such that $a^n \neq 0$ and every left $R$-homomorphism from $Ra^n$ to $M$ extends to one from $R$ to $M$. Clearly, left nil-injective modules and left $YJ$-injective modules [6] are all Wnil-injective.

Theorem 3.5. The following conditions are equivalent for a ring $R$.

(1) $R$ is reduced.
(2) $R$ is n-regular and $N(R)$ is a quasi-ideal of $R$ such that $aN(R) = N(R)a$ for all $a \in N(R)$. 

[7]; and left (resp. right) $C2$ rings are left (resp. right) GC2; strongly regular rings are left and right GC2.
(3) $R$ is left MC2 and $NI$ such that every simple singular left $R$–module is $\text{Wnil}$–injective.

(4) $R$ is abelian and $N(R)$ forms a right ideal of $R$ whose simple singular left $R$–modules are $\text{Wnil}$–injective.

(5) $R$ is $ZT$ and for any $a \in N(R)$, $l(Ra) = Re, e^2 = e \in R$.

**Proof.** $(1) \Rightarrow (i), i = 2, 3, 4, 5$ are clear.

$(2) \Rightarrow (1)$ By $(2), a = aba$ for all $a \in N(R)$. Since $N(R)$ is a quasi-ideal of $R$ and $a \in N(R), bab \in N(R)$. Thus $ab = abab = a(bab) \in aN(R) = N(R)a$ and so $a = aba \in N(R)a^2$. This implies $R$ is reduced.

$(3) \Rightarrow (1)$ Let $a \in R$ such that $a^2 = 0$. We claim that $RaR + l(a) = R$. If not, there exists a maximal left ideal $M$ containing $RaR + l(a)$. If $M$ is not essential in $R$, then $M = l(e), e^2 = e \in R$. Since $R$ is left MC2 ring, $R$ is semiprime by [8, Corollary 3.6]. Since $eaR \in RaR \subseteq M = l(e), eaRe = 0$. Hence $eaRea = 0$ and so $ea = 0$ because $R$ is semiprime. Thus $e \in l(a) \subseteq M = l(e)$, which is a contradiction. Hence $M$ is essential in $R$, and so $R/M$ is $\text{Wnil}$–injective. This implies there exists $b \in R$ such that $1 - ab \in M$ and so $1 \in M$ because $ab \in RaR$, which is a contradiction. So $RaR + l(a) = R$ and then $a = ya$ for some $y \in RaR$. Since $R$ is $NI$ and $a \in N(R), y \in N(R)$. Hence $y^n = 0$ for some positive integer $n$. So $a = ya = y^2a = \cdots = y^na = 0$.

$(4) \Rightarrow (1)$ Let $a \in R$ with $a^2 = 0$. If $a \neq 0$, then $l(a) \neq R$, so there exists a left ideal $L$ of $R$ such that $l(a) \oplus L$ is essential in $R$. If $l(a) \oplus L \neq R$, there exists a maximal left ideal $M$ of $R$ containing $l(a) \oplus L$. Clearly, $M$ is essential left ideal of $R$, by hypothesis, $R/M$ is $\text{Wnil}$–injective. So there exists $c \in R$ such that $1 - ac \in M$. Since $N(R)$ is a right ideal of $R, ac \in N(R)$, so $1 - ac$ is invertible. Hence $M = R$, which is a contradiction. This shows $l(a) \oplus L = R$. Let $l(a) = Re, e^2 = e \in R$. Clearly, $a = ac = ea = 0$, which is a contradiction. So $a = 0$.

$(5) \Rightarrow (1)$ Let $a^2 = 0$. By $(5), l(Ra) = Re$. Let $x \in l(a)$. Then $xa = 0$, so $xRa = 0$ because $R$ is $ZT$. Hence $x \in l(Ra)$, this shows that $l(a) = l(Ra)$ and so $l(a) = Re$. Since $R$ is $ZT, R$ is abelian. So $a = 0$. □

**Theorem 3.6.** The following conditions are equivalent for a ring $R$.

$(1)$ $R$ is reduced.

$(2)$ $R$ is $ZC$ and every essential maximal left ideal of $R$ is $\text{Wnil}$–injective.

$(3)$ $R$ is semiprime left nonsingular and for any $a \in N(R), Ra$ is an ideal of $R$.

$(4)$ $R$ is semiprime left nonsingular and for any $a \in N(R), Ra$ is a left annihilator of a left ideal of $R$. 

The following conditions are equivalent for a ring $R$:

(1) $R$ is regular.

(2) Every left $R$–module is $Wnil$–injective.

(3) Every cyclic left $R$–module is $Wnil$–injective.

(4) $R$ is left $Wnil$–injective and left $NPP$.

Proof. (1) $\Rightarrow$ (4) $\Rightarrow$ (3) and (1) $\Rightarrow$ (2) are trivial.

(2) $\Rightarrow$ (1) Let $a \in R$ with $a^2 = 0$ and $L$ a left ideal of $R$ such that $l(a) \oplus L$ is essential left ideal of $R$. If $l(a) \oplus L \neq R$, then there exists an essential maximal left ideal $M$ of $R$ containing $l(a) \oplus L$. By hypothesis, $R_M$ is $Wnil$–injective. So the inclusion map $Ra \hookrightarrow M$ can be extended to $R \longrightarrow M$, this implies $a = am$ for some $m \in M$. Since $R$ is $ZC$, $a = ma$. So $1 - m \in l(a) \subseteq M$, which is a contradiction. So $l(a) \oplus L = R$. Then, clearly, $a = 0$ because $R$ is abel.

(3) $\Rightarrow$ (1) Let $a^2 = 0$ and $L$ a left ideal of $R$ such that $l(a) \cap L = 0$. Since $Ra$ is an ideal of $R$, $aL \subseteq Ra$. Hence $aL \subseteq l(a) \cap L = 0$, so $(La)^2 = 0$. Since $R$ is semiprime, $La = 0$. So $L \subseteq l(a)$ and then $L = 0$. Therefore $l(a)$ is an essential left ideal of $R$. But $R$ is left nonsingular, so $a = 0$.

4. $n$–regular rings

In [8, Theorem 2.18], we have shown that a ring $R$ is $n$–regular if and only if every left $R$–module is $nil$–injective. Since $nil$–injective modules are $Wnil$–injective, we can generalize this theorem as follows:

**Theorem 4.1.** The following conditions are equivalent for a ring $R$.

(1) $R$ is $n$–regular.

(2) Every left $R$–module is $Wnil$–injective.

(3) Every cyclic left $R$–module is $Wnil$–injective.

(4) $R$ is left $Wnil$–injective and left $NPP$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4) are trivial.

(3) $\Rightarrow$ (1) Let $a \in N(R)$. By (3), $R_R$ is $Wnil$–injective. If $a^2 = 0$, then the identity map $I : Ra \longrightarrow Ra$ can be extended to $R \longrightarrow R$, this implies there exists $b \in R$ such that $a = aba$. If $a^2 \neq 0$, then there exists a positive integer $n$ such that $a^n \neq 0$ and any left $R$–homomorphism $Ra^n \longrightarrow Ra$ can be extended to $R \longrightarrow Ra$. Set $f : Ra^n \longrightarrow Ra$ is the inclusion map, then, clearly, $f = ca, c \in R$. So $a^n = f(a^n) = a^nca$. Let $d = a^{n-1} - a^{n-1}ca$. Then $d^2 = 0$. By the above proof, we can obtain that $d = a^{n-1} - a^{n-1}ca$ is regular element of $R$. By [1, Lemma 2.1], $a^{n-1} = a^{n-1}da$ for some $d \in R$. Repeating the above-mentioned process, we can obtain $v \in R$ such that $a = avv$.

(4) $\Rightarrow$ (1) Let $0 \neq a \in N(R)$. Since $R$ is left $Wnil$–injective, there exists a positive integer $n$ such that $a^n \neq 0$ and $rl(a^n) = a^nR$. Since $R$ is left $NPP$ and $a^n \in N(R)$, $l(a^n) = R(1 - e), e^2 = e \in R$. Hence $eR = r(R(1 - e)) = rl(a^n) = a^nR$. This implies that $a^n$ is a regular element of $R$. If $a^2 = 0$, the argument above shows
that $a$ is a regular element. So, by [1, Theorem 2.2], even if $a^2 \neq 0$, $a$ is also a regular element of $R$.

It is well known that $R$ is von Neumann regular if and only if every essential left ideal of $R$ is $YJ$–injective. And note that the direct summand of a nil–injective (resp. $Wnil$–injective) module is nil–injective (resp. $Wnil$–injective). So we can give the following theorem:

**Theorem 4.2.** The following conditions are equivalent for a ring $R$.

1. $R$ is $n$–regular.
2. Every essential left ideal of $R$ is nil–injective as left $R$–module.
3. Every essential left ideal of $R$ is $Wnil$–injective as left $R$–module.
4. Every direct product (or sum) of cyclic left $R$–modules is nil–injective.
5. Every direct product (or sum) of cyclic left $R$–modules is $Wnil$–injective.
6. $R$ is left nil–injective and cyclic singular left $R$–modules are nil–injective.

**Proof.** (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (6) $\Leftrightarrow$ (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follows from Theorem 4.1 and [8, Theorem 2.18].

(3) $\Rightarrow$ (1) Let $a \in R$ with $a^2 = 0$. Clearly there exists a left ideal $L$ of $R$ such that $Ra \oplus L$ is essential left ideal of $R$. By (3), $Ra \oplus L$ is $Wnil$–injective, so $Ra$ is $Wnil$–injective. Hence, by the proof of (3) $\rightarrow$ (1) in Theorem 4.1, we have $a = aba$ for some $b \in R$. 

A right $R$–module $M$ is called $Nflat$ if for any $a \in N(R)$, the map $1_M \otimes i : M \otimes_R Ra \rightarrow M \otimes_R R$ is monic, where $i : Ra \rightarrow R$ is the inclusion map. Clearly, flat modules are $Nflat$.

By definition, we know that every module over any reduced ring is $Nflat$. Since there exists a reduced ring $R$ which is not von Neumann regular, there exists a module over $R$ which is not flat. So there exists a $Nflat$ module which is not flat. The following proposition is trivial.

**Proposition 4.3.** (1) The direct sum $\bigoplus_{i \in I} M_i$ of left $R$–modules $\{M_i | i \in I\}$ is $Nflat$ if and only if each $M_i$ is $Nflat$.

(2) If $\{M_i | i \in I\}$ is a direct system of $Nflat$ modules, then the direct limit of these modules is also $Nflat$.

(3) If every finitely generated submodule of a right $R$–module $M$ is $Nflat$, then $M$ is $Nflat$.

(4) If $M_R$ is a module such that every cyclic submodule of $M$ is contained in a $Nflat$ submodule then $M$ is $Nflat$. 
Let $R$ and $S$ be rings and $B$ an $(S,R)$–bimodule. Then for any left $R$–module $A$ and left $S$–module $C$, we have a left $\mathbb{Z}$–module isomorphism map:

$$\tau_{A,C} : \text{Hom}_S(B \otimes_R A, C) \rightarrow \text{Hom}_R(A, \text{Hom}_S(B, C))$$

where $\tau_{A,C}(h) : A \rightarrow \text{Hom}_S(B, C)$

$$a \mapsto \tau_{A,C}(h)(a)$$

satisfies $\tau_{A,C}(h)(a)(b) = h(b \otimes a)$ for all $b \in B$.

**Theorem 4.4.** Let $R$ and $S$ be rings, $B$ an $(S,R)$–bimodule. If $B_R$ is $N$flat, $C$ is injective left $S$–module, then as a left $R$–module, $\text{Hom}_S(B,C)$ is $\text{nil}$–injective.

**Proof.** Let $a \in N(R)$ and $f : Ra \rightarrow \text{Hom}_S(B,C)$ be any left $R$–homomorphism. Since $B_R$ is $N$flat, $1_B \otimes i : B \otimes_R Ra \rightarrow B \otimes_R R$ is monic. Since $S_C$ is injective, $(1_B \otimes i)^* : \text{Hom}_S(B \otimes_R R,C) \rightarrow \text{Hom}_S(B \otimes_R Ra,C)$ is epic. Since we have the following commutating diagram:

$$\begin{array}{cccc}
\text{Hom}_S(B \otimes_R C) & \xrightarrow{\tau_{R,C}} & \text{Hom}_R(R, \text{Hom}_S(B,C)) \\
(1_B \otimes i)^* & & & i^* \\
\text{Hom}_S(B \otimes Ra, C) & \xrightarrow{\tau_{Ra,C}} & \text{Hom}_R(Ra, \text{Hom}_S(B,C))
\end{array}$$

$i^* : \text{Hom}_R(R, \text{Hom}_S(B,C)) \rightarrow \text{Hom}_R(Ra, \text{Hom}_S(B,C))$ is epic. Hence there exists a left $R$–homomorphism $h : R \rightarrow \text{Hom}_S(B,C)$ such that $i^*(h) = f$, that is $hi = f$ or equivalently, $h|_{Ra} = f$. This shows that $\text{Hom}_S(B,C)$ is $\text{nil}$–injective as a left $R$–module. \qed

**Theorem 4.5.** Right $R$–module $B$ is $N$flat if and only if $B^* \overset{\text{def}}{=} \text{Hom}_Z(B, \mathbb{Q}/\mathbb{Z})$ is $\text{nil}$–injective, where $\mathbb{Q}$ is the field of real numbers.

**Proof.** Let $B$ be $N$flat. Since $\mathbb{Q}/\mathbb{Z}$ is an injective left $\mathbb{Z}$–module, $B^*$ is $\text{nil}$–injective as a left $R$–module by Theorem 4.4.

Converse, assume that $B^*$ is a $\text{nil}$–injective left $R$–module. Let $a \in N(R)$. We show that $1_B \otimes i : B \otimes Ra \rightarrow B \otimes R$ is monic.

Since we have the following commutating diagram:

$$\begin{array}{cccc}
\text{Hom}_Z(B \otimes_R R, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\tau_{R,\mathbb{Q}/\mathbb{Z}}} & \text{Hom}_R(R, \text{Hom}_Z(B, \mathbb{Q}/\mathbb{Z})) \\
(1_B \otimes i)^* & & & i^* \\
\text{Hom}_Z(B \otimes Ra, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\tau_{Ra,\mathbb{Q}/\mathbb{Z}}} & \text{Hom}_R(Ra, \text{Hom}_Z(B, \mathbb{Q}/\mathbb{Z}))
\end{array}$$

where $\tau_{R,\mathbb{Q}/\mathbb{Z}}$ and $\tau_{Ra,\mathbb{Q}/\mathbb{Z}}$ are $\mathbb{Z}$–isomorphism, $(1_B \otimes i)^*$ is epic if and only if $i^*$ is epic.
Since $B^* = \text{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z})$ is nil–injective left $R$–module, $i^*$ is epic. Hence $(1_B \otimes i)^*$ is also epic. Since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator in $\mathbb{Z}$–module category, $(1_B \otimes i)$ is a monic. This shows that $B_R$ is $Nflat$. □

Theorem 4.6. (1) Let $B$ be an $Nflat$ right $R$–module and $a \in N(R)$. Then there exists a unique $\mathbb{Z}$–module isomorphism $\theta : B \otimes Ra \longrightarrow Ba$ satisfies $\theta(b \otimes ra) = bra$ for all $b \in B$ and $r \in R$.

(2) Let $B$ be a right $R$–module and there exists a right $R$–short exact sequence:

$$0 \longrightarrow K \overset{J}{\longrightarrow} F \overset{g}{\longrightarrow} B \longrightarrow 0$$

where $F$ is $Nflat$. Then $B_R$ is $Nflat$ if and only if $K \cap Fa = Ka$ for all $a \in N(R)$.

(3) Let $M_R$ be $Nflat$ and $U$ a submodule of $M_R$. Then $M/U$ is $Nflat$ if and only if $Ua = U \cap Ma$ for all $a \in N(R)$.

(4) Let $I$ be a right ideal of $R$. Then $R/I$ is $Nflat$ right $R$–module if and only if $Ia = I \cap Ra$ for all $a \in N(R)$.

Proof. (1) Let $f : B \times Ra \longrightarrow Ba$ satisfy $f((b, ra)) = bra, b \in B, r \in R$. Clearly, $f$ is an $R$–tensorial mapping, so there exists a unique $\mathbb{Z}$–homomorphism

$$\theta : B \otimes Ra \longrightarrow Ba, \quad b \otimes ra \longmapsto bra$$

such that the following diagram is commutative:

$$\begin{array}{ccc}
B \times Ra & \xrightarrow{h} & B \otimes Ra \\
\downarrow f & & \downarrow \theta \\
Ba & \xrightarrow{I} & Ba
\end{array}$$

where $I : Ba \longrightarrow Ba$ is the identity mapping and $h : B \times Ra \longrightarrow B \otimes Ra, b \times ra \longmapsto b \otimes ra$.

Clearly, $\theta(b \otimes ra) = bra$ and $\theta$ is epic. Since $B_R$ is $Nflat$, $1_B \otimes i : B \otimes Ra \longrightarrow B \otimes R$ is monic. Since $\psi : B \otimes R \longrightarrow B, b \otimes 1 \longmapsto b$ is a $\mathbb{Z}$–isomorphism, $\theta = \psi(1_B \otimes i)$ is monic. Hence $\theta$ is an isomorphism.

(2) Since $\otimes Ra$ is right exact, there is an exact sequence:

$$K \otimes Ra \overset{j \otimes 1}{\longrightarrow} F \otimes Ra \overset{g \otimes 1}{\longrightarrow} B \otimes Ra \longrightarrow 0.$$

Since $F_R$ is $Nflat$, by (1), there exists a unique $\mathbb{Z}$–isomorphism $\rho : F \otimes Ra \longrightarrow Fa$ satisfying $\rho(x \otimes ra) = xra$ for all $x \in F$ and $r \in R$. So there is a $\mathbb{Z}$–epic mapping $(g \otimes 1)\rho^{-1} : Fa \longrightarrow B \otimes Ra$. Since $\text{Ker}((g \otimes 1)\rho^{-1}) = Ka$, there is a
Z–isomorphism $\gamma : Fa/Ka \longrightarrow B \otimes Ra$ satisfying $\gamma(xa + Ka) = g(x) \otimes a$ for all $x \in F$.

On the other hand, $\delta : Ba \longrightarrow Fa/(K \cap Fa)$ defined by $\delta(ba) = xa + (K \cap Fa)$, where $g(x) = b, x \in F, b \in B$ is $Z$–isomorphism. Hence we obtain $Z$–homomorphism $\sigma = \delta \theta \gamma : Fa/Ka \longrightarrow Fa/K \cap Fa$ satisfying $\sigma(xa + Ka) = xa + (K \cap Fa)$, $x \in F$.

Since $Ka \subseteq K \cap Fa$, $\sigma$ is a $Z$–isomorphism mapping if and only if $Ka = K \cap Fa$.

The if part: Assume that $B_R$ is $N_{flat}$, By (1), $\theta : B \otimes Ra \longrightarrow Ba$ is a $Z$–isomorphism mapping, so $Ka = K \cap Fa$.

The only if part: Since $Ka = K \cap Fa$ for all $a \in N(R)$, $\theta : B \otimes Ra \longrightarrow Ba$ is a $Z$–isomorphism mapping. By the following commutating diagram:

\[
\begin{array}{ccc}
B \otimes Ra & \xrightarrow{1_b \otimes i} & B \otimes R \\
\downarrow \theta & & \downarrow \psi \\
Ba & \xrightarrow{i} & B
\end{array}
\]

where $i : Ba \hookrightarrow B$ is the inclusion mapping, we have that $1_B \otimes i$ is monic. Hence $B_R$ is $N_{flat}$.

(3) and (4) follow from (2). \qed

**Theorem 4.7.** The following conditions are equivalent for a ring $R$.

(1) $R$ is $n$–regular.

(2) Every right $R$–module is $N_{flat}$.

(3) Every cyclic right $R$–module is $N_{flat}$.

**Proof.** (2) $\Rightarrow$ (3) is trivial.

(1) $\Rightarrow$ (2) Let $M$ be any right $R$–module. Then there is a right $R$–short exact sequence $0 \longrightarrow K \xrightarrow{i} F \xrightarrow{\pi} M \longrightarrow 0$ where $F_R$ is free. For any $a \in N(R)$, we always have $Ka \subseteq K \cap Fa$. Let $x \in K \cap Fa$. Then $x = za$ for some $z \in F$. Since $R$ is $n$–regular and $a \in N(R)$, $a = aba$ for some $b \in R$. Set $e = ba$, then $a = ae$ and $e = e^2 = ba \in Ra$. Clearly, $x = za = zae = xe \in Ka$. This shows that $Ka = K \cap Fa$ for all $a \in N(R)$ and so $M_R$ is $N_{flat}$.

(3) $\Rightarrow$ (1) Let $a \in N(R)$. Since $R/aR$ is a cyclic right $R$–module, $R/aR$ is $N_{flat}$ by (3). In terms of the following right $R$–short exact sequence

\[
0 \longrightarrow aR \xrightarrow{i} R \xrightarrow{\pi} R/aR \longrightarrow 0
\]

we have $aRa = aR \cap Ra$. So $a \in aR \cap Ra = aRa$. Thus $R$ is $n$–regular. \qed
Call a ring \( R \) left (resp. right) SNF if every simple left (resp. right) \( R \)-module is \( N \text{flat} \). By Theorem 4.7, \( n \)-regular rings are SNF. Call a ring \( R \) is left (resp. right) quasi-duo if every maximal left (resp. right) ideal of \( R \) is an ideal. A ring \( R \) is called MELT (resp. MERT) if every essential maximal left (resp. right) ideal of \( R \) is an ideal. A ring \( R \) is called left SF if every simple left \( R \)-module is flat.

Clearly, a left SF ring is left SNF, but the converse is not true. Because there exists a reduced MELT ring \( R \) which is not von Neumann regular, there exists a reduced MELT ring \( R \) which is not left SF by [10, Theorem 1]. On the other hand, by Theorem 4.7, reduced rings are left SNF, so there exists a left SNF ring which is not left SF.

**Theorem 4.8.** \( R \) is \( n \)-regular if and only if \( R \) is right SNF and every maximal submodule of any cyclic right \( R \)-module is \( N \text{flat} \).

**Proof.** The necessity follows from Theorem 4.7.

The sufficiency: Let \( a \in N(R) \). Then \( aR \neq R \), so there exists a maximal right ideal \( M \) of \( R \) such that \( aR \subseteq M \). Since \( (R/aR)/(M/aR) \cong R/M \), \( M/aR \) is a maximal submodule of cyclic right \( R \)-module \( R/aR \). So \( M/aR \) is \( N \text{flat} \) by hypothesis. Since \( M \) is a maximal submodule of cyclic right \( R \)-module \( R \), \( M \) is \( N \text{flat} \). In terms of Theorem 4.6 and the following right \( R \)-short exact sequence:

\[
0 \rightarrow aR \overset{j}{\rightarrow} M \overset{\pi}{\rightarrow} M/aR \rightarrow 0
\]

we have \( aRa = aR \cap Ma \) because \( a \in N(R) \). Since \( R \) is right \( SNF \) ring and \( R/M \) is simple right \( R \)-module, \( R/M \) is \( N \text{flat} \). Hence, by Theorem 4.6, \( Ma = M \cap Ra \), so \( a \in M \cap Ra = Ma \). Thus \( a \in Ma \cap aR = aRa \), obtaining that \( R \) is \( n \)-regular. \( \square \)

**Theorem 4.9.** (1) Let \( R \) be left quasi-duo. Then \( R \) is reduced if and only if \( R \) is right \( SNF \).

(2) Let \( R \) be right \( SF \). Then

(a) If \( R \) is MELT, then \( R \) is left nonsingular;

(b) If \( R \) is left MC2 and MELT, then \( R \) is semiprime and right nonsingular.

(3) Let \( R \) be right \( SNF \). If \( r(M) \) is essential in \( R_R \) for all maximal right ideal \( M \) of \( R \), then \( R \) is reduced.

(4) \( R \) is reduced if and only \( R \) is \( ZC \) and right \( SNF \).

**Proof.** (1) The if part is clear by Theorem 4.7.

The only if part: Let \( a \in R \) with \( a^2 = 0 \). If \( a \neq 0 \), then there exists a maximal left ideal \( M \) containing \( l(a) \). Since \( R \) is left quasi-duo, \( M \) is an ideal of \( R \). So there exists a maximal right ideal \( L \) of \( R \) such that \( M \subseteq L \). Since \( R/L \) is a simple right

$x \in M$ give a contradiction. Hence can show that $a = 0$ and so $R$ is reduced.

(2) (a) If $Z_l(R) \neq 0$, then there exists a $0 \neq a \in Z_l(R)$ such that $a^2 = 0$. So there exists a maximal left ideal $M$ of $R$ containing $l(a)$. Clearly, $M$ is an essential left ideal. Since $R$ is $MELT$, $M$ is an ideal of $R$. By the proof of (1), we can obtain a contradiction. Hence $Z_l(R) = 0$.

(b) First, we show that $R$ is semiprime. Let $a \in R$ satisfy $aRa = 0$. If $a \neq 0$, then there exists a maximal left ideal $M$ of $R$ containing $l(a)$. If $M$ is not an essential left ideal of $R$, then $M = l(e)$ for some $e^2 = e \in R$. Since $RaR \subseteq l(a)$, $aRe = 0$. Since $R$ is left MC2, $eRa = 0$. So $e \in l(a) \subseteq M = l(e)$, which is a contradiction. Hence $M$ is an essential left ideal. The rest proof is similar to (1).

Next, we show that $Z_r(R) = 0$. If not, there exists a $0 \neq a \in Z_r(R)$ such that $a^2 = 0$. We claim that $Z_r(R) + l(a) = R$. If not, there exists a maximal left ideal $M$ of $R$ containing $Z_r(R) + l(a)$. Since $R$ is left MC2, similar to the proof of (a), we can show that $M$ is an essential left ideal. By the proof proceeding of (1), we shall give a contradiction. Hence $Z_r(R) + l(a) = R$. Let $1 = z + x$, where $z \in Z_r(R)$ and $x \in l(a)$. Then $a = za$ and so $(1 - z)a = 0$. Since $z \in Z_r(R)$ and $r(z) \cap r(1 - z) = 0$, $r(1 - z) = 0$. Hence $a = 0$, which is a contradiction. This shows that $R$ is right nonsingular.

(3) Let $a \in R$ satisfy $a^2 = 0$. If $a \neq 0$, then $r(a) \subseteq M$ where $M$ is a maximal right ideal of $R$. Since $R$ is right SNF, $R/M$ is an $N\text{flat}$ right $R$–module. By Theorem 4.6, $a = xa$ for some $x \in M$, so $a \in r(1 - x)$. Since $r(M) \subseteq r(x)$ and $r(M)$ is an essential right ideal of $R$, $x \in Z_r(R)$. Hence $r(1 - x) = 0$, which implies that $a = 0$, which is a contradiction. Thus $a = 0$.

(4) Let $a \in R$ satisfy $a^2 = 0$. If $a \neq 0$, then, similar to the proof of (3), there exists $x \in M$ such that $a = xa$ where $M$ is a maximal right ideal of $R$ containing $r(a)$. Since $R$ is $ZC$, $a = ax$. Hence $1 - x \in r(a) \subseteq M$ and so $1 \in M$, which is a contradiction. Thus $a = 0$.

\[\Box\]

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