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SOME RESULTS ON GQP-INJECTIVE MODULES

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Abstract. Let R be a ring. In this note we study some properties of GQP-injective $R-$modules, some results on GP-injective rings and QP-injective modules are extended to these modules. Some new properties of GP-injective rings are obtained.

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1. Introduction

Throughout R is an associative ring with identity and modules are unitary. Recall that a right $R-$module M is called QP-injective [6, 7] if for every M-cyclic submodule K of M, any R-homomorphism from K to M extends to an endomorphism of M. And a right $R-$module M is called GQP-injective [10] if for every $0 \neq s \in S = end(M_R)$, there exists a positive integer n such that $s^n \neq 0$ and any R-homomorphism from s^nM to M extends to an endomorphism of M. Clearly, QP-injective modules are GQP-injective, R is right P-injective [4] if and only if R_R is QP-injective, R is right GP-injective [2] if and only if R_R is GQP-injective. Since GP-injective rings need not be P-injective [3], so GQP-injective modules need not be QP-injective. Following Albu and Wisbauer [1], a module M_R is called Kasch if any simple module in $\sigma[M]$ embeds in M, where $\sigma[M]$ is the category consisting of all M-subgenerated right $R-$modules. It is easy to see that a ring R is right Kasch if and only if R_R is Kasch. In this note we study some properties of GQP-injective modules , especially GQP-injective Kasch modules. Some results on GP-injective rings and QP-injective modules in articles [2,11] are obtained as corollaries and some new results on GP-injective rings are obtained as well.

As usual, we denote the socle and the Jacobson radical of a module N by $Soc(N)$ and $Rad(N)$ respectively. The Goldie dimension and the length of a module N are denoted by $G(N)$ and $c(N)$ respectively. If the Goldie dimension of a module N is finite, then we call N finite dimensional. Let M be a right $R-$module with $S = end(M_R)$ and let $X \subseteq M$ and $Y \subseteq S$, then we write $l_S(X) = \{s \in S \mid sx = 0, \forall x \in X\}$ and $r_M(Y) = \{m \in M \mid ym = 0, \forall y \in Y\}$. And we write $L \trianglelefteq {}_S S$ if L is an essential left ideal of S.

2. GQP-injective Modules

Proposition 2.1. *If M_R is a finitely generated GQP-injective Kasch module with $S = end(M_R)$, then*

(1) $l_S(RadM) \trianglelefteq {}_S S$.

(2) $Soc({}_S S) \trianglelefteq {}_S S$.

(3) For any $s \in S$, Ss is a minimal left ideal of S if and only if $s(M)$ is a simple submodule of M.

Proof. (1) If $0 \neq s \in S$, then there exists a positive integer n such that $s^n \neq 0$ and any R-homomorphism from $s^n M$ to M extends to an endomorphism of M by the GQP-injectivity of M. Choose a maximal submodule T of the right $R-$module $s^n M$. Since M is Kasch, there exists a monomorphism $f : s^n M/T \to M$. Define $g : s^n M \to M$ by $g(x) = f(x + T)$. As M is GQP-injective, $g = s'|s^n M$ for some $s' \in S$. Take $y \in M$ such that $s^n y \overline{\in} T$. Then $s' s^n y = g(s^n y) = f(s^n y + T) \neq 0$, and thus $s' s^n \neq 0$. If $s^n(RadM) \nsubseteq T$, then $S^n(RadM) + T = M$. But $s^n(RadM) << s^n M$ because M is finitely generated, so $T = s^n M$, a contradiction. Hence $s^n(RadM) \subseteq T$. Thus, $(s' s^n)(RadM) = g(s^n(RadM)) = f(0) = 0$, whence $0 \neq s' s^n \in Ss^n \cap l_S(RadM)$. This implies that $l_S(RadM) \trianglelefteq {}_S S$.

(2) Let $0 \neq s \in S$. Since M_R is GQP-injective, there exists a positive integer n such that $s^n \neq 0$ and $l_S(Ker(s^n)) = Ss^n$ by [10,Theorem 3]. Let $Ker(s^n) \subseteq T$ for some maximal submodule T of M, then $Ss \supseteq Ss^n = l_S(Ker(s^n)) \supseteq l_S(T)$. But $l_S(T)$ is minimal by [10, Theorem 12], so $Soc({}_S S) \cap Ss \neq 0$, and hence $Soc({}_S S) \trianglelefteq {}_S S$.

(3) If Ss is minimal, then by [10, Theorem 12], $Ker(s)$ is maximal, and so $s(M) \cong M/Ker(s)$ is simple. Conversely, suppose that $s(M)$ is simple. For any $0 \neq ts \in Ss$, since M_R is GQP-injective, there exists a positive integer n such that $(ts)^n \neq 0$ and any R-homomorphism from $(ts)^n M$ to M extends to an endomorphism of M. Now we define $\varphi : s(M) \to (ts)^n M$ such that $\varphi(sm) = (ts)^n m$ for all $m \in M$, then φ is an isomorphism. Let $i : s(M) \to M$ be the inclusion map and let $\psi = i\varphi^{-1}$. Then ψ is a homomorphism from $(ts)^n M$ to M with $\psi((ts)^n m) = sm$ for all $m \in M$, and so there exists $v \in S$ such that $v(ts)^n m = sm$ for all $m \in M$. It means that $v(ts)^n = s$ and then $Ss = S(ts)$. Therefore, Ss is minimal. □

Corollary 2.2. *If R is a right GP-injective Kasch ring with $J = J(R)$, then*

(1) [2, Lemma 2.2(1), Theorem 2.3(1)] *For any $x \in R$, Rx is a minimal left ideal if and only if xR is a minimal right ideal.*

(2) [2, Theorem 2.3(2)] $Soc(_RR) = Soc(R_R) \trianglelefteq {}_RR$.

(3) [2, Theorem 2.3(4)] $l_R(J) \trianglelefteq {}_RR$.

Theorem 2.3. *Let M_R be a finitely generated GQP-injective Kasch module with $S = end(M_R)$. Then $M/RadM$ is semisimple if and only if S is left finite dimensional. In this case, $Soc(_SS) = l_S(RadM)$, and $G(_SS) = c(_SSoc(S_S)) = c(M/RadM)$*

Proof. ($\Rightarrow$) The case $M = 0$ is trivial. If $M \neq 0$, then $M/RadM \neq 0$ because M is finitely generated. As $M/RadM$ is semisimple, by [11, Lemma 8], there exist maximal submodules $T_1, T_2, \cdots, T_n$ such that $M/RadM \cong \oplus_{i=1}^n M/T_i$. Hence, by [11, Lemma 7] and [10,Theorem 12], $l_S(RadM) \cong {}_SHom_R(M/RadM, {}_SM_R) \cong {}_SHom_R(\oplus_{i=1}^n M/T_i, {}_SM_R) \cong \oplus_{i=1}^n l_S(T_i)$ is an n–generated semisimple module. This implies that $l_S(RadM) = Soc(_SS) \trianglelefteq {}_SS$ by Proposition 2.1, and therefore S is left finite dimensional and $G(_SS) = n = c(_SSoc(_SS))$.

($\Leftarrow$) See [11, Proposition 6]. □

Our next result improves [2, Theorem 2.8]

Corollary 2.4. *Let R be right GP-injective and right Kasch. Then R is semilocal if and only if R is left finite dimensional. In this case, $Soc(_RR) = Soc(R_R)$, and $G(_RR) = c(_RSoc(_RR)) = c(\overline{R}_R)$, where $\overline{R} = R/J(R)$*

Proof. This is immediate from Theorem 2.3 and Corollary 2.2. □

Proposition 2.5. *Let M_R be a GQP-injective module with $S = end(M_R)$. Then*

(1) If $s, t \in S$ and $sM \cong tM$ are simple, then $Ss \cong St$.

(2) If M_R is a self-generator, then $Soc(M_R) \subseteq Soc(_SM)$.

Proof. (1) By hypotheses, there exists a positive integer n such that $s^n \neq 0$ and any R-homomorphism from s^nM to M extends to an endomorphism of M. Since sM is simple, $s^nM = sM$. Let $\sigma : sM \to tM$ be an isomorphism, then σ extends to an endomorphism τ of M. Let $\phi : St \to Ss$ be defined by $\phi(ut) = u\tau s$. Then ϕ is well defined since $(\tau s)M \subseteq t(M)$. Now it is routine to verify that ϕ is an isomorphism.

(2) Since M_R is a self-generator, every simple submodule K of M_R has the form $s(M)$ for some $s \in S$, thus by the proof of Proposition 2.1(3), Ss is simple. This

follows that $SsM \cong Ss$ and hence SsM is a simple left S- module. Therefore, $K \subseteq Soc(M_R)$, and (2) follows. □

Let $S = end(M_R)$, following [5], we write $W(S) = \{s \in S \mid ker(s) \text{ is essential in } M\}$.

Lemma 2.6. *Let M_R be GQP-injective which is a self-generator with $S = End(M_R)$. If $s \notin W(S)$, then the inclusion $Ker(s) \subset Ker(s - sts)$ is strict for some $t \in S$.*

Proof. If $s \notin W(S)$, then $Ker(s) \cap K = 0$ for some nonzero submodule K of M, and so $Ker(s) \cap s'(M) = 0$ for some $0 \neq s' \in S$ because M_R is a self-generator. Clearly, $ss' \neq 0$. Since M_R is GQP-injective, there exists a positive integer n such that $(ss')^n \neq 0$ and $l_S(Ker(ss')^n) = S(ss')^n$. Thus, $s'(ss')^{n-1} \in l_S(Ker(s'(ss')^{n-1})) = l_S(Ker((ss')^n)) = S(ss')^n$. Write $s'(ss')^{n-1} = t(ss')^n$, then $(1 - ts)s'(ss')^{n-1} = 0$ and hence $(s - sts)s'(ss')^{n-1} = 0$. It is obvious that $Ker(s) \subseteq Ker(s - sts)$. Note that $(s'(ss')^{n-1})M$ is contained in $Ker(s - sts)$ but not contained in $Ker(s)$, the inclusion $Ker(s) \subseteq Ker(s - sts)$ is strict. □

Theorem 2.7. *Let M_R be GQP-injective which is a self-generator with $S = end(M_R)$. Then the following conditions are equivalent.*

(1) S is right perfect.

(2) For any sequence $\{s_1, s_2, \cdots\} \subseteq S$, the chain $Ker(s_1) \subseteq Ker(s_2 s_1) \subseteq \cdots$ terminates.

Proof. By using [10, Theorem 5], Lemma 2.6 and [12, Lemma 2.8] , one can complete the proof in a similar way to that of [12, Theorem 2.9]. □

Following [8], a module M_R is said to be *GC2* if for any $N \leq M$ with $N \cong M$, N is a direct summand of M.

Proposition 2.8. *Let M be a right R-module with $S = end(M_R)$. Then the following conditions are equivalent.*

(1) M_R is $GC2$.

(2) If $Ker(s) = 0$, $s \in S$, then $S = Ss$.

Proof. (1)⇒(2) Let s be given as in (2). Then the mapping $\sigma : sM \to M; sm \mapsto m$ is an R-isomorphism. By (1), sM is a direct summand of M, so σ can be extended to an endomorphism t of M. It then follows that $1 = ts \in Ss$.

(2)⇒(1) Suppose that N is a submodule of M and $N \cong M$. Let $f : M \to N$ be an isomorphism and let $i : N \to M$ be the inclusion mapping, writing $s = if$, then $N = s(M)$ and $Ker(s) = 0$. So by (2), $1 = ts$ for some $t \in S$. This follows that $(st)^2 = st$ and $sM = (st)M$. Whence N is a direct summand of M. □

Theorem 2.9. *If M_R is a GQP-injective module, then it is GC2.*

Proof. Let $s \in S = end(M_R)$ with $Ker(s) = 0$. Then $Ker(s^k) = 0$ for each positive integer k. Since M_R is GQP-injective, there exists a positive integer n such that $s^n \neq 0$ and $l_S(Ker(s^n)) = Ss^n$. Which implies that $S = Ss^n$ and then $S = Ss$. □

Since the endomorphism ring of a finite dimensional GC2 module is semilocal by [9, Lemma 1.1], we have immediately the following:

Corollary 2.10. *Let M_R be a GQP-injective module with $S = end(M_R)$. If M_R is finite dimensional, then S is semilocal.*

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