International Electronic Journal of Algebra
Volume 4 (2008) 53-62

DUALITY FOR PARTIAL GROUP ACTIONS

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Received: 24 September 2007; Revised: 4 February 2008
Communicated by Derya Keskin Tütüncü

Abstract. Given a finite group G acting as automorphisms on a ring $\mathcal{A}$, the skew group ring $\mathcal{A} * G$ is an important tool for studying the structure of G-stable ideals of $\mathcal{A}$. The ring $\mathcal{A} * G$ is G-graded, i.e. G coacts on $\mathcal{A} * G$. The Cohen-Montgomery duality says that the smash product $\mathcal{A}*G\#k[G]^*$ of $\mathcal{A}*G$ with the dual group ring $k[G]^*$ is isomorphic to the full matrix ring $M_n(\mathcal{A})$ over $\mathcal{A}$, where n is the order of G. In this note we show how much of the Cohen-Montgomery duality carries over to partial group actions in the sense of R.Exel. In particular we show that the smash product $(\mathcal{A} *_\alpha G)\#k[G]^*$ of the partial skew group ring $\mathcal{A} *_\alpha G$ and $k[G]^*$ is isomorphic to a direct product of the form $K \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$ where $\mathbf{e}$ is a certain idempotent of $M_n(\mathcal{A})$ and K is a subalgebra of $(\mathcal{A} *_\alpha G)\#k[G]^*$. Moreover $\mathcal{A} *_\alpha G$ is shown to be isomorphic to a separable subalgebra of $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$. We also look at duality for infinite partial group actions.

Mathematics Subject Classification (2000): 16S35, 16W22
Keywords: Partial Group actions, Cohen-Montgomery duality

1. Introduction

Let k be a commutative unital ring and $\mathcal{A}$ a untial k-algebra. Given a finite group G acting as k-linear automorphisms on $\mathcal{A}$, Cohen and Montgomery showed in [1] that the smash product $\mathcal{A} * G\#k[G]^*$ of the skew group ring $\mathcal{A} * G$ and the dual group ring $k[G]^* = \mathrm{Hom}(k[G], k)$ is isomorphic to the full matrix ring $M_n(\mathcal{A})$ over $\mathcal{A}$, where n is the order of G.

The notion of a partial group action on a k-algebra $\mathcal{A}$ has been introduced by R.Exel in the study of C^*-algebras (see [4]). One says that G *acts partially on* $\mathcal{A}$

This work was carried out as part of the project *Interacções entre álgebras e co-álgebras* between the Universidade do Porto and Universidade Federal do Rio Grande do Sul and Universidade de São Paulo financed through GRICES (Portugal) and CAPES (Brazil) and was partially supported by Centro de Matemática da Universidade do Porto (CMUP), financed by FCT (Portugal) through the programs POCTI (Programa Operacional Ciência, Tecnologia, Inovação) and POSI (Programa Operacional Sociedade da Informação), with national and European community structural funds.

by a family $\{\alpha_g : D_{g^{-1}} \to D_g\}_{g\in G}$ if for all $g \in G$, D_g is an ideal of $\mathcal{A}$ and α_g is an isomorphism of k-algebras such that for all $g, h \in G$:

(i) $D_e = \mathcal{A}$ and α_e is the identity map of $\mathcal{A}$;
(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for all $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

The partial skew group ring of $\mathcal{A}$ and G is defined to be the projective left $\mathcal{A}$-module $\mathcal{A} *_\alpha G = \bigoplus_{g\in G} D_g$ whose multiplication will be defined in the next section. Since $\mathcal{A} *_\alpha G$ is naturally G-graded, the question arises how much of the Cohen-Montgomery duality carries over to partial group actions.

As in [3] we will assume that the ideals D_g are generated by central idempotents, i.e. $D_g = \mathcal{A}1_g$ with central idempotent $1_g \in \mathcal{A}$ for all $g \in G$. For any $g \in G$ we define the following endomorphism $\beta_g : A \to A$ of $\mathcal{A}$ by

$$\beta_g(a) = \alpha_g(a1_{g^{-1}}) \quad \forall a \in \mathcal{A}$$

This map gives rise to a k-linear map $k[G] \otimes A \to A$ with

$$g \otimes a \mapsto g \cdot a := \beta_g(a) = \alpha_g(a1_{g^{-1}})$$

for all $g \in G, a \in \mathcal{A}$.

Lemma 1.1. *With the notation above we have that*

(1) β_g *are k-algebra endomorphisms of $\mathcal{A}$ for all $g \in G$, i.e.*

$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \forall a, b \in \mathcal{A}.$$

(2) $g \cdot (h \cdot a) = ((gh) \cdot a)1_g$ *for all $g, h \in G$ and $a \in \mathcal{A}$.*
(3) $(g \cdot a)b = g \cdot (a(g^{-1} \cdot b))$ *for all $a, b \in \mathcal{A}$ and $g \in G$.*

Proof. (1) follows since the α_g are algebra homomorphisms and the idempotents 1_g are central, i.e. for all $a, b \in \mathcal{A}$:

$$\beta_g(ab) = \alpha_g(ab1_{g^{-1}}) = \alpha_g(a1_{g^{-1}}b1_{g^{-1}}) = \alpha_g(a1_{g^{-1}})\alpha_g(b1_{g^{-1}}) = \beta_g(a)\beta_g(b).$$

(2) follows from [3, 2.1(ii)]:

$$\alpha_g(\alpha_h(a1_{h^{-1}})1_{g^{-1}}) = \alpha_{gh}(a1_{h^{-1}g^{-1}})1_g$$

what expressed by β yields the statement of (2).

(3) Using (1), (2) and the fact that $\beta_e = id$ and that the image of β_g is $D_g = A1_g$ we have that

$$g \cdot (a(g^{-1} \cdot b)) = (g \cdot a)(g \cdot (g^{-1} \cdot b)) = (g \cdot a)b1_g = (g \cdot a)b.$$

□

Obviously we also have $g\cdot 1 = \alpha_g(1_{g^{-1}}) = 1_g$ and $g\cdot(g^{-1}\cdot a) = ((gg^{-1})\cdot a)1_g = a1_g$ for all $a \in \mathcal{A}$ and $g \in G$ using property (2). Moreover using the fact that α_g is bijective and 1_g central we have for all $a \in \mathcal{A}$ and $g \in G$ that $g \cdot a = 0$ if and only if $a \in \mathcal{A}(1 - 1_g)$.

2. Grading of the partial skew group ring

The partial skew group ring is the projective left $\mathcal{A}$-module $\mathcal{A} *_\alpha G = \bigoplus_{g\in G} D_g$. We will write an element of $\mathcal{A} *_\alpha G$ as a finite sum of elements $\sum_{g\in G} a_g\overline{g}$ where $a_g \in D_g = \mathcal{A}1_g$ and $\overline{g}$ is a placeholder for the g-th component. $\mathcal{A} *_\alpha G$ becomes an associative k-algebra by the product:

$$(a\overline{g})(b\overline{h}) = \alpha_g(\alpha_{g^{-1}}(a)b)\overline{gh}$$

for all $g, h \in G$ and $a \in D_g$ and $b \in D_h$. Using our "$\cdot$"-notation we see easily

$$(a\overline{g})(b\overline{h}) = a(g \cdot b)\overline{gh}.$$

The algebra $\mathcal{A} *_\alpha G$ is naturally G-graded where the homogeneous elements are those in $\{D_g\}_{g\in G}$, i.e. $D_gD_h \subseteq D_{gh}$ by definition of the multiplication in $\mathcal{A} *_\alpha G$. Thus $\mathcal{A} *_\alpha G$ becomes a $k[G]$-comodule algebra. Note that the G-grading is strong, in the sense that $D_gD_h = D_{gh}$ if and only if $D_g = \mathcal{A}$ for all $g \in G$, i.e. the G-action is global (since if $D_gD_h = D_{gh}$ for all $g, h \in G$, then

$$\mathcal{A}1_g1_{g^{-1}} = D_gD_{g^{-1}} = D_{gg^{-1}} = D_e = \mathcal{A},$$

thus 1_g is an invertible central idempotent and hence equals 1, i.e. $D_g = \mathcal{A}$). Known results on graded rings can be applied to the G-grading of $\mathcal{A} *_\alpha G$. and we will point out some of those results now. Recall that a graded ring is called graded semiprime, if it has no non-zero nilpotent graded ideals.

Theorem 2.1. *Let G be a finite group acting partially on $\mathcal{A}$.*

1. *$\mathcal{A}$ is semiprime if and only if $\mathcal{A} *_\alpha G$ is graded semiprime.*
2. *If $\mathcal{A}$ is $|G|$-torsion free, then $\mathcal{A}$ is semiprime if and only if $\mathcal{A} *_\alpha G$ is semiprime.*
3. *If $P \subsetneq Q$ are prime ideals in $\mathcal{A} *_\alpha G$, then $P \cap A \subsetneq Q \cap A$ are primes in $\mathcal{A}$.*
4. *If P is a prime in $\mathcal{A} *_\alpha G$, then there are $k \leq |G|$ primes $p_1, \ldots, p_k$ in $\mathcal{A}$ minimal over $P\cap A$, and moreover $P\cap A = p_1\cap\cdots\cap p_k$. The set $\{p_1, \ldots, p_k\}$ is uniquely determined by P.*
5. *Given any prime p of $\mathcal{A}$, there exists a prime P of $\mathcal{A} *_\alpha G$ so that p is minimal over $P \cap A$. There are at most $m \leq |G|$ such primes $P_1, \ldots, P_m$ of $\mathcal{A} *_\alpha G$.*

Proof. (1) follows from [1, 2.9], if we show that the grading of the partial skew group ring is *non-degenerated*. The grading of a G-graded ring $\mathcal{A} = \bigoplus_{g\in G} A_g$ is called *non-degenerated* if for any $g \in G$ and $0 \neq a_g \in A_g$ also $a_g A_{g^{-1}} \neq 0 \neq A_{g^{-1}} a_g$ (see [1, Lemma 2.5]). Take any $0 \neq a_g = a\overline{g} \in A_g = D_g\overline{g}$ of the partial skew group ring $\mathcal{A} *_\alpha G$. Then

$$0 \neq a\overline{e} = (a\overline{g})\left(1_{g^{-1}}\overline{g}^{-1}\right) \in a_g A_{g^{-1}} \quad \text{and}$$

$$0 \neq \alpha_g^{-1}(a)\overline{e} = 1_{g^{-1}}\left(g^{-1}\cdot a\right)\overline{e} = \left(1_{g^{-1}}\overline{g}^{-1}\right)(a\overline{g}) \in A_{g^{-1}}a_g.$$

Hence the G-grading of $\mathcal{A} *_\alpha G$ is non-degenerated.

(2) follows from [1, 5.5]; (3) follows from [1, 7.1]; (4)+(5) follow from [1, 7.3]. □

3. Duality for partial actions of finite groups

Assume G to be finite, then $k[G]^*$ becomes a Hopf algebra with projective basis $p_g \in k[G]^*$ where $p_g(h) = \delta_{g,h}$ for all $g, h \in H$. The multiplication is defined as $p_g * p_h = \delta_{g,h} p_g$ and the identity element of $k[G]^*$ is $1 = \sum_{h\in H} p_h$. Now $\mathcal{A} *_\alpha G$ becomes a $k[G]^*$-module algebra by

$$p_h \triangleright (a\overline{g}) = \delta_{g,h} a\overline{g}$$

for all $g, h \in G$ and $a_g \in D_g$. The multiplication of the smash product $(\mathcal{A} *_\alpha G)\#k[G]^*$ is defined as

$$(a\overline{g}\#p_h)(b\overline{k}\#p_l) = \sum_{s\in G}(a\overline{g})[p_s\triangleright(b\overline{k})]\#p_{s^{-1}h}*p_l = (a\overline{g})(b\overline{k})\#p_{k^{-1}h}*p_l = a(g\cdot b)\overline{gk}\#\delta_{h,kl}p_l.$$

The identity element of $\mathcal{B} = \mathcal{A} *_\alpha G\#k[G]^*$ is $\sum_{h\in G} 1\overline{e}\#p_h$. In the case of global actions Cohen and Montgomery proved in [1] that $\mathcal{A} * G\#k[G]^* \simeq M_n(\mathcal{A})$ where $n = |G|$ and $M_n(\mathcal{A})$ denotes the ring of $n \times n$-matrices over $\mathcal{A}$. We will index the matrices of $M_n(\mathcal{A})$ by elements of G and denote by $E_{g,h}$ the elementary matrix that has the value 1 in the g-th row and the h-th column and zero elsewhere.

Proposition 3.1. *Let G be a finite group of n elements, acting partially on a k-algebra $\mathcal{A}$ and consider the k-algebra $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^*$. The map*

$$\Phi : \mathcal{B} \longrightarrow M_n(\mathcal{A}) \quad \textit{with}$$

$$\sum_{g,h} a_{g,h}\overline{g}\#p_h \mapsto \sum_{g,h} h^{-1}\cdot(g^{-1}\cdot a_{g,h})E_{gh,h}$$

is a k-algebra homomorphism.

Proof. First note that for any $g,h,k \in G$ and $a \in D_g, b \in D_h$ we have, using Lemma refproperties(2) in the 2nd, 4th and 6th line and Lemma 1.1(1) in the 3rd line:

$$\begin{aligned}
k^{-1}\cdot((gh)^{-1}\cdot(a(g\cdot b))) &= k^{-1}\cdot\big(((gh)^{-1}\cdot a)((gh)^{-1}\cdot(g\cdot b))\big)\\
&= \big[k^{-1}\cdot((gh)^{-1}\cdot a)\big]\big[k^{-1}\cdot(h^{-1}\cdot b)\big]\\
&= ((ghk)^{-1}\cdot a)((hk)^{-1}\cdot b)1_{k^{-1}}\\
&= ((ghk)^{-1}\cdot a)1_{(hk)^{-1}}((hk)^{-1}\cdot b)1_{k^{-1}}\\
&= ((hk)^{-1}\cdot(g^{-1}\cdot a))(k^{-1}\cdot(h^{-1}\cdot b))
\end{aligned}$$

Thus we showed:

$$k^{-1}\cdot((gh)^{-1}\cdot(a(g\cdot b))) = ((hk)^{-1}\cdot(g^{-1}\cdot a))(k^{-1}\cdot(h^{-1}\cdot b)) \tag{1}$$

For any $a\overline{g}\#p_h, b\overline{k}\#p_l \in (\mathcal{A} *_\alpha G)\#k[G]^*$ we have, using equation (1):

$$\begin{aligned}
\Phi((a\overline{g}\#p_h)(b\overline{k}\#p_l)) &= \Phi(a(g\cdot b)\overline{gk}\#\delta_{h,kl}p_l)\\
&= l^{-1}\cdot((gk)^{-1}\cdot(a(g\cdot b)))E_{gkl,l}\delta_{h,kl}\\
&= ((kl)^{-1}\cdot(g^{-1}\cdot a))(l^{-1}\cdot(k^{-1}\cdot b))E_{gh,h}E_{kl,l}\delta_{h,kl}\\
&= (h^{-1}\cdot(g^{-1}\cdot a))E_{gh,h}\,(l^{-1}\cdot(k^{-1}\cdot b))E_{kl,l}\\
&= \Phi(a\overline{g}\#p_h)\Phi(b\overline{k}\#p_l)
\end{aligned}$$

Hence Φ is an algebra homomorphism. □

Note that Φ restricted to $\mathcal{A} *_\alpha G$ is injective, i.e. $\mathcal{A} *_\alpha G$ can be considered a subalgebra of $M_n(\mathcal{A})$. In general $\mathrm{Ker}(\Phi)$ is non-trivial, unless the partial action is a global action. Recall the partial order on the boolean algebra $B(\mathcal{A})$ of central idempotents of $\mathcal{A}$: for any $e,f \in B(\mathcal{A}) : e \leq f \Leftrightarrow e = ef$. For our situation of a partial group action G on $\mathcal{A}$ set for any $g \in G$:

$$\Lambda_g = \{h \in G \mid 1_g \not\leq 1_{gh}\}$$

Proposition 3.2. $\mathrm{Ker}(\Phi) = \bigoplus_{g\in G}\bigoplus_{h\in\Lambda_g}\mathcal{A}(1-1_{gh})1_g\overline{g}\#p_h$.

Proof. Suppose $\gamma = \sum_{g,h} a_{g,h}\overline{g}\#p_h \in \mathrm{Ker}(\Phi)$, then $h^{-1}\cdot(g^{-1}\cdot a_{g,h}) = 0$ for all $g,h \in G$. Thus $(g^{-1}\cdot a_{g,h}) \in \mathcal{A}(1-1_h)\cap D_{g^{-1}} = \mathcal{A}(1-1_h)1_{g^{-1}}$. Hence

$$a_{g,h} = g\cdot(g^{-1}\cdot a_{g,h}) \in \mathcal{A}g\cdot(1-1_h) = \mathcal{A}(1_g - 1_g 1_{gh}),$$

i.e. $\gamma \in \bigoplus_{g,h}\mathcal{A}(1-1_{gh})1_g\overline{g}\#p_h = \bigoplus_{g\in G}\bigoplus_{h\in\Lambda_g}\mathcal{A}(1-1_{gh})1_g\overline{g}\#p_h$. The other inclusion follows because $\Phi((g\cdot(1-1_h))\overline{g}\#p_h) = h^{-1}\cdot(g^{-1}\cdot(g\cdot(1-1_h)))E_{gh,h} = h^{-1}\cdot((1-1_h)1_g)E_{gh,h} = 0$. □

Hence the kernel depends on the partial order of the central idempotent 1_g. In particular $\Lambda_e = \emptyset$ means $1 = 1_g$ for all $g \in G$.

Note that the inclusion of $\mathcal{A} *_\alpha G$ into $(\mathcal{A} *_\alpha G)\#k[G]^*$ is given by $a\overline{g} \mapsto \sum_{h\in G} a\overline{g}\#p_h$ for all $g \in G$ and $a \in D_g$. If $\sum_{h\in G} a\overline{g}\#p_h \in \mathrm{Ker}(\Phi)$, then $a \in \mathcal{A}(1-1_{gh})1_g$ for all $h \in G$. In particular for $h = e$ we have $a \in \mathcal{A}(1-1_g)1_g = 0$. Hence Φ restricted to $\mathcal{A} *_\alpha G$ is injective.

We will describe the image of Φ. By definition of Φ, the image of an arbitrary element $\gamma = \sum_{g,h} a_{g,h}\overline{g}\#p_h$ is

$$\Phi(\gamma) = \sum_{g,h} ((gh)^{-1} \cdot a_{g,h})1_{(gh)^{-1}}1_{h^{-1}}E_{gh,h} = \left(b_{r,s}1_{r^{-1}}1_{s^{-1}}\right)_{r,s\in G}$$

with $b_{r,s} = r^{-1} \cdot a_{rs^{-1},s}$ for all $r, s \in G$.

Proposition 3.3. *The image of Φ consists of all matrices of the form $\left(b_{g,h}1_{g^{-1}}1_{h^{-1}}\right)_{g,h\in G}$ for any matrix $(b_{g,h})$ of elements of $\mathcal{A}$. In particular $\mathrm{Im}(\Phi) = \mathbf{e}M_n(A)\mathbf{e}$, where $\mathbf{e}$ is the idempotent $\sum_{g\in G} 1_{g^{-1}}E_{g,g}$.*

Proof. We saw already that an element of the image of Φ is of the given form. Note that by definition of partial group action we have

$$D_g \cap D_{gh} = \alpha_g(D_{g^{-1}} \cap D_h)$$

for all $g, h \in G$. Hence also

$$D_{g^{-1}} \cap D_{h^{-1}} = \alpha_{g^{-1}}(D_g \cap D_{gh^{-1}})$$

holds for all $g, h \in G$. Thus for all $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ such that

$$b1_{g^{-1}}1_{h^{-1}} = \alpha_{g^{-1}}(a1_{gh^{-1}}1_g) = g^{-1} \cdot (a1_{gh^{-1}}).$$

This implies that

$$\begin{aligned} \Phi(a1_g1_{gh^{-1}}\overline{gh^{-1}}\#p_h) &= h^{-1} \cdot ((hg^{-1}) \cdot (a1_g1_{gh^{-1}}))E_{g,h} \\ &= g^{-1} \cdot (a1_g1_{gh^{-1}}))1_{h^{-1}}E_{g,h} \\ &= b1_{g^{-1}}1_{h^{-1}}E_{g,h} \end{aligned}$$

Hence given any matrix $(b_{g,h})$ there are elements $a_{g,h}$ such that

$$\Phi\left(\sum_{g,h} a_{g,h}1_g1_{gh^{-1}}\overline{gh^{-1}}\#p_h\right) = \sum_{g,h} b_{g,h}1_{g^{-1}}1_{h^{-1}}E_{g,h} = \left(b_{g,h}1_{g^{-1}}1_{h^{-1}}\right)_{g,h\in G}.$$

This shows that $\mathrm{Im}(\Phi)$ consists of all matrices of the given form and hence is equal to $\mathbf{e}M_n(A)\mathbf{e}$. Note that $\mathbf{e}$ is the image of the identity element of $\mathcal{B}$. □

The last Propositions yield our main result in this section

Theorem 3.4. $(\mathcal{A} *_\alpha G) \# k[G]^* \simeq \mathrm{Ker}(\Phi) \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

Proof. The kernel of Φ is an ideal and a direct summand of $\mathcal{B} = (\mathcal{A} *_\alpha G) \# k[G]^*$. To see this we first show that the left $\mathcal{A}$-module $I = \bigoplus_{g,h\in G} \mathcal{A}1_{gh}1_g\overline{g}\#p_h$ is a two-sided ideal of $\mathcal{B}$. For any $x\overline{k}\#p_l \in \mathcal{B}$ and $a1_{gh}1_g\overline{g}\#p_h \in I$ we have

$$\begin{aligned}(a1_{gh}1_g\overline{g}\#p_h)(b\overline{k}\#p_l) &= a1_{gh}1_g(g\cdot b1_k)\overline{gk}\#\delta_{h,kl}p_l = a(g\cdot b)\delta_{h,kl}1_{gkl}1_{gk}\overline{gk}\#p_l \in I.\\ (b\overline{k}\#p_l)(a1_{gh}1_g\overline{g}\#p_h) &= b(k\cdot a1_{gh}1_g)\overline{kg}\#\delta_{k,gh}p_h = b(g\cdot a)\delta_{h,kl}1_{kgh}1_{kg}\overline{kg}\#p_h \in I.\end{aligned}$$

Since $I \oplus \mathrm{Ker}(\Phi) = \mathcal{B}$ and both direct summands are two-sided ideals we have $\mathcal{B} = I \times \mathrm{Ker}(\Phi)$ (ring direct product). Moreover $\Phi(I) = \mathbf{e}M_n(\mathcal{A})\mathbf{e} = \mathrm{Im}(\Phi)$. This implies $\mathcal{B} \simeq \mathrm{Ker}(\Phi) \times \mathbf{e}M_n(\mathcal{A})\mathbf{e}$. □

Note that Φ embedds $\mathcal{A} *_\alpha G$ into the Pierce corner $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$.

Corollary 3.5. *$\mathcal{A} *_\alpha G$ is isomorphic to a separable subalgebra of $\mathbf{e}M_n(\mathcal{A})\mathbf{e}$.*

Proof. Recall that the subalgebra $\mathcal{A} *_\alpha G$ sits into $\mathcal{B}$ by $a\overline{g} \mapsto \sum_{h\in G} a\overline{g}\#p_h$. The right action of $\mathcal{A} *_\alpha G$ on $\mathcal{B}$ is given by

$$(x\overline{k}\#p_l)\cdot a\overline{g} = (x\overline{k}\#p_l)\left(\sum_{h\in G} a\overline{g}\#p_h\right) = (x\overline{k})(a\overline{g})\#p_{g^{-1}l}$$

The left action is given by

$$a\overline{g}\cdot(x\overline{k}\#p_l) = \left(\sum_{h\in G} a\overline{g}\#p_h\right)(x\overline{k}\#p_l) = (a\overline{g})(x\overline{k})\#p_l$$

The element

$$f = \sum_{g\in G} \overline{e}\#p_g \otimes \overline{e}\#p_g \in \mathcal{B} \otimes_{\mathcal{A}*_\alpha G} \mathcal{B}$$

is $\mathcal{A} *_\alpha G$-centralising, i.e. for all $a\overline{h} \in \mathcal{A} *_\alpha G$ we have

$$fa\overline{h} = \sum_{g\in G} \overline{e}\#p_g \otimes a\overline{h}\#p_{h^{-1}g} = \sum_{g\in G} a\overline{h}\#p_{h^{-1}g} \otimes \overline{e}\#p_{h^{-1}g} = a\overline{h}f$$

Since also $\mu(f) = \overline{e}\#\sum_{g\in G} p_g = 1_{\mathcal{B}}$ we have that f is a seperability idempotent for $\mathcal{B}$ over $\mathcal{A} *_\alpha G$. Hence $\mathbf{e}M_n(\mathcal{A})\mathbf{e} \simeq \Phi(\mathcal{B})$ is separable over $\Phi(\mathcal{A} *_\alpha G) \simeq \mathcal{A} *_\alpha G$. □

4. Trivial partial actions

The easiest example of partial actions arise from (central) idempotents in a k-algebra $\mathcal{A}$. Suppose that $\mathcal{A}$ admits a non-zero central idempotent, i.e. there exist algebras R, S such that $\mathcal{A} = R \times S$ as algebras. For any group G set $D_g = R \times 0$ and $\alpha_g = id_{D_g}$ for all $g \neq e$ and $D_e = \mathcal{A}$ and $\alpha_e = id_{\mathcal{A}}$. Then $\{\alpha_g \mid g \in G\}$ is a partial action of G on $\mathcal{A}$. The partial skew group ring turns out to be $\mathcal{A} *_\alpha G \simeq R[G] \times S$, where $R[G]$ denotes the group ring of R and G. Note that $0 \times S$ is in the zero-component of the G-grading on $\mathcal{A} *_\alpha G$. If G is finite, say of order n, then a short calculation (using Cohen-Montgomery duality, Proposition 3.2 and Theorem 3.4) shows that $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^*$ is isomorphic to $M_n(R) \times S^n$ where S^n denotes the direct product of n copies of S. Depending on the rings R and S, $\mathcal{B}$ might or might not be Morita equivalent to $\mathcal{A}$. For instance if $R = S = k$ is a field, then any progenerator P for $\mathcal{A}$ has the form $k^r \times k^s$ for numbers $r, s \geq 1$. Thus $\mathrm{End}_k(P) \simeq M_r(k) \times M_s(k)$, whose center is isomorphic to $k^2 = \mathcal{A}$. On the other hand $\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^* \simeq M_n(k) \times k^n$ has center k^{n+1}, i.e. $\mathcal{B}$ will be Morita equivalent to $\mathcal{A}$ if and only if G is trivial.

On the other hand, there are algebras which satisfy (as algebras) $\mathcal{A}^n \simeq \mathcal{A} \simeq M_n(\mathcal{A})$ for any n. To give an example, let R be the ring of sequences of elements of a field k, i.e. $R = k^{\mathbb{N}}$ with componentwise multiplication and addition. The function $\mathbf{e}$ with $\mathbf{e}(2n) = 1$ and $\mathbf{e}(2n+1) = 0$ for all n defines an idempotent of R. The map $\Psi : \mathbf{e}R \to R$ with $\Psi(\mathbf{e}f)(n) = f(2n)$ is a ring isomorphism. Analogously we can show that $(1-\mathbf{e})R \simeq R$. Hence $R^2 \simeq R$. Now take $\mathcal{A} = \mathrm{End}_k(S)$, where $S = R^{(\mathbb{N})}$ denotes the countable infinite free R-module. Using again $\mathbf{e}$ we have that

$$\mathbf{e}\mathcal{A} \simeq (1-\mathbf{e})\mathcal{A} \simeq \mathcal{A} = (\mathbf{e}\mathcal{A}) \times ((1-\mathbf{e})\mathcal{A}) \simeq \mathcal{A} \times \mathcal{A} \simeq \cdots \simeq \mathcal{A}^n$$

for any $n \geq 2$. Moreover for any partition of $\mathbb{N}$ into n infinite disjoint subsets $\Lambda_1, \ldots, \Lambda_n$, we have that

$$S = R^{(\mathbb{N})} \simeq R^{(\Lambda_1)} \oplus \cdots \oplus R^{(\Lambda_n)} \simeq S^n.$$

Hence $\mathcal{A} = \mathrm{End}_k(S) \simeq \mathrm{End}_k(S^n) \simeq M_n(\mathcal{A})$. Applying the double skew group ring construction again we conclude that

$$\mathcal{B} = (\mathcal{A} *_\alpha G)\#k[G]^* \simeq M_n(\mathbf{e}\mathcal{A}) \times ((1-\mathbf{e})\mathcal{A})^n \simeq \mathcal{A} \times \mathcal{A} \simeq \mathcal{A}.$$

5. Infinite partial group actions

Following Quinn [6] we define Φ in case of G being infinite as a map from $\mathcal{A} *_\alpha G$ to the ring of row and column finite matrices. Let $M_G(\mathcal{A})$ be the subring of $\mathrm{End}_k(\mathcal{A}^{|G|})$ consisting of row and column finite matrices $(a_{g,h})_{g,h \in G}$ indexed by elements of G with entries in $\mathcal{A}$, i.e. for any $g \in G$ the sets $\{a_{gh} | h \in G\}$ and $\{a_{hg} | h \in G\}$ are finite. Let $E_{g,h}$ be, as above, those matrices that are 1 in the (g,h)th component and zero elsewhere. Note that $E_{g,h}E_{r,s} = \delta_{h,r}E_{g,s}$. Then define $\Phi : \mathcal{A} *_\alpha G \to M_G(\mathcal{A})$ by

$$a\overline{g} \mapsto \sum_{h \in G} h^{-1} \cdot (g^{-1} \cdot a) E_{gh,h}$$

for any $a\overline{g} \in \mathcal{A} *_\alpha G$. Note that the (infinite) sum on the right side makes sense in $M_G(\mathcal{A})$. As above one checks that Φ is an algebra homomorphism.

Proposition 5.1. *Let G be any group acting partially on $\mathcal{A}$. Then $\mathcal{A} *_\alpha G$ is isomorphic to a subalgebra of $\mathbf{e}M_G(A)\mathbf{e}$ where $M_G(\mathcal{A})$ denotes the ring of row and column finite matrices indexed by elements of G and with entries in $\mathcal{A}$. The element $\mathbf{e}$ is the idempotent $\sum_{g\in G} 1_{g^{-1}} E_{g,g}$.*

Proof. For all $a\overline{g}, b\overline{h} \in \mathcal{A} *_\alpha G$ we have using equation (1) in the 4th line:

$$\begin{aligned}
\Phi(a\overline{g})\Phi(b\overline{h}) &= \left(\sum_{k\in G} k^{-1}\cdot(g^{-1}\cdot a)E_{gk,k}\right)\left(\sum_{l\in G} l^{-1}\cdot(h^{-1}\cdot b)E_{hl,l}\right) \\
&= \sum_{k,l\in G} (k^{-1}\cdot(g^{-1}\cdot a))(l^{-1}\cdot(h^{-1}\cdot b))E_{gk,k}E_{hl,l} \\
&= \sum_{l\in G} ((hl)^{-1}\cdot(g^{-1}\cdot a))(l^{-1}\cdot(h^{-1}\cdot b))E_{ghl,l} \\
&= \sum_{l\in G} l^{-1}\cdot((gh)^{-1}\cdot(a(g\cdot b)))E_{ghl,l} \\
&= \Phi(a(g\cdot b)\overline{g}h) \\
&= \Phi((a\overline{g})(b\overline{h}))
\end{aligned}$$

Hence Φ is an algebra homomorphism. Since

$$\Phi(a\overline{g}) = 0 \Leftrightarrow (\forall h\in G) : h^{-1}\cdot(g^{-1}\cdot a) = 0 \Rightarrow g\cdot(g^{-1}\cdot a) = a1_g = 0 \Rightarrow a = 0,$$

we have that Φ is injective. Moreover $\Phi(a\overline{g}) \in \mathbf{e}M_G(A)\mathbf{e}$ as above. □

References

[1] M. Cohen and S. Montgomery, *Group-graded rings, smash products, and group actions.*, Trans. Amer. Math. Soc., 282(1) (1984), 237–258.

[2] W. Cortes and M. Ferrero, *Partial skew polynomial rings: prime and maximal ideals.*, Comm. Algebra, 35(4) (2007), 1183–1199.

[3] M. Dokuchaev, M. Ferrero and A. Paques, *Partial actions and Galois theory.*, J. Pure Appl. Algebra, 208(1) (2007), 77–87.

[4] R. Exel, *Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences*, J. Funct. Anal., 122 (1994), 361-401.

[5] S. Montgomery, *Hopf algebras and their actions on rings.*, CBMS Regional Conference Series in Mathematics, 82. AMS (1993)

[6] D. Quinn, *Group-graded rings and duality.* Trans. Amer. Math. Soc., 292(1) (1985), 155–167.

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