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REPRESENTATIONS OF LIE ALGEBRAS ARISING FROM POLYTOPES

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Abstract. We present an extremely elementary construction of the simple Lie algebras over $\mathbb{C}$ in all of their minuscule representations, using the vertices of various polytopes. The construction itself requires no complicated combinatorics and essentially no Lie theory other than the definition of a Lie algebra; in fact, the Lie algebras themselves appear as by-products of the construction.

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Introduction

The simple Lie algebras over the complex numbers are objects of key importance in representation theory and mathematical physics. These algebras fall into four infinite families (A_n, B_n, C_n, D_n) and five exceptional types (E_6, E_7, E_8, F_4 and G_2). The classical (i.e., non-exceptional) types of Lie algebras are easily defined in terms of Lie algebras of matrices; such representations are called the natural representations of the Lie algebras. However, it is not so easy to give similar descriptions of the exceptional algebras in a way that makes it easy to carry out calculations with them. Another natural question is whether one can give easy descriptions of other representations of the classical Lie algebras, such as the spin representations of algebras of types B_n and D_n, which are traditionally constructed in terms of Clifford algebras (see [2, §13.5]).

There are several combinatorial approaches to the representation theory of the simple Lie algebras over $\mathbb{C}$. Two of these include Littelmann's description of representations in terms of paths, and the crystal basis approach of Kashiwara and the Kyoto school. Both these approaches are very versatile but can be combinatorially complicated. Recent work of the author shows how to construct certain Lie algebra representations using combinatorial structures called "full heaps", whose theory is developed in [7,8]. This in turn builds on work of Stembridge [14] on minuscule elements and their heaps, and on work of Wildberger [15] on constructing minuscule representations for simply laced simple Lie algebras over $\mathbb{C}$ (although the paper [15] does not contain proofs). The approach of the present paper grew out of an attempt to explain the full heap representations in as simple a way as possible, and it does not require any complicated combinatorial constructions. The simple Lie algebras of types E_8, F_4 and G_2 have no minuscule representations and do not appear to fit directly into our framework, so we do not consider them here. It may well be possible to treat type G_2 by modifying the arguments here, just as the methods of [15] may be adapted to treat type G_2 in an ad hoc way (see [16]).

The polytopes we consider in this paper are convex subsets of $\mathbb{R}^n$ whose vertices (i.e., 0-skeletons) have integer coordinates; such polytopes are sometimes called "lattice polytopes". These include the hypercube, the hyperoctahedron (which is the dual of the hypercube) and the polytopes known as 2_{21} and 3_{21} in Coxeter's notation [5]; the latter two polytopes have 27 and 56 vertices respectively. All these polytopes are highly symmetrical, and the symmetry groups have been known for a long time. The reason that these polytopes are relevant in Lie theory is that the set of weights for the minuscule representations of simple Lie algebras over $\mathbb{C}$ form the vertices of one of the aforementioned polytopes. This is not obvious, but it is not a complete surprise either: Manivel [13, Introduction] for example mentions in passing that the weights of the 56-dimensional representation of $\mathfrak{e}_7$ correspond to the vertices of 3_{21}.

Our approach in this paper is to start with the vertices of the polytope and use them to construct representations of Lie algebras *without* first constructing the Lie algebras themselves. All the minuscule representations of simple Lie algebras over $\mathbb{C}$ may be constructed in this way, and the construction is remarkably simple. In §2, we introduce the notion of a "minuscule system", which involves two subsets of $\mathbb{R}^n$, denoted by Ψ and Δ. The set Ψ is said to be a minuscule system with respect to the simple system Δ if two conditions are satisfied (see Definition 2.1). These conditions are very elementary and easy to check, and whenever they hold, the set Δ defines a set of linear operators on a vector space with dimension $|\Psi|$ (Definition 2.2). If one makes a judicious choice of Ψ and Δ, then these linear operators turn out to be the representations of the Chevalley generators of a simple Lie algebra over $\mathbb{C}$ acting in one of its minuscule representations with respect to an obvious basis (the basis can be shown to be the crystal basis in the sense of [12], by adapting the argument of [7, §8]). We will show that all minuscule representations can be constructed in exactly this way. The dimension of the space containing Ψ and Δ is in some cases much smaller than the dimension of the representation being constructed and the dimension of the corresponding Lie algebra.

In all our examples here, Ψ and Δ are finite sets, and the set Δ is recognizable as either the set of simple roots for a simple Lie algebra, or as the set of simple roots together with $\alpha_0 = -\theta$, where θ is the highest root. In the latter case, we obtain finite dimensional representations of certain derived affine Kac–Moody algebras. Formulating the results in terms of affine algebras can be more natural, as the affine algebras have a greater degree of symmetry. Another advantage is that it is easier to see how the modules behave under restriction; for example, the 56-dimensional module for the Lie algebra of type E_7, after inflation to a module for the derived affine algebra, can be restricted to a module for the Lie algebra of type A_7 which is the direct sum of two nonisomorphic 28-dimensional irreducible submodules. Once this observation is made, our approach here to $\mathfrak{e}_7$ is seen to be very natural.

The layout of this paper is as follows. In §1, we recall some of the basic theory of representations of Lie algebras. Minuscule systems are defined in §2, and developed in §3. Our main result is Theorem 3.2, and the linear operators used in it are defined in Definition 2.2. Sections 4–6 are devoted to examples of minuscule systems. §4 describes the representations of $\mathfrak{e}_6$ and $\mathfrak{e}_7$ arising from the polytopes 2_{21} and 3_{21} respectively. §5 describes various representations arising from the hypercube, including the spin representations of the Lie algebras of types B_n and D_n. The minuscule representations of type A_{n-1} are obtained by restricting the spin representation in type B_n to a subalgebra. §6 describes representations arising from the hyperoctahedron, including the natural representations of the Lie algebras of types C_n and D_n. In §7, we explore connections with algebraic geometry. We give some concluding remarks are given in §8, including a discussion of the construction of Chevalley bases for the Lie algebra.

1. Background on Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field k equipped with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the *Lie bracket*) satisfying the conditions

$$[x, x] = 0,$$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0,$$

for all $x, y, z \in \mathfrak{g}$. (These conditions are known respectively as *antisymmetry* and the *Jacobi identity*.)

If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras over a field k, then a *homomorphism* of Lie algebras from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is a k-linear map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}_1$. An *isomorphism* of Lie algebras is a bijective homomorphism.

If V is any vector space over k then the Lie algebra $\mathfrak{gl}(V)$ is the k-vector space of all k-linear maps $T : V \to V$, equipped with the Lie bracket satisfying

$$[T, U] := T \circ U - U \circ T,$$

where $\circ$ is composition of maps.

A *representation* of a Lie algebra $\mathfrak{g}$ over k is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ for some k-vector space V. In this case, we call V a (left) *module* for the Lie algebra $\mathfrak{g}$ (or a $\mathfrak{g}$-module, for short) and we say that V *affords* ρ. If $x \in \mathfrak{g}$ and $v \in V$, we write $x.v$ to mean $\rho(x)(v)$. The *dimension* of a module (or of the corresponding representation) is the dimension of V. If ρ is the zero map, then the representation ρ and the module V are said to be *trivial*.

A *submodule* of a $\mathfrak{g}$-module V is a k-subspace W of V such that $x.w \in W$ for all $x \in \mathfrak{g}$ and $w \in W$. If V has no submodules other than itself and the zero submodule, then V is said to be *irreducible*.

If V_1 and V_2 are $\mathfrak{g}$-modules, then a k-linear map $f : V_1 \to V_2$ is called a *homomorphism of* $\mathfrak{g}$*-modules* if $f(x.v) = x.f(v)$ for all $x \in \mathfrak{g}$ and $v \in V_1$. An *isomorphism of* $\mathfrak{g}$*-modules* is an invertible homomorphism of $\mathfrak{g}$-modules.

A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. If, furthermore, we have $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ (or, equivalently, $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$) then $\mathfrak{h}$ is said to be an *ideal* of $\mathfrak{g}$. If $\mathfrak{g}$ has no ideals other than itself and the zero ideal, then $\mathfrak{g}$ is said to be *simple*. The *derived algebra*, $\mathfrak{g}'$, of $\mathfrak{g}$ is the subalgebra generated by all elements $\{[x_1, x_2] : x_1, x_2 \in \mathfrak{g}\}$. It can be shown that $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$.

Definition 1.1. Let A be an $n \times n$ matrix with integer entries. Following [11, §1.1], we call $A = (a_{ij})$ a *generalized Cartan matrix* if it satisfies the following three properties:

(i) $a_{ii} = 2$ for all $1 \leq i \leq n$;
(ii) $a_{ij} \leq 0$ for all $i \neq j$;
(iii) $a_{ij} = 0 \Rightarrow a_{ji} = 0$.

We call the matrix A *symmetrizable* if there exists an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$.

The next result is a well known presentation for the derived algebra of a Kac–Moody algebra corresponding to a symmetrizable Cartan matrix.

Theorem 1.2. *Let A be a symmetrizable generalized Cartan matrix. The derived Kac–Moody algebra $\mathfrak{g} = \mathfrak{g}'(A)$ corresponding to A is the Lie algebra over $\mathbb{C}$ generated by the elements $\{e_i, f_i, h_i : i \in \Delta\}$ subject to the defining relations*

$$
\begin{aligned}
[h_i, h_j] &= 0,\\
[h_i, e_j] &= A_{ij} e_j,\\
[h_i, f_j] &= -A_{ij} f_j,\\
[e_i, f_j] &= \delta_{ij} h_i,\\
\underbrace{[e_i, [e_i, \cdots [e_i,}_{1-A_{ij}\ times} e_j] \cdots]] &= 0,\\
\underbrace{[f_i, [f_i, \cdots [f_i,}_{1-A_{ij}\ times} f_j] \cdots]] &= 0,
\end{aligned}
$$

where δ is the Kronecker delta.

Proof. This is a special case of [11, Theorem 9.11]. □

Remark 1.3. In this paper, we are mostly interested the case where A is of finite type (as defined in [11, §4.3]). In this case, the resulting algebra $\mathfrak{g}$ is simple.

Suppose for the rest of §1 that $\mathfrak{g}$ is an algebra satisfying the hypotheses of Theorem 1.2. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ spanned by the elements $\{h_i : i \in \Delta\}$. Let $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ be the dual vector space of $\mathfrak{h}$, and let $\{\omega_i : i \in \Delta\}$ be the basis of $\mathfrak{h}^*$ dual to $\{h_i : i \in \Delta\}$. Let V be a $\mathfrak{g}$-module. An element $v \in V$ is called a *weight vector* of *weight* $\lambda \in \mathfrak{h}^*$ if for all $h \in \mathfrak{h}$, we have $h.v = \lambda(h)v$. The weights ω_i are known as *fundamental weights*. If the weight vector v is annihilated by the action of all of the elements e_i (respectively, all of the elements f_i), then we call v a *highest weight vector* (respectively, a *lowest weight vector*).

The following result is well known.

Proposition 1.4. (i) *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. If λ is a non-negative $\mathbb{Z}$-linear combination of the fundamental weights ω_i then up to isomorphism there is a unique finite dimensional irreducible $\mathfrak{g}$-module $L(\lambda)$ of the form $\mathfrak{g}.v_\lambda$, where v_λ is of weight λ and is the unique nonzero highest weight vector of $L(\lambda)$. The modules $L(\lambda)$ are pairwise nonisomorphic and exhaust all finite dimensional irreducible modules of $\mathfrak{g}$.*

(ii) *Suppose that V is a finite dimensional $\mathfrak{g}$-module containing a nonzero highest weight vector v_λ of weight λ, and that* $\dim(V) = \dim(L(\lambda))$. *Then* $V \cong L(\lambda)$.

Proof. Part (i) is a special case of [2, Theorem 10.21]. For part (ii), it follows from the proof of [2, Proposition 10.13] that any $\mathfrak{g}$-module $\mathfrak{g}.v_\lambda$ generated by a highest weight vector of weight λ is a quotient of the Verma module $M(\lambda)$. The Verma module has a unique maximal submodule, $J(\lambda)$ (see [2, Theorem 10.9]) and we have $L(\lambda) = M(\lambda)/J(\lambda)$ by definition. It follows that $\mathfrak{g}.v_\lambda$ has a quotient module isomorphic to $L(\lambda)$. The assumption about dimensions allows this only if $V \cong L(\lambda)$ (and $\mathfrak{g}.v_\lambda = V$). □

If λ is a fundamental weight, the corresponding module $L(\lambda)$ is called a *fundamental module*. If, furthermore, λ has the property that

$$2\frac{\lambda.\mathbf{a}}{\mathbf{a}.\mathbf{a}} \leq 1$$

for all positive roots $\mathbf{a}$, then λ and its associated module and representation are said to be *minuscule*. (See [1, 2.11.15] for more details of the definition.) The purpose of this paper is to provide a uniform and very elementary construction of these modules. We now list the minuscule modules, their weights and their dimensions; more information on this may be found in [2, §13]. Our indexing of the weights in this paper is based on that of Kac [11], and in some cases, this differs from Carter's notation in [2].

For the simple Lie algebra of type A_n, all the fundamental modules

$$L(\omega_1), \ldots, L(\omega_n)$$

are minuscule, and we have

$$\dim(L(\omega_i)) = \binom{n+1}{i}.$$

In this case, $L(\omega_1)$ is the natural module, and $L(\omega_i)$ is the i-th exterior power of $L(\omega_1)$.

For the simple Lie algebra of type B_n (for $n \geq 2$), the only minuscule module is the spin module, $L(\omega_n)$, which has dimension 2^n.

For the simple Lie algebra of type C_n (for $n \geq 2$), the only minuscule module is the natural module, $L(\omega_1)$, which has dimension $2n$.

For the simple Lie algebra of type D_n (for $n \geq 4$), there are three minuscule modules. These are the natural module $L(\omega_1)$, of dimension $2n$, and the two spin modules $L(\omega_{n-1})$ and $L(\omega_n)$, each of which has dimension 2^{n-1}.

The simple Lie algebra of type E_6 has two minuscule modules, $L(\omega_1)$ and $L(\omega_5)$, each of which has dimension 27.

The simple Lie algebra of type E_7 has one minuscule module, $L(\omega_6)$, which has dimension 56.

The simple Lie algebras of types E_8, F_4 and G_2 have no minuscule modules.

2. Minuscule systems

Definition 2.1. Let Ψ and Δ be subsets of vectors in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, where $\mathbb{R}^n$ is equipped with the usual scalar product and $0 \notin \Delta$. We say that Ψ is a *minuscule system with respect to the simple system* Δ if the following conditions are satisfied for every $\mathbf{v} \in \Psi$ and $\mathbf{a} \in \Delta$.

(i) We have $2\mathbf{v}.\mathbf{a} = c\mathbf{a}.\mathbf{a}$ for some $c = c(\mathbf{v}, \mathbf{a}) \in \{-1, 0, +1\}$.

(ii) Let $c = c(\mathbf{v}, \mathbf{a})$ be as in (i). Then we have $\mathbf{v} + \mathbf{a} \in \Psi$ if and only if $c = -1$, and we have $\mathbf{v} - \mathbf{a} \in \Psi$ if and only if $c = 1$. (In particular, if $c = 0$, then neither vector $\mathbf{v} \pm \mathbf{a}$ lies in Ψ.)

Definition 2.2. Let Ψ be a minuscule system with respect to the simple system Δ, and let k be a field. We define V_Ψ to be the k-vector space with basis $\{b_\mathbf{v} : \mathbf{v} \in \Psi\}$. For each $\mathbf{a} \in \Delta$, we define k-linear endomorphisms $E_\mathbf{a}$, $F_\mathbf{a}$, $H_\mathbf{a}$ of V_Ψ by specifying their effects on basis elements, as follows:

$$E_\mathbf{a}(b_\mathbf{v}) = \begin{cases} b_{\mathbf{v}+\mathbf{a}} & \text{if } \mathbf{v} + \mathbf{a} \in \Psi; \\ 0 & \text{otherwise;} \end{cases}$$

$$F_\mathbf{a}(b_\mathbf{v}) = \begin{cases} b_{\mathbf{v}-\mathbf{a}} & \text{if } \mathbf{v} - \mathbf{a} \in \Psi; \\ 0 & \text{otherwise;} \end{cases}$$

$$H_\mathbf{a}(b_\mathbf{v}) = c(\mathbf{v}, \mathbf{a}) b_\mathbf{v} = 2\frac{\mathbf{v}.\mathbf{a}}{\mathbf{a}.\mathbf{a}} b_\mathbf{v}.$$

Definition 2.3. Let Ψ be a minuscule system with respect to the simple system Δ. We define the *generalized Cartan matrix*, A, of Δ to be the $|\Delta| \times |\Delta|$ matrix whose $(\mathbf{a}, \mathbf{b})$ entry is given by

$$A_{\mathbf{a},\mathbf{b}} = 2\frac{\mathbf{a}.\mathbf{b}}{\mathbf{a}.\mathbf{a}}.$$

Although we have apparently given two meanings to the term "generalized Cartan matrix" (the above meaning and Definition 1.1), they coincide in all the examples of this paper. A formulation very similar to Definition 2.3 may be found in [11, §2.3].

3. Results on minuscule systems

The following lemma is the key ingredient for our main result, and links the Serre presentation given in Theorem 1.2 to the linear operators arising from the polytope.

Lemma 3.1. *Using the notation of Definition 2.2, we have the following identities in* $\mathrm{End}(V_\Psi)$*, where* $\mathbf{a}, \mathbf{b} \in \Delta$*:*

$$H_{\mathbf{a}} \circ E_{\mathbf{b}} = E_{\mathbf{b}} \circ H_{\mathbf{a}} + A_{\mathbf{a},\mathbf{b}} E_{\mathbf{b}}, \tag{1}$$

$$H_{\mathbf{a}} \circ E_{\mathbf{a}} = E_{\mathbf{a}} = -E_{\mathbf{a}} \circ H_{\mathbf{a}}, \tag{2}$$

$$H_{\mathbf{a}} \circ F_{\mathbf{b}} = F_{\mathbf{b}} \circ H_{\mathbf{a}} - A_{\mathbf{a},\mathbf{b}} F_{\mathbf{b}}, \tag{3}$$

$$H_{\mathbf{a}} \circ F_{\mathbf{a}} = -F_{\mathbf{a}} = -F_{\mathbf{a}} \circ H_{\mathbf{a}}, \tag{4}$$

$$E_{\mathbf{a}} \circ F_{\mathbf{b}} = F_{\mathbf{b}} \circ E_{\mathbf{a}} = 0 \textit{ if } A_{\mathbf{a},\mathbf{b}} < 0, \tag{5}$$

$$E_{\mathbf{a}} \circ F_{\mathbf{b}} = F_{\mathbf{b}} \circ E_{\mathbf{a}} \textit{ if } A_{\mathbf{a},\mathbf{b}} = 0, \tag{6}$$

$$E_{\mathbf{a}} \circ E_{\mathbf{a}} = 0, \tag{7}$$

$$E_{\mathbf{a}} \circ E_{\mathbf{b}} = E_{\mathbf{b}} \circ E_{\mathbf{a}} \textit{ if } A_{\mathbf{a},\mathbf{b}} = 0, \tag{8}$$

$$E_{\mathbf{a}} \circ E_{\mathbf{b}} \circ E_{\mathbf{a}} = 0 \textit{ if } A_{\mathbf{a},\mathbf{b}} = -1, \tag{9}$$

$$F_{\mathbf{a}} \circ F_{\mathbf{a}} = 0, \tag{10}$$

$$F_{\mathbf{a}} \circ F_{\mathbf{b}} = F_{\mathbf{b}} \circ F_{\mathbf{a}} \textit{ if } A_{\mathbf{a},\mathbf{b}} = 0, \tag{11}$$

$$F_{\mathbf{a}} \circ F_{\mathbf{b}} \circ F_{\mathbf{a}} = 0 \textit{ if } A_{\mathbf{a},\mathbf{b}} = -1, \tag{12}$$

Proof. We prove (1) by acting each side of the equation on a basis vector $b_{\mathbf{v}}$. If $E_{\mathbf{b}}(b_{\mathbf{v}}) = 0$, then both sides are trivial, so we may assume this is not the case, meaning that $\mathbf{v} + \mathbf{b} \in \Psi$. It follows that, in the notation of Definition 2.1, we have $c(\mathbf{v}, \mathbf{b}) = -1$ and $c(\mathbf{v} + \mathbf{b}, \mathbf{b}) = 1$. In turn, this means that

$$H_{\mathbf{a}} \circ E_{\mathbf{b}}(b_{\mathbf{v}}) = H_{\mathbf{a}}(b_{\mathbf{v}+\mathbf{b}}) = 2\frac{(\mathbf{v}+\mathbf{b}).\mathbf{a}}{\mathbf{a}.\mathbf{a}} b_{\mathbf{v}+\mathbf{b}}$$

and that

$$E_{\mathbf{b}} \circ H_{\mathbf{a}}(b_{\mathbf{v}}) = 2\frac{\mathbf{v}.\mathbf{a}}{\mathbf{a}.\mathbf{a}} E_{\mathbf{b}}(b_{\mathbf{v}}) = 2\frac{\mathbf{v}.\mathbf{a}}{\mathbf{a}.\mathbf{a}} b_{\mathbf{v}+\mathbf{b}}.$$

Subtracting, we have

$$(H_{\mathbf{a}} \circ E_{\mathbf{b}} - E_{\mathbf{b}} \circ H_{\mathbf{a}})(b_{\mathbf{v}}) = 2\frac{\mathbf{b}.\mathbf{a}}{\mathbf{a}.\mathbf{a}} b_{\mathbf{v}+\mathbf{b}} = A_{\mathbf{a},\mathbf{b}} E_{\mathbf{b}}(b_{\mathbf{v}}),$$

which proves (1).

If $\mathbf{b} = \mathbf{a}$, then the above argument shows that

$$H_{\mathbf{a}} \circ E_{\mathbf{a}}(b_{\mathbf{v}}) = H_{\mathbf{a}}(b_{\mathbf{v}+\mathbf{a}}) = c(\mathbf{v}+\mathbf{a}, \mathbf{a}) b_{\mathbf{v}+\mathbf{a}} = E_{\mathbf{a}}(b_{\mathbf{v}}).$$

Part (2) follows from this and the fact that $A_{\mathbf{a},\mathbf{a}} = 2$.

The proof of (3) (respectively, (4)) follows by adapting the argument used to prove (1) (respectively, (2)).

We now prove that $E_{\mathbf{a}} \circ F_{\mathbf{b}} = 0$ if $A_{\mathbf{a},\mathbf{b}} < 0$. By (1) and (2), we have

$$\begin{aligned} E_{\mathbf{a}} F_{\mathbf{b}} &= H_{\mathbf{a}} E_{\mathbf{a}} F_{\mathbf{b}} \\ &= E_{\mathbf{a}} H_{\mathbf{a}} F_{\mathbf{b}} + 2 E_{\mathbf{a}} F_{\mathbf{b}} \\ &= E_{\mathbf{a}} F_{\mathbf{b}} H_{\mathbf{a}} + (2 - A_{\mathbf{a},\mathbf{b}}) E_{\mathbf{a}} F_{\mathbf{b}}. \end{aligned}$$

Rearranging, this gives

$$E_{\mathbf{a}} F_{\mathbf{b}} H_{\mathbf{a}} = (A_{\mathbf{a},\mathbf{b}} - 1) E_{\mathbf{a}} F_{\mathbf{b}}.$$

Suppose that $E_{\mathbf{a}} F_{\mathbf{b}} \neq 0$, and let $b_{\mathbf{v}}$ be a basis element for which $E_{\mathbf{a}} \circ F_{\mathbf{b}}(b_{\mathbf{v}}) \neq 0$. This implies that $H_{\mathbf{a}}(b_{\mathbf{v}}) = (A_{\mathbf{a},\mathbf{b}} - 1) b_{\mathbf{v}}$, but this is a contradiction to Definition 2.1 (i), because $c(\mathbf{v}, \mathbf{a}) = A_{\mathbf{a},\mathbf{b}} - 1 \leq -2$. This shows that $E_{\mathbf{a}} \circ F_{\mathbf{b}} = 0$, and the proof that $F_{\mathbf{b}} \circ E_{\mathbf{a}} = 0$ is very similar, proving (5).

We next turn to (6). Let us first suppose that $E_{\mathbf{a}} \circ F_{\mathbf{b}}(b_{\mathbf{v}}) \neq 0$ for some basis element $\mathbf{v}$. (This means that $E_{\mathbf{a}} \circ F_{\mathbf{b}}(b_{\mathbf{v}}) = b_{\mathbf{v}-\mathbf{b}+\mathbf{a}}$ and that $\mathbf{v} - \mathbf{b} + \mathbf{a} \in \Psi$.) By (2) and (3), we have

$$\begin{aligned} E_{\mathbf{a}} \circ F_{\mathbf{b}}(b_{\mathbf{v}}) &= -E_{\mathbf{a}} \circ H_{\mathbf{a}} \circ F_{\mathbf{b}}(b_{\mathbf{v}}) \\ &= -E_{\mathbf{a}} \circ F_{\mathbf{b}} \circ H_{\mathbf{a}}(b_{\mathbf{v}}). \end{aligned}$$

It follows that $H_{\mathbf{a}}(b_{\mathbf{v}}) = -b_{\mathbf{v}}$, and that $c(\mathbf{v}, \mathbf{a}) = -1$. In turn, this implies that $\mathbf{v} + \mathbf{a} \in \Psi$ and $E_{\mathbf{a}}(b_{\mathbf{v}}) = b_{\mathbf{v}+\mathbf{a}} \neq 0$. Since $\mathbf{v} - \mathbf{b} + \mathbf{a} \in \Psi$, we have

$$F_{\mathbf{b}} \circ E_{\mathbf{a}}(b_{\mathbf{v}}) = b_{\mathbf{v}+\mathbf{a}-\mathbf{b}} = E_{\mathbf{a}} \circ F_{\mathbf{b}}(b_{\mathbf{v}}).$$

It follows that if $E_{\mathbf{a}} \circ F_{\mathbf{b}} \neq 0$, then $E_{\mathbf{a}} \circ F_{\mathbf{b}} = F_{\mathbf{b}} \circ E_{\mathbf{a}}$. The converse statement also follows by a similar argument. This in turn implies that $E_{\mathbf{a}} \circ F_{\mathbf{b}} = 0$ if and only if $F_{\mathbf{b}} \circ E_{\mathbf{a}} = 0$, which completes the proof of (6).

The proofs of (8) and (11) follow the same line of argument as the proof of (6).

To prove (7), we show that $E_{\mathbf{a}} \circ E_{\mathbf{a}}(b_{\mathbf{v}}) = 0$ for all basis elements $b_{\mathbf{v}}$. As before, we may reduce to the case where $E_{\mathbf{a}}(b_{\mathbf{v}}) \neq 0$, meaning that $\mathbf{v} + \mathbf{a} \in \Psi$, $c(\mathbf{v}, \mathbf{a}) = -1$ and $c(\mathbf{v} + \mathbf{a}, \mathbf{a}) = 1$. The latter fact implies that $E_{\mathbf{a}}(b_{\mathbf{v}+\mathbf{a}}) = 0$, which completes the proof. The proof of (10) follows the same argument.

We now prove (9). As in the proof of (7), the proof reduces to showing that

$$E_{\mathbf{a}} \circ E_{\mathbf{b}} \circ E_{\mathbf{a}}(b_{\mathbf{v}}) = 0$$

in the case where $c(\mathbf{v}, \mathbf{a}) = -1$. Using (1) and (2), we then have

$$\begin{aligned} E_{\mathbf{a}} \circ E_{\mathbf{b}} \circ E_{\mathbf{a}}(b_{\mathbf{v}}) &= -E_{\mathbf{a}} \circ H_{\mathbf{a}} \circ E_{\mathbf{b}} \circ E_{\mathbf{a}}(b_{\mathbf{v}}) \\ &= -E_{\mathbf{a}} \circ E_{\mathbf{b}} \circ (H_{\mathbf{a}} \circ E_{\mathbf{a}}(b_{\mathbf{v}})) + E_{\mathbf{a}} \circ E_{\mathbf{b}} \circ E_{\mathbf{a}}(b_{\mathbf{v}}) \\ &= 0, \end{aligned}$$

as required. The proof of (12) follows the same argument as the proof of (9). □

We are now ready to state our main result.

Theorem 3.2. *Let Ψ be a minuscule system with respect to the simple system Δ, and let A be the generalized Cartan matrix of Δ. Assume that A is a symmetrizable generalized Cartan matrix in the sense of Definition 1.1, and let $\mathfrak{g}$ be the corresponding derived Kac–Moody algebra. Then the $\mathbb{C}$-vector space V_Ψ has the structure of a $\mathfrak{g}$-module, where e_i (respectively, f_i, h_i) acts via the endomorphism E_i (respectively, F_i, H_i).*

Proof. We need to show that the defining relations of Theorem 1.2 are satisfied.

Since the operators $H_{\mathbf{a}}$ are simultaneously diagonalizable with respect to the basis $\{b_{\mathbf{v}} : \mathbf{v} \in \Psi\}$, they commute, and so we have $[h_i, h_j] = 0$.

Lemma 3.1 (1) establishes the relations between the h_i and the e_j, and Lemma 3.1 (3) establishes the relations between the h_i and the f_j. Lemma 3.1 (5) and (6) prove that $[e_i, f_j] = 0$ if $i \neq j$.

We now prove that $[e_i, f_i] = h_i$, for which we need to show that

$$E_i \circ F_i - F_i \circ E_i = H_i.$$

It is enough to evaluate each side of the equation on a basis element $b_{\mathbf{v}}$. If $c(\mathbf{v}, i) = 0$ then all terms act as zero. If $c(\mathbf{v}, i) = 1$ then $E_i \circ F_i(b_{\mathbf{v}}) = b_{\mathbf{v}}$, $H_i(b_{\mathbf{v}}) = b_{\mathbf{v}}$, and $F_i \circ E_i(b_{\mathbf{v}}) = 0$, thus satisfying the equation. The case $c(\mathbf{v}, i) = -1$ is dealt with by a similar argument.

Next we prove that the Serre relation

$$\underbrace{[e_i, [e_i, \cdots [e_i,}_{1-A_{ij} \text{ times}} e_j] \cdots]] = 0$$

is satisfied. If $A_{ij} = 0$, this states that $[e_i, e_j] = 0$, which is immediate from Lemma 3.1 (8). If $A_{ij} = -1$, this states that

$$[e_i, [e_i, e_j]] = 0,$$

or in other words,

$$E_i \circ E_i \circ E_j - E_i \circ E_j \circ E_i + E_j \circ E_i \circ E_i = 0,$$

which is immediate from Lemma 3.1 (7) and (9). The only other possibility is that $A_{ij} \leq -2$. In this case, every term of the corresponding identity in terms of E_i and E_j involves an $E_i \circ E_i$, which is zero by Lemma 3.1 (7), and this completes the proof.

A similar argument shows that the Serre relation involving the f_i is also satisfied. □

The following result provides some methods of constructing new minuscule systems from known ones, and these will be useful in the sequel.

Proposition 3.3. *Let $\Psi \subset \mathbb{R}^n$ be a minuscule system with respect to the simple system Δ. Let Ψ' and Δ' be nonempty subsets of Ψ and Δ, respectively.*

(i) *Suppose that for every $\mathbf{v} \in \Psi'$ and $\mathbf{a} \in \Delta'$, the following conditions are satisfied.*
   (a) *If $c(\mathbf{v}, \mathbf{a}) = -1$ then $\mathbf{v} + \mathbf{a} \in \Psi'$.*
   (b) *If $c(\mathbf{v}, \mathbf{a}) = 1$ then $\mathbf{v} - \mathbf{a} \in \Psi'$.*
   Then Ψ' is a minuscule system with respect to Δ'.

(ii) *If $\Psi' = \Psi$ and $\emptyset \neq \Delta' \subset \Delta$ then Ψ' is a minuscule system with respect to Δ'.*

(iii) *Let $\mathbf{n} \in \mathbb{R}^n$ and $l \in \mathbb{R}$. Suppose that the sets*

$$\Psi(\mathbf{n}, l) = \{\mathbf{v} \in \Psi : \mathbf{v}.\mathbf{n} = l\}$$

and

$$\Delta(\mathbf{n}) = \{\mathbf{a} \in \Delta : \mathbf{a}.\mathbf{n} = 0\}$$

are nonempty. Then $\Psi(\mathbf{n}, l)$ is a minuscule system with respect to the simple system $\Delta(\mathbf{n})$.

Proof. Definition 2.1 applied to Ψ' and Δ' follows immediately from the hypotheses of (i). Part (ii) is an immediate consequence of (i).

Part (iii) follows from (i) and the observation that if $\mathbf{v} \in \Psi(\mathbf{n}, l)$ and $\mathbf{a} \in \Delta(\mathbf{n})$ then $(\mathbf{v} \pm \mathbf{a}).\mathbf{n} = l \pm 0 = l$. □

Definition 3.4. If Ψ' and Δ' satisfy the hypotheses of Proposition 3.3 (i), we will call the pair (Ψ', Δ') a *minuscule subsystem* of (Ψ, Δ).

We now explain how minuscule systems associated with a Lie algebra also support actions of the corresponding Weyl group.

Definition 3.5. Let $n \in \mathbb{N}$ and $0 \neq \alpha \in V = \mathbb{R}^n$. The *reflection* s_α associated to α is the linear map $s_\alpha : V \to V$ given by

$$s_\alpha(\mathbf{v}) = \mathbf{v} - 2\frac{\mathbf{v}.\alpha}{\alpha.\alpha}\alpha.$$

If $\Psi \subset \mathbb{R}^n$ is a minuscule system with respect to the simple system Δ, then we define the Weyl group $W = W_{\Psi,\Delta}$ of (Ψ, Δ) to be the group of automorphisms of $\mathbb{R}^n$ generated by the set $\{s_\mathbf{a} : \mathbf{a} \in \Delta\}$.

It is not hard to check that this agrees with the usual notion of the Weyl group associated to a simple Lie algebra over $\mathbb{C}$ (see [11, (1.1.2), §3.7]). It is well known [10, §1.1] that the Weyl group action respects the scalar product on V.

Proposition 3.6. *If Ψ is a minuscule system with respect to the simple system Δ, then $W = W_{\Psi,\Delta}$ acts on Ψ.*

Proof. It is enough to show that if $\mathbf{a} \in \Delta$ and $\mathbf{v} \in \Psi$, then $s_{\mathbf{a}}(\mathbf{v}) \in \Psi$. By the definitions of $s_{\mathbf{a}}$ and $c = c(\mathbf{v}, \mathbf{a})$, we have $s_{\mathbf{a}}(\mathbf{v}) = \mathbf{v} - c\mathbf{a}$, which lies in Ψ by Definition 2.1. □

4. The Hesse polytope

In §4, we introduce some examples of minuscule systems related to the polytope known in Coxeter's notation as 3_{21}. This polytope does not have a consistent name in the literature; we will follow Conway and Sloane in calling 3_{21} the *Hesse polytope*, as this name does not appear to have any other connotations. The Hesse polytope has 56 vertices, whose coordinates are given by the set Ψ^{E_7} of Definition 4.1. Note that we have multiplied Conway and Sloane's coordinates for the vertices by 4, in order to make them integers and to retain compatibility with du Val's coordinates [6, §7].

The *Schläfli polytope*, which is called 2_{21} in Coxeter's notation, also plays a role in the examples of this section involving the Lie algebra of type E_6. It has 27 vertices, whose coordinates can be given by either of the sets $\Psi(\mathbf{n}, \pm 8)$ appearing in Proposition 4.3. More details on the inclusion of the Schläfli polytope in the Hesse polytope may be found in [3, §9].

Definition 4.1. Let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_7 \in \mathbb{R}^8$ be such that ε_i has a 1 in position $i+1$, and zeros elsewhere. For $0 \leq i, j \leq 7$, define the vector $\mathbf{v}_{i,j} = \mathbf{v}_{\{i,j\}} = \mathbf{v}_{j,i} \in \mathbb{R}^8$ by

$$\mathbf{v}_{i,j} := 4(\varepsilon_i + \varepsilon_j) - \left(\sum_{i=0}^{7} \varepsilon_i \right).$$

(For example, we have $\mathbf{v}_{0,1} = (3, 3, -1, -1, -1, -1, -1, -1)$.) Let Ψ^{E_7} consist of the 56 vectors $\{\pm \mathbf{v}_{i,j} : 1 \leq i < j \leq 8\}$.

It is convenient for later purposes to introduce the sets $K_0 = \{0, 1, 2, 3\}$ and $K_7 = \{4, 5, 6, 7\}$.

Lemma 4.2. *Let Ψ^{E_7} be as in Definition 4.1, and let*

$$\Delta^{E_7^{(1)}} = \{\alpha_0, \alpha_1, \dots, \alpha_7\},$$

where $\alpha_i = 4(\varepsilon_i - \varepsilon_{i+1})$ if $0 \leq i < 7$, and $\alpha_7 = (-2,-2,-2,-2,2,2,2,2)$. Then Ψ^{E_7} is a minuscule system with respect to the simple system $\Delta^{E_7^{(1)}}$.

Proof. Suppose first that $\mathbf{a} = \alpha_i$ for some $i < 7$, and let $\mathbf{v} \in \Psi^{E_7}$. Write $\mathbf{v} = \sum_{j=0}^{7} \lambda_j \varepsilon_j$. The proof is a case by case check according to the values of λ_i and λ_{i+1}. There are three cases to check.

The first possibility is that $\lambda_i = \lambda_{i+1}$. This implies that $\mathbf{v}.\alpha_i = 0$. The coefficients of ε_i and of ε_{i+1} in $\mathbf{v} + \mathbf{a}$ differ by 8, which means that $\mathbf{v} + \mathbf{a} \not\in \Psi$, and a similar argument shows that $\mathbf{v} - \mathbf{a} \not\in \Psi$. The conditions of Definition 2.1 are therefore satisfied.

The second possibility is that $(\lambda_i, \lambda_{i+1}) \in \{(-3,1),(-1,3)\}$, that is, $\lambda_{i+1} = \lambda_i + 4$. This implies that, $\mathbf{v}.\mathbf{a} = -16$ and $\mathbf{a}.\mathbf{a} = 32$. This satisfies Definition 2.1 (i) with $c = -1$. In this case, $\mathbf{v} - \mathbf{a} \not\in \Psi$, because the coefficients of ε_i and ε_{i+1} in $\mathbf{v} - \mathbf{a}$ do not lie in the set $\{\pm 3, \pm 1\}$. However, the vector $\mathbf{v} + \mathbf{a}$ is obtained from $\mathbf{v}$ by exchanging the coefficients of ε_i and ε_{i+1}, which means that $\mathbf{v} + \mathbf{a} \in \Psi$. This satisfies Definition 2.1 (ii).

The third possibility is that $(\lambda_i, \lambda_{i+1}) \in \{(3,-1),(1,-3)\}$, that is, $\lambda_{i+1} = \lambda_i - 4$. An analysis like that of the previous paragraph shows that $c = 1$, $\mathbf{v} + \mathbf{a} \not\in \Psi$, and $\mathbf{v} - \mathbf{a} \in \Psi$, as required.

It remains to show that Definition 2.1 is satisfied with $\mathbf{a} = \alpha_7$. To check this, we use the sets K_0, K_7 of Definition 4.1. Let $\mathbf{v} = \pm \mathbf{v}_{i,j}$. As before, there are three cases to check.

The first possibility is that $\{i,j\} \not\subseteq K_l$ for some $l \in \{0,1\}$. (Informally, this means that the two occurrences of 3 (or -3) in $\mathbf{v}$ do not occur in the same half of the vector.) This implies that $\mathbf{v}.\alpha_7 = 0$. Furthermore, neither of the two vectors $\mathbf{v} \pm \alpha_7$ lies in Ψ, because in each of them, one of the basis vectors ε_i appears with coefficient ± 5. Definition 2.1 is therefore satisfied in this case.

The second possibility is that either $\mathbf{v} = +\mathbf{v}_{i,j}$ with $\{i,j\} \subset K_0$, or that $\mathbf{v} = -\mathbf{v}_{i,j}$ with $\{i,j\} \subset K_7$. In each case, $\mathbf{v}.\alpha_7 = -16$ and $\mathbf{v} + \alpha_7 \in \Psi$. However, in each case, we have $\mathbf{v} - \alpha_7 \not\in \Psi$, because two basis vectors appear in $\mathbf{v} - \alpha_7$ with coefficient ± 5. Since $\alpha_7.\alpha_7 = 32$, Definition 2.1 is satisfied with $c = -1$.

The third possibility is that either $\mathbf{v} = +\mathbf{v}_{i,j}$ with $\{i,j\} \subset K_7$, or that $\mathbf{v} = -\mathbf{v}_{i,j}$ with $\{i,j\} \subset K_0$. An analysis like that of the above paragraph shows that Definition 2.1 is satisfied with $c = 1$, $\mathbf{v} - \alpha_7 \in \Psi$, and $\mathbf{v} + \alpha_7 \not\in \Psi$. This completes the proof. $\square$

Proposition 4.3. *Let $\Psi = \Psi^{E_7}$ be as in Definition 4.1, and let $\Delta = \Delta^{E_7^{(1)}}$ be as in Lemma 4.2.*

(i) *The 56-dimensional $\mathbb{C}$-vector space V_Ψ has the structure of a $\mathfrak{g}$-module, where $\mathfrak{g}$ is the derived affine Kac–Moody algebra of type $E_7^{(1)}$.*

(ii) *Let $\Psi' = \Psi$ and $\Delta' = \Delta \backslash \{\alpha_0\}$. Then Ψ' is a minuscule system with respect to the simple system Δ', and $V_{\Psi'}$ is a module for the simple Lie algebra $\mathfrak{e}_7$ over $\mathbb{C}$ of type E_7. It is an irreducible module with highest weight vector $-\mathbf{v}_{0,7}$ and lowest weight vector $\mathbf{v}_{0,7}$ (as in Definition 4.1).*

(iii) *Let $\mathbf{n} = \mathbf{v}_{0,7}$. Then we have a disjoint union*

$$\Psi = \Psi(\mathbf{n}, 24) \,\dot\cup\, \Psi(\mathbf{n}, 8) \,\dot\cup\, \Psi(\mathbf{n}, -8) \,\dot\cup\, \Psi(\mathbf{n}, -24).$$

For $l \in \{24, 8, -8, -24\}$, $\Psi(\mathbf{n}, l)$ is a minuscule system with respect to the simple system $\Delta(\mathbf{n}) = \Delta \backslash \{\alpha_0, \alpha_6\}$, and $V_{\Psi(\mathbf{n},l)}$ is a module for the simple Lie algebra $\mathfrak{e}_6$ over $\mathbb{C}$ of type E_6. The two modules $V_{\Psi(\mathbf{n},\pm 24)}$ are trivial one-dimensional modules for $\mathfrak{e}_6$, whereas the two modules $V_{\Psi(\mathbf{n},\pm 8)}$ are nonisomorphic 27-dimensional irreducible modules. The module $V_{\Psi(\mathbf{n},8)}$ has highest weight $\mathbf{v}_{1,7}$ and lowest weight $\mathbf{v}_{0,6}$. The module $V_{\Psi(\mathbf{n},-8)}$ has highest weight $-\mathbf{v}_{0,6}$ and lowest weight $-\mathbf{v}_{1,7}$.

(iv) *For $l \in \{24, 8, -8, -24\}$, $\Psi(\mathbf{n}, l)$ is a minuscule system with respect to the simple system $\Delta(\mathbf{n}) \cup \{\alpha\}$, where $\alpha = 4(\varepsilon_7 - \varepsilon_0)$. This makes $V_{\Psi(\mathbf{n},l)}$ into a module for the derived affine Kac–Moody algebra $\mathfrak{g}$ of type $E_6^{(1)}$.*

Proof. By Lemma 4.2, Ψ is a minuscule system with respect to the simple system Δ. One may check directly (using [11, §2.3]) that the associated matrix A is the symmetrizable generalized Cartan matrix of type $E_7^{(1)}$ of [11]. Theorem 3.2 then establishes (i).

For (ii), we know that Ψ' is a minuscule system with respect to the simple system Δ' by Proposition 3.3 (ii). The matrix A in this case is the symmetrizable (generalized) Cartan matrix of type E_7 of [11]. It follows from Theorem 3.2 that $V_{\Psi'}$ is a module for $\mathfrak{e}_7$. Direct checks show that $-\mathbf{v}_{0,7}$ is annihilated by all the operators E_i, $\mathbf{v}_{0,7}$ is annihilated by all the operators F_i, and $-\mathbf{v}_{0,7}$ is annihilated by all the operators H_i except H_6, in which case we have $H_6(-\mathbf{v}_{0,7}) = -\mathbf{v}_{0,7}$. Since $V_{\Psi'}$ has the same dimension as $L(\omega_6)$ and contains a highest weight vector of weight ω_6, the modules $V_{\Psi'}$ and $L(\omega_6)$ are isomorphic and irreducible by Proposition 1.4.

We next establish the decomposition of Ψ described in (iii). We have $\Psi(\mathbf{n}, 24) = \{\mathbf{v}_{0,7}\}$ and $\Psi(\mathbf{n}, -24) = \{-\mathbf{v}_{0,7}\}$. The set $\Psi(\mathbf{n}, 8)$ consists of the vectors

$$\{\mathbf{v}_{0,i} : 1 \leq i \leq 6\} \cup \{\mathbf{v}_{i,7} : 1 \leq i \leq 6\} \cup \{-\mathbf{v}_{i,j} : 1 \leq i < j \leq 6\},$$

and we have $\Psi(\mathbf{n}, -8) = -\Psi(\mathbf{n}, 8)$. It is easy to check that Ψ is the disjoint union of these four sets. Proposition 3.3 (iii) shows that $\Psi(\mathbf{n}, l)$ is a minuscule system with respect to $\Delta(\mathbf{n})$, and Theorem 3.2 shows that the modules $V_{\Psi(\mathbf{n},l)}$ are modules for $\mathfrak{e}_6$ (after the generalized Cartan matrix has been recognized as symmetrizable of type E_6). The assertions about dimensions and weight vectors are easy to check.

A quick calculation shows that

$$H_i.\mathbf{v}_{1,7} = \begin{cases} \mathbf{v}_{1,7} & \text{if } i = 1, \\ 0 & \text{if } i \in \{2, 3, 4, 5, 7\} \end{cases}$$

and

$$H_i.(-\mathbf{v}_{0,6}) = \begin{cases} -\mathbf{v}_{0,6} & \text{if } i = 5, \\ 0 & \text{if } i \in \{1, 2, 3, 4, 7\}. \end{cases}$$

This shows that $V_{\Psi(\mathbf{n},8)}$ (respectively, $V_{\Psi(\mathbf{n},-8)}$ has the same dimension as, and a nonzero weight vector of the same weight as $L(\omega_1)$ (respectively, $L(\omega_5)$). Proposition 1.4 now shows that the two modules $V_{\Psi(\mathbf{n},\pm 8)}$ are irreducible and nonisomorphic.

To prove (iv), we need to check that Definition 2.1 is satisfied with $\mathbf{a} = \alpha$. This follows by imitating the case analysis for the case $i < 7$ in Lemma 4.2, using the fact that $\mathbf{n}.\alpha = 0$. □

5. The hypercube

In §5, we consider examples relating to the polytope known as the the *hypercube* or *measure polytope*; in Coxeter's notation it is denoted γ_n. The set Ψ defined in Lemma 5.1 is our standard set of coordinates for the 2^n vertices of the hypercube.

We will show how the hypercube may be used to construct the spin representations of the simple Lie algebras of types B_n and D_n. By passing to appropriate subsystems, we obtain all the fundamental representations of the simple Lie algebra of type A_n as a by-product.

Lemma 5.1. *Let $n \geq 3$, let $\varepsilon_0, \ldots, \varepsilon_{n-1} \in \mathbb{R}^n$ be the usual basis for $\mathbb{R}^n$, and let Ψ be the set of 2^n vectors of the form*

$$(\pm 2, \pm 2, \ldots, \pm 2).$$

Let $\Delta = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, where $\alpha_0 = -4(\varepsilon_0 + \varepsilon_1)$, $\alpha_n = 4\varepsilon_{n-1}$, and $\alpha_i = 4(\varepsilon_{i-1} - \varepsilon_i)$ for $0 < i < n$. Then Ψ is a minuscule system with respect to the simple system Δ.

Proof. We check that Definition 2.1 holds for each of the α_i in turn. Let $\mathbf{v} = \sum_{j=0}^{n-1} \lambda_j \varepsilon_j \in \Psi$.

Suppose first that $0 < i < n$. The proof is a case by case check according to the values of λ_i and λ_{i+1}. There are three cases to check, and we omit the details because the cases are almost identical to those in the first part of the argument proving Lemma 4.2.

Next, suppose that $i = 0$. There are three cases to check, according to the values of λ_0 and λ_1. If $\lambda_0 = \lambda_1 = +2$, then $\mathbf{v} + \alpha_0 \in \Psi$, $\mathbf{v} - \alpha_0 \notin \Psi$, and $2\mathbf{v}.\alpha_0 = -32 = -\alpha_0.\alpha_0$, giving $c = c(\mathbf{v}, \alpha_0) = -1$ as required. If $\lambda_0 = \lambda_1 = -2$, then $\mathbf{v} - \alpha_0 \in \Psi$, $\mathbf{v} + \alpha_0 \notin \Psi$, and $2\mathbf{v}.\alpha_0 = 32 = \alpha_0.\alpha_0$, giving $c = 1$ as required. If $\lambda_0 \neq \lambda_1$, then neither vector $\mathbf{v} \pm \alpha_0$ lies in Ψ, and $2\mathbf{v}.\alpha_0 = 0$, giving $c = 0$. Definition 2.1 is therefore satisfied in all three cases.

Finally, suppose that $i = n$. There are two cases to check, according to the value of λ_{n-1}. If $\lambda_{n-1} = +2$ then $\mathbf{v} - \alpha_n \in \Psi$ and $\mathbf{v} + \alpha_n \notin \Psi$. We also have $2\mathbf{v}.\alpha_n = 16 = \alpha_n.\alpha_n$, giving $c = c(\mathbf{v}, \alpha_n) = 1$, thus satisfying Definition 2.1. If $\lambda_{n-1} = -2$ then $\mathbf{v}+\alpha_n \in \Psi$ and $\mathbf{v}-\alpha_n \notin \Psi$. We also have $2\mathbf{v}.\alpha_n = -16 = -\alpha_n.\alpha_n$, giving $c = -1$, thus satisfying Definition 2.1 and completing the proof. $\square$

We may now state an analogue of Proposition 4.3.

Proposition 5.2. *Maintain the notation of Definition 5.1. Let* $\mathbf{j} = \sum_{j=0}^{n-1} \varepsilon_j$ *and*

$$S = \{2n - 4j : 0 \leq j \leq n\}.$$

(i) *The 2^n-dimensional $\mathbb{C}$-vector space V_Ψ has the structure of a $\mathfrak{g}$-module, where $\mathfrak{g}$ is the derived affine Kac–Moody algebra of type $B_n^{(1)}$.*

(ii) *Let $\Psi' = \Psi$ and $\Delta' = \Delta \backslash \{\alpha_0\}$. Then Ψ' is a minuscule system with respect to the simple system Δ', and $V_{\Psi'}$ is a module for the simple Lie algebra $\mathfrak{b}_n$ over $\mathbb{C}$ of type B_n. It is an irreducible module with highest weight vector $2\mathbf{j}$ and lowest weight vector $-2\mathbf{j}$, and affords the spin representation of $\mathfrak{b}_n$.*

(iii) *We have a disjoint union*

$$\Psi = \bigcup_{j=0}^{n} \Psi(\mathbf{j}, 2n - 4j).$$

For $l \in S$, $\Psi(\mathbf{j}, l)$ is a minuscule system with respect to the simple system

$$\Delta(\mathbf{j}) = \Delta \backslash \{\alpha_0, \alpha_n\},$$

and $V_{\Psi(\mathbf{j},l)}$ is a module for the simple Lie algebra $\mathfrak{a}_{n-1}$ over $\mathbb{C}$ of type A_{n-1}. The two modules $V_{\Psi(\mathbf{j},\pm 2n)}$ are trivial one-dimensional modules for $\mathfrak{a}_{n-1}$, *and the other modules $V_{\Psi(\mathbf{j},l)}$ satisfy*

$$V_{\Psi(\mathbf{j},2n-4j)} \cong L(\omega_{n-j}).$$

The module $V_{\Psi(\mathbf{j},2n-4j)}$ has highest weight

$$-2\mathbf{j} + 4\left(\sum_{i=0}^{n-j-1} \varepsilon_i\right)$$

and lowest weight

$$2\mathbf{j} - 4\left(\sum_{i=0}^{j-1} \varepsilon_i\right).$$

(iv) *For $l \in S$, $\Psi(\mathbf{j}, l)$ is a minuscule system with respect to the simple system $\Delta(\mathbf{j}) \cup \{\alpha\}$, where $\alpha = 4(\varepsilon_{n-1} - \varepsilon_0)$. This makes $V_{\Psi(\mathbf{j},l)}$ into a module for the derived affine Kac–Moody algebra $\mathfrak{g}$ of type $A_{n-1}^{(1)}$.*

Proof. Using Lemma 5.1 in place of Lemma 4.2, the proof of (i) follows the same argument as the proof of Proposition 4.3 (i).

The proof of (ii) now follows by copying the argument of Proposition 4.3 (ii). In this case, the module turns out to be $L(\omega_n)$.

It is easily checked that $\Psi(\mathbf{j}, 2n-4j)$ consists precisely of the vectors in Ψ that have j occurrences of -2, from which the first assertion of (iii) follows. Proposition 3.3 (iii) shows that $\Psi(\mathbf{j}, 2n-4j)$ is a minuscule system with respect to $\Delta(\mathbf{j})$, and Theorem 3.2 shows that the modules $V_{\Psi(\mathbf{j},2n-4j)}$ are modules for $\mathfrak{a}_{n-1}$ (after the generalized Cartan matrix has been recognized as symmetrizable of type A_{n-1}). The assertions about dimensions and weight vectors are easy to check. If $j \neq \pm n$ and $\mathbf{v}$ is the highest weight vector $\mathbf{v}$ of $V_{\Psi(\mathbf{j},2n-4j)}$, then we have $H_i.\mathbf{v} = 0$ unless $i = n-j$, in which case $H_i.\mathbf{v} = \mathbf{v}$. The required isomorphism now follows from Proposition 1.4.

To prove (iv), we may copy the argument of Proposition 4.3 (iv) to check that Definition 2.1 is satisfied with $\mathbf{a} = \alpha$. (Note that $\mathbf{j}.\alpha = 0$.) $\square$

Lemma 5.3. *Let $n \geq 4$, let $\varepsilon_0, \ldots, \varepsilon_{n-1} \in \mathbb{R}^n$ be the usual basis for $\mathbb{R}^n$, and let Ψ be as in Lemma 5.1. Let Ψ_D^+ (respectively, Ψ_D^-) be the subset of Ψ whose vectors contain an even (respectively, odd) number of occurrences of -2.*

Let $\Delta_D = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha'_n\}$, where $\alpha'_n = 4(\varepsilon_{n-2} + \varepsilon_{n-1})$ and the other vectors α_i are as in Lemma 5.1.

Then $\Psi = \Psi_D^+ \mathbin{\dot\cup} \Psi_D^-$ is a minuscule system with respect to the simple system Δ_D, and both (Ψ_D^+, Δ_D) and (Ψ_D^-, Δ_D) are minuscule subsystems of (Ψ, Δ_D).

Proof. Most of the work for checking that Ψ is a minuscule system with respect to Δ_D is done in the proof of Lemma 5.1. The only extra criterion to check is that Definition 2.1 holds for $\mathbf{a} = \alpha'_n$. This follows by making appropriate sign changes to the argument used to check Definition 2.1 for $\mathbf{a} = \alpha_0$ as in the proof of Lemma 5.1.

Letting $\mathbf{j}$ be as in Proposition 5.2, we see that $\mathbf{v} \in \Psi$ lies in Ψ_D^+ if the integer $\mathbf{v}.\mathbf{j}$ is a multiple of 8, and $\mathbf{v}$ lies in Ψ_D^- otherwise. We observe that each $\mathbf{a} \in \Delta_D$ has the property that $\mathbf{a}.\mathbf{j}$ is a multiple of 8. We now apply Proposition 3.3 (i), which proves that $(\Psi_D^\pm, \Delta_D)$ are minuscule subsystems. □

Proposition 5.4. *Maintain the notation of 5.1–5.3.*

(i) *Each of the 2^{n-1}-dimensional $\mathbb{C}$-vector spaces $V_{\Psi_D^\pm}$ has the structure of a $\mathfrak{g}$-module, where $\mathfrak{g}$ is the derived affine Kac–Moody algebra of type $D_n^{(1)}$.*

(ii) *Let $\Psi^\pm = \Psi_D^\pm$ and $\Delta^\pm = \Delta_D \backslash \{\alpha_0\}$. Then each of the two sets $\Psi^\pm$ is a minuscule system with respect to each of the simple systems $\Delta^\pm$ respectively, and each of the two spaces $V_{\Psi^\pm}$ is a module for the simple Lie algebra $\mathfrak{d}_n$ over $\mathbb{C}$ of type D_n. The modules are nonisomorphic and both irreducible, and they afford the two spin representations of $\mathfrak{d}_n$. The highest weight vector of V_{Ψ^+} (respectively, V_{Ψ^-}) is $2\mathbf{j}$ (respectively, $2\mathbf{j} - 4\varepsilon_{n-1}$). The lowest weight vectors of $V_{\Psi^\pm}$ are $-2\mathbf{j}$ and $-2\mathbf{j} + 4\varepsilon_{n-1}$, where the assignment of vectors to modules depends on whether n is even or odd.*

Proof. Using Lemma 5.3 in place of Lemma 4.2, the proof of (i) follows the same argument as the proof of Proposition 4.3 (i).

The first assertion of (ii) follows by using Lemma 5.3 and copying the argument of Proposition 4.3 (ii). The operators H_i (for $1 \leq i < n-1$) all act as zero on $2\mathbf{j}$ and $2\mathbf{j} - 4\varepsilon_{n-1}$. The operator H_{n-1} (corresponding to α_{n-1}) acts as zero on $2\mathbf{j}$ and acts as the identity on $2\mathbf{j} - 4\varepsilon_{n-1}$. The operator H_n (corresponding to α'_n) acts as the identity on $2\mathbf{j}$ and as zero on $2\mathbf{j} - 4\varepsilon_{n-1}$. The second assertion is then proved by adapting the corresponding argument in Proposition 4.3 (iii). □

6. The hyperoctahedron

In §6, we consider examples relating to the polytope known as the the *hyperoctahedron* or *cross polytope*; in Coxeter's notation it is denoted β_n. The set Ψ defined in Lemma 6.1 is our standard set of coordinates for the $2n$ vertices of the hyperoctahedron.

We will show how to use the hyperoctahedron to construct the remaining two types of minuscule representations, namely the natural representations for Lie algebras of types C_n and D_n.

Lemma 6.1. *Let $n \geq 4$, let $\varepsilon_0, \dots, \varepsilon_{n-1} \in \mathbb{R}^n$ be the usual basis for $\mathbb{R}^n$, and let*

$$\Psi = \{\pm 4\varepsilon_i : 0 \leq i \leq n-1\}.$$

Let Δ_D be as in Lemma 5.3. Then Ψ is a minuscule system with respect to Δ_D.

Proof. We check Definition 2.1, treating each vector $\mathbf{a} \in \Delta_D$ in turn. Suppose first that $\mathbf{a} = \alpha_i$ for some $1 \leq i \leq n-1$, and let $\mathbf{v} \in \Psi$.

Define ε_j to be the unique basis element such that $\mathbf{v} = \pm 4\varepsilon_j$. If $j \notin \{i-1, i\}$ then we have $c = c(\mathbf{v}, \mathbf{a}) = 0$ and neither vector $\mathbf{v} \pm \mathbf{a}$ lies in Ψ, satisfying Definition 2.1 (ii). If $\mathbf{v} \in \{4\varepsilon_{i-1}, -4\varepsilon_i\}$ then $\mathbf{v} - \mathbf{a} \in \Psi$, $2\mathbf{v}.\mathbf{a} = 32 = \mathbf{a}.\mathbf{a}$, giving $c = 1$ as required. The other possibility is that $\mathbf{v} \in \{-4\varepsilon_{i-1}, 4\varepsilon_i\}$, in which case $\mathbf{v} + \mathbf{a} \in \Psi$, $2\mathbf{v}.\mathbf{a} = -32 = -\mathbf{a}.\mathbf{a}$, giving $c = -1$ as required.

Now suppose $\mathbf{a} = \alpha'_n$. In this case, if $j \notin \{n-2, n-1\}$ then we have $c = c(\mathbf{v}, \mathbf{a}) = 0$ and neither vector $\mathbf{v} \pm \mathbf{a}$ lies in Ψ. If $\mathbf{v} \in \{4\varepsilon_{n-2}, 4\varepsilon_{n-1}\}$ then $\mathbf{v} - \mathbf{a} \in \Psi$, $\mathbf{v} + \mathbf{a} \notin \Psi$, and $2\mathbf{v}.\mathbf{a} = 32 = \mathbf{a}.\mathbf{a}$, giving $c = 1$ as required. The other possibility is that $\mathbf{v} \in \{-4\varepsilon_{n-2}, -4\varepsilon_{n-1}\}$, in which case $\mathbf{v} + \mathbf{a} \in \Psi$, $\mathbf{v} - \mathbf{a} \notin \Psi$, and $2\mathbf{v}.\mathbf{a} = -32 = -\mathbf{a}.\mathbf{a}$, giving $c = -1$ as required.

Finally, suppose $\mathbf{a} = \alpha_0$. In this case, if $j \notin \{0, 1\}$ then we have $c = c(\mathbf{v}, \mathbf{a}) = 0$ and neither vector $\mathbf{v} \pm \mathbf{a}$ lies in Ψ. If $\mathbf{v} \in \{-4\varepsilon_0, -4\varepsilon_1\}$ then $\mathbf{v} - \mathbf{a} \in \Psi$, $\mathbf{v} + \mathbf{a} \notin \Psi$, and $2\mathbf{v}.\mathbf{a} = 32 = \mathbf{a}.\mathbf{a}$, giving $c = 1$ as required. The other possibility is that $\mathbf{v} \in \{4\varepsilon_{i-1}, 4\varepsilon_i\}$, in which case $\mathbf{v} + \mathbf{a} \in \Psi$, $\mathbf{v} - \mathbf{a} \notin \Psi$, and $2\mathbf{v}.\mathbf{a} = -32 = -\mathbf{a}.\mathbf{a}$, giving $c = -1$ as required. □

Proposition 6.2. *Maintain the notation of Lemma 6.1.*

(i) *The $2n$-dimensional $\mathbb{C}$-vector space V_Ψ has the structure of a $\mathfrak{g}$-module, where $\mathfrak{g}$ is the derived affine Kac–Moody algebra of type $D_n^{(1)}$.*

(ii) *Let $\Psi' = \Psi$ and $\Delta' = \Delta \backslash \{\alpha_0\}$. Then Ψ' is a minuscule system with respect to the simple system Δ', and $V_{\Psi'}$ is a module for the simple Lie algebra $\mathfrak{d}_n$ over $\mathbb{C}$ of type D_n. It is an irreducible module with highest weight vector $4\varepsilon_0$ and lowest weight vector $-4\varepsilon_0$, and affords the natural representation of $\mathfrak{d}_n$.*

Proof. Using Lemma 6.1 in place of Lemma 4.2, the proof of (i) follows the same argument as the proof of Proposition 4.3 (i).

The first assertion of (ii) follows by using Lemma 6.1 and copying the argument of Proposition 4.3 (ii). The operators H_i (for $1 < i \leq n$, where H_n corresponds to α'_n) all act as zero on $4\varepsilon_0$. The operator H_1 acts as the identity on $4\varepsilon_0$. The second assertion is then proved by adapting the corresponding argument in Proposition 4.3 (ii). □

Lemma 6.3. *Let $n \geq 2$, let Ψ be as in Lemma 6.1, and let*

$$\Delta_C = \{\alpha_1, \ldots, \alpha_{n-1}\} \cup \{\alpha''_0, \alpha''_n\},$$

where α_i is as in Lemma 5.1 for $1 \leq i \leq n-1$, $\alpha''_0 = -8\varepsilon_0$ and $\alpha''_n = 8\varepsilon_{n-1}$. Then Ψ is a minuscule system with respect to Δ_C.

Proof. We check Definition 2.1, treating each vector $\mathbf{a} \in \Delta_C$ in turn. The only cases not already covered by Lemma 6.1 are the cases where $\mathbf{a} \in \{\alpha''_0, \alpha''_n\}$. Let $\mathbf{v} \in \Psi$, and define ε_j to be the unique basis element such that $\mathbf{v} = \pm 4\varepsilon_j$.

Suppose that $\mathbf{a} = \alpha''_n$. If $j \neq n-1$ then we have $c = c(\mathbf{v}, \mathbf{a}) = 0$ and neither vector $\mathbf{v} \pm \mathbf{a}$ lies in Ψ, satisfying Definition 2.1. If $\mathbf{v} = \pm 4\varepsilon_{n-1}$ then $\mathbf{v} \mp \mathbf{a} \in \Psi$, $\mathbf{v} \pm \mathbf{a} \notin \Psi$, and $2\mathbf{v}.\mathbf{a} = \pm 64 = \pm \mathbf{a}.\mathbf{a}$, giving $c = \pm 1$ as required.

The other possibility is that $\mathbf{a} = \alpha''_0$. If $j \neq 0$ then we have $c = c(\mathbf{v}, \mathbf{a}) = 0$ and neither vector $\mathbf{v} \pm \mathbf{a}$ lies in Ψ, satisfying Definition 2.1. If $\mathbf{v} = \pm 4\varepsilon_0$ then $\mathbf{v} \pm \mathbf{a} \in \Psi$, $\mathbf{v} \mp \mathbf{a} \notin \Psi$, and $2\mathbf{v}.\mathbf{a} = \mp 64 = \mp \mathbf{a}.\mathbf{a}$, giving $c = \mp 1$ and completing the proof. □

Proposition 6.4. *Maintain the notation of Lemma 6.3.*

(i) *The $2n$-dimensional $\mathbb{C}$-vector space V_Ψ has the structure of a $\mathfrak{g}$-module, where $\mathfrak{g}$ is the derived affine Kac–Moody algebra of type $C_n^{(1)}$.*

(ii) *Let $\Psi' = \Psi$ and $\Delta' = \Delta \backslash \{\alpha''_0\}$. Then Ψ' is a minuscule system with respect to the simple system Δ', and $V_{\Psi'}$ is a module for the simple Lie algebra $\mathfrak{c}_n$ over $\mathbb{C}$ of type C_n. It is an irreducible module with highest weight vector $4\varepsilon_0$ and lowest weight vector $-4\varepsilon_0$, and affords the natural representation of $\mathfrak{c}_n$.*

Proof. The proof is the same as the proof of Proposition 6.2, using Lemma 6.3 in place of Lemma 6.1. □

7. Lines on Del Pezzo surfaces

In §7, we revisit the examples of §4 involving the exceptional Lie algebras $\mathfrak{e}_6$ and $\mathfrak{e}_7$. We will highlight the close link between the representation theory and the combinatorial algebraic geometry associated with configurations of lines on Del Pezzo surfaces. For more details on the latter, the reader is referred to [9, §V.4].

Lemma 7.1. *Let Ψ and Δ be as in Proposition 4.3, and let K_0 and K_7 be as in Definition 4.1. The action of the generators $\{s_{\mathbf{a}} : \mathbf{a} \in \Delta\}$ of the Weyl group $W = W_{\Psi,\Delta}$ on Ψ are as follows. If $\mathbf{a} = \alpha_i$ with $0 \leq i \leq 6$, then*

$$s_{\mathbf{a}}(\pm \mathbf{v}_{j,k}) = s_{\mathbf{a}}(\pm \mathbf{v}_{s_i(j),s_i(k)}),$$

where s_i is the simple transposition $(i, i+1)$. We have $s_{\alpha_7}(\pm \mathbf{v}_{i,j}) = \pm \mathbf{v}_{i,j}$ unless $\{i,j\} \subset K_k$ for some $k \in \{0,7\}$, in which case we have

$$s_{\alpha_7}(\pm \mathbf{v}_{\{i,j\}}) = \mp \mathbf{v}_{K_k \backslash \{i,j\}}.$$

The action of W on Ψ is transitive.

Proof. The formulae for the action of the $s_{\mathbf{a}}$ are obtained by a routine case by case check.

The action of the s_{α_i} for $0 \leq i \leq 6$ makes it clear that the vectors $\{+\mathbf{v}_{i,j} : 0 \leq i < j \leq 7\}$ are W-conjugate to each other, as are the vectors $\{-\mathbf{v}_{i,j} : 0 \leq i < j \leq 7\}$. The fact that $s_{\alpha_7}(+\mathbf{v}_{0,1}) = -\mathbf{v}_{2,3}$ completes the proof. □

Remark 7.2. The transformations induced by s_{α_7} described in the preceding proof are sometimes known as *bifid transformations* (see Example 3 of [13, §4]).

Lemma 7.3. *Let Ψ and Δ be as in Proposition 4.3. The diagonal action of the Weyl group W on $\Psi \times \Psi$ has four orbits, each of which consists of a set*

$$\{(\mathbf{v}_1, \mathbf{v}_2) : \mathbf{v}_1, \mathbf{v}_2 \in \Psi \text{ and } |\mathbf{v}_1 - \mathbf{v}_2| = D\}$$

for some fixed number D. More explicitly, the orbits are as follows:

(i) $\{(\mathbf{v}, \mathbf{v}) : \mathbf{v} \in \Psi\}$, *corresponding to $D = 0$;*
(ii) $\{(\pm\mathbf{v}_{i,j}, \pm\mathbf{v}_{i,k}) : |\{i,j,k\}| = 3\} \cup \{(\pm\mathbf{v}_{i,j}, \mp\mathbf{v}_{k,l}) : |\{i,j,k,l\}| = 4\}$, *corresponding to $D = \sqrt{32}$,*
(iii) $\{(\pm\mathbf{v}_{i,j}, \mp\mathbf{v}_{i,k}) : |\{i,j,k\}| = 3\} \cup \{(\pm\mathbf{v}_{i,j}, \pm\mathbf{v}_{k,l}) : |\{i,j,k,l\}| = 4\}$, *corresponding to $D = \sqrt{64}$,*
(iv) $\{(\mathbf{v}, -\mathbf{v}) : \mathbf{v} \in \Psi\}$, *corresponding to $D = \sqrt{96}$.*

Proof. The assertions about D are easy to check. This other assertions, which are also not difficult to prove, are a restatement of [4, (4.1)]. □

Proposition 7.4. *The 56 elements of Ψ are in natural bijection with the 56 lines of the Del Pezzo surface of degree 2; more precisely, if $\mathbf{v}_1, \mathbf{v}_2 \in \Psi$ are distinct points with $|\mathbf{v}_1 - \mathbf{v}_2| = \sqrt{32D}$, then $D - 1$ is the intersection number of the lines corresponding to $\mathbf{v}_1$ and $\mathbf{v}_2$. In particular, pairs of points at distance $\sqrt{32}$ correspond to skew lines on the Del Pezzo surface.*

The 27 *elements of* $\Psi(\mathbf{n}, 8)$ *(defined in Proposition 4.3 (iii)) are*

$$\{\mathbf{v}_{0,i} : 1 \leq i \leq 6\} \cup \{-\mathbf{v}_{i,j} : 1 \leq i < j \leq 6\} \cup \{\mathbf{v}_{i,7} : 1 \leq i \leq 6\}.$$

These are in natural bijection with the 27 *lines of the Del Pezzo surface of degree* 1*: in Hartshorne's notation* [9, Theorem V.4.9]*, we identify* E_i *with* $\mathbf{v}_{0,i}$*,* F_{ij} *with* $-\mathbf{v}_{i,j}$ *and* G_i *with* $\mathbf{v}_{i,7}$*. The intersection number is defined as in the case of the* 56 *lines.*

Proof. The assertions about the Del Pezzo surface of degree 2 are proved in [6, p28], where it is shown that

$$|\mathbf{v}_1 - \mathbf{v}_2|^2 = d^2(x+1),$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are two distinct points of Ψ, d is the minimal nontrivial distance between two points, and x is the intersection number of the pair of lines corresponding to $\mathbf{v}_1$ and $\mathbf{v}_2$. (The precise link with the polytopes 2_{21} and 3_{21} is given on [6, p33].)

It is easily checked that the 27 elements of $\Psi(\mathbf{n}, 8)$ are as listed. By the result mentioned above, the only possible intersection numbers for two distinct lines on the Del Pezzo surface of degree 3 are 0 (meaning the lines are skew) and 1 (meaning the lines are incident). Since no two elements of $\Psi(\mathbf{n}, 8)$ are at distance $\sqrt{96}$, it remains to check that two distinct points of $\Psi(\mathbf{n}, 8)$ are at distance $\sqrt{32}$ if and only if the corresponding lines are skew, and this follows from the rules given in [9, Remark V.4.10.1]. $\square$

Note that, because $\Psi(\mathbf{n}, -8) = -\Psi(\mathbf{n}, 8)$, the two 27-dimensional representations of $\mathfrak{e}_6$ are interchangeable in this context.

The next result explains how to recover the root system of type E_7 from the set E of directed edges of the polytope 3_{21}.

Proposition 7.5. *Maintain the notation of Lemma 7.3. Let*

$$E = \{(\mathbf{v}_1, \mathbf{v}_2) : \mathbf{v}_1, \mathbf{v}_2 \in \Psi \text{ and } |\mathbf{v}_1 - \mathbf{v}_2| = \sqrt{32}\}$$

has size 1512*. The vectors* $E' = \{\mathbf{v}_1 - \mathbf{v}_2 : (\mathbf{v}_1, \mathbf{v}_2) \in E\}$ *form a root system of type* E_7*, and each of the* 126 *roots occurs with multiplicity* 12 *in* E*.*

Proof. If $\mathbf{v}_1 = \mathbf{v}_{0,1}$ then one checks directly that there are 27 vectors $\mathbf{v}_2$ such that $(\mathbf{v}_1, \mathbf{v}_2) \in E$. The fact (Lemma 7.1) that W acts transitively on Ψ implies by Lemma 7.1 that E has size $|\Psi| \times 27 = 1512$.

For the second assertion, note that $\mathbf{v}_{0,1} - \mathbf{v}_{0,2} = \alpha_1$. The (additive) action of W on Ψ induces an action on E', and Lemma 7.3 (ii) shows that W acts transitively on E'. Note that if $A_{ij} = A_{ji} = -1$, then $s_i s_j(\alpha_i) = \alpha_j$. This implies that all the roots α_i are conjugate under the action of the Weyl group, and then [11, §5.1] shows that the orbit $W.\alpha_1$ consists precisely of the root system of type E_7. By transitivity of the action of W on the root system, each root in E' occurs with the same multiplicity, and by [2, Appendix] there are 126 roots of type E_7. Since $1512/126 = 12$, the proof is completed. □

Proposition 7.6. *Let Ψ' and Δ' be as in Proposition 4.3, and let $V_{\Psi'}$ be the corresponding 56-dimensional representation of the Lie algebra $\mathfrak{e}_7$. If $\mathbf{v} \in \Psi'$ and $x \in \mathfrak{e}_7$, then we have*

$$x.\mathbf{v} = \sum_{\mathbf{u} \in \Psi''} \lambda_{\mathbf{u}} \mathbf{u},$$

where $\Psi'' = \{\mathbf{v}\} \cup \{\mathbf{u} : |\mathbf{v} - \mathbf{u}| = \sqrt{32}\}$. In other words, if $\lambda_{\mathbf{u}} \neq 0$, then either $\mathbf{u} = \mathbf{v}$ or the lines on the Del Pezzo surface of degree 2 corresponding to $\mathbf{u}$ and $\mathbf{v}$ are skew.

A similar result holds for either of the 27-dimensional representations of $\mathfrak{e}_6$ and the Del Pezzo surface of degree 3.

Proof. By [2, §4.1], we have

$$\mathfrak{e}_7 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{h}$ is a 7-dimensional Cartan subalgebra, Φ is the root system for $\mathfrak{e}_7$, and the subspaces $\mathfrak{g}_\alpha$ are one-dimensional. We identify $\mathfrak{e}_7$ with the algebra of operators on the 56-dimensional module V as described in Proposition 4.3 (ii).

With these identifications, if $\alpha = \alpha_i$ for $i \neq 0$, then $\mathfrak{g}_\alpha$ (respectively, $\mathfrak{g}_{-\alpha}$) is spanned by the Lie algebra element $E_i = E_{\alpha_i}$ (respectively, $F_i = F_{\alpha_i}$). The Cartan subalgebra $\mathfrak{h}$ has as a basis the operators $H_i = H_{\alpha_i}$ for $1 \leq i \leq 7$.

It is possible to extend this to a basis for $\mathfrak{e}_7$ in which (a) the subspace $\mathfrak{g}_\alpha$ for α a positive root is spanned by a vector of the form

$$[\cdots[[E_{i_1}, E_{i_2}]E_{i_3}]\cdots E_{i_m}]$$

where $\alpha = \sum_{j=1}^m \alpha_{i_j}$ and (b) the subspace $\mathfrak{g}_\alpha$ for α a negative root is spanned by a vector of the form

$$[\cdots[[F_{i_1}, F_{i_2}]F_{i_3}]\cdots F_{i_m}]$$

where $-\alpha = \sum_{j=1}^m \alpha_{i_j}$. (See [7, Proposition 5.4 (ii), (iv)] or [11, (7.8.5)] for more details.)

It follows that if $b_{\mathbf{v}}$ is a basis element of V, $\alpha \in \Phi$ and $g_\alpha \in \mathfrak{g}_\alpha$, then $g_\alpha.b_{\mathbf{v}} = \lambda b_{\mathbf{v}+\alpha}$ for some scalar λ (meaning that $g_\alpha.b_{\mathbf{v}} = 0$ if $\mathbf{v} + \alpha \notin \Psi$). If $\lambda \neq 0$, then

Proposition 7.5 shows that the distance from $\mathbf{v}$ to $\mathbf{v}+\alpha$ is $\sqrt{32}$. By Proposition 7.4, we see that $\mathbf{v}$ and $\mathbf{v}+\alpha$ correspond to skew lines. It follows easily from the definition of the H_i that if $h \in H$ then $h.b_{\mathbf{v}} = \lambda b_{\mathbf{v}}$ for some scalar λ.

Combining these observations proves the assertions about the 56-dimensional representation. The argument can be easily adapted to work for the 27-dimensional representation, because the root system of type E_6 embeds naturally into the root system of type E_7. □

8. Concluding remarks

In the various constructions presented above for irreducible modules for simple Lie algebras, we did not provide self-contained proofs that the modules constructed were irreducible. However, this was done only to save space, and it is not hard to give an elementary field-independent proof that these modules are irreducible.

One application of the polytope approach to minuscule representations is that one can describe the crystal graph of each of the irreducible modules that arises from the construction directly in terms of the polytope. To do this, one starts with the vertices of Ψ, and for each element $\mathbf{a} \in \Delta$ corresponding to a simple root of the simple Lie algebra, one connects two vertices $\mathbf{v}_1$ and $\mathbf{v}_2$ of Ψ by an edge labelled $\mathbf{a}$ if $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{a}$. It is not hard to show that this produces a realization of the crystal graph, with the extra property that two edges are parallel if and only if they have the same label.

In the cases where the pair (Ψ, Δ) corresponds to a representation of a simple Lie algebra, the elements of Ψ may be partially ordered by stipulating that $\mathbf{v}_1 \leq \mathbf{v}_2$ if $\mathbf{v}_2 - \mathbf{v}_1$ is a positive linear combination of elements of Δ; this corresponds to the usual partial order on the weights of a representation. The resulting partial order on Ψ makes Ψ into a distributive lattice under the operations of greatest lower bound and least upper bound. (This is not a priori obvious, but follows, for example, from the full heaps approach; see [7, Corollary 2.2].) It would be interesting to know whether there is an easy way to define the meet and join operations directly from the data (Ψ, Δ).

It may be tempting to think that one can describe a basis for each of the simple Lie algebras described in this paper by including operators $E_{\mathbf{a}}$ and $F_{\mathbf{a}}$ for *every* positive root $\mathbf{a}$. However, such an algebra of operators would not be closed under the Lie bracket (except in trivial cases) and what is needed instead is to modify the definition of these new operators to introduce sign changes in certain places. This information may easily be computed by hand, because there is a dictionary between the vertices of polytopes in this paper and certain heaps appearing in [7,15]. Associated to each heap is a number ± 1, called the "parity" of the heap, which may easily be computed by hand. One may then introduce sign changes to the operators $E_{\mathbf{a}}$ and $F_{\mathbf{a}}$ according to the parity of the heaps involved. In the simply laced case, a definition of parity is given in [15]. A more versatile and general definition is given in [7, Definition 4.3, Definition 6.3], which can be used to produce the Chevalley basis corresponding to an arbitrary orientation of the Dynkin diagram as described by Kac in [11, (7.8.5), (7.9.3)]. We plan to give full details of this construction elsewhere.

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