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JCP-INJECTIVE RINGS

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Abstract. As a generalization of right p-injective rings, we introduce the notion of right Jcp-injective rings, i.e. for any right nonsingular element c of R and any right R-homomorphism $g : cR \to R$, there exists $m \in R$ such that $g(ca) = mca$ for all $a \in R$. Some important results which are known for right p-injective rings are shown to hold for right Jcp-injective rings.

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1. Introduction and Preliminaries

Throughout this paper, R denotes an associative ring with identity, and all modules are unitary. We write $_RM$ and M_R to indicate a left and right R-module, respectively. For any nonempty subset X of a ring R, $r(X)$ and $l(X)$ denote the right annihilator of X and the left annihilator of X, respectively. If $X = \{a\}$, we usually abbreviate it to $l(a)$ and $r(a)$. As usual, $J(R) = J$, Z_l (Z_r), S_l (S_r) and $N(R)$ stand for the Jacobson radical of R, the left (right) singular ideal of R and the left (right) socle of R and the set of all nilpotent elements of R, respectively. $N|M$ will mean that submodule N is a direct summand of M.

A ring R is called right Jcp-injective if for each $a \in R \setminus Z_r$, any homomorphism from aR to R can be extended to one of R into R. Clearly, right p-injective rings are right Jcp-injective. In section 2, Theorem 2.1 gives some characterizations of right Jcp-injective rings. Example 2.4 points out that there exists a right Jcp-injective ring which is not right p-injective. In this section, we also consider some conditions for a right Jcp-injective ring being right p-injective.

(von Neumann) regular rings have been studied extensively by many authors (for example, [5], [6] and [9]). It is well known that a ring R is regular if and only if every

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right R-module is p-injective. In Theorem 2.9, we give some new characterizations of von Neumann regular rings.

Call a right R-module M *nil*-injective [15] if for each $a \in N(R)$, any right R-homomorphism $aR \to M$ can be extended to $R \to M$. If R is *nil*-injective as a right R-module, then we call R a right *nil*-injective ring. Theorem 3.7 shows that if R is a right Jcp-injective ring such that every simple singular right R-module is *nil*-injective, then R is a right p-injective, semiprimitive and left and right nonsingular ring.

In Section 4, we consider right weakly injective rings and obtain the following equivalent conditions for a right weakly injective ring R: (1) R is right self-injective; (2) R is right Jcp-injective; (3) R is right weakly Gnp-injective. This generalizes many known results which appears in [11] and [3].

In Section 5, we give some characterizations of division rings and semisimple artinian rings.

2. Right Jcp-injective Rings

A right R-module M is Jcp-injective if for each $a \in R \setminus Z_r$, every right R-homomorphism from aR to M can be extended to one of R into M. If R_R is Jcp-injective, we call R is a right Jcp-injective ring. Clearly, right p-injective rings are right Jcp-injective.

Theorem 2.1. *The following conditions are equivalent for a ring R.*

(1) R is right Jcp-injective.

(2) $lr(a) = Ra$ for each $a \notin Z_r$.

(3) $r(a) \subseteq r(b), a, b \in R$ and $a \notin Z_r$ implies that $Rb \subseteq Ra$.

(4) $l(bR \cap r(a)) = l(b) + Ra$ for $a, b \in R$ with $ab \notin Z_r$.

Proof. (1) $\Rightarrow$ (2). Clearly, $Ra \subseteq lr(a)$ for all $a \notin Z_r$. Now let $x \in lr(a)$ and $f : aR \to R$ defined by $f(ar) = xr$. Then f is a well defined right R-homomorphism because $r(a) \subseteq r(x)$. By (1), $f = c\cdot$ for some $c \in R$. Hence $x = f(a) = ca \in Ra$, which implies that $lr(a) \subseteq Ra$. Consequently, $lr(a) = Ra$.

(2) $\Rightarrow$ (3) If $r(a) \subseteq r(b)$ with $a, b \in R$ and $a \notin Z_r$, then $Rb \subseteq lr(b) \subseteq lr(a)$. Since $a \notin Z_r$, by (2), $lr(a) = Ra$. Hence $Rb \subseteq Ra$.

(3) $\Rightarrow$ (4) Clearly, $l(b) + Ra \subseteq l(bR \cap r(a))$. Now let $x \in l(bR \cap r(a))$. Then $r(ab) \subseteq r(xb)$. Since $ab \notin Z_r$, $Rxb \subseteq Rab$ by (3). So $xb = cab$ for some $c \in R$. Hence $x - ca \in l(b)$, as required.

(4) $\Rightarrow$ (1) Let $a \notin Z_r$ and $f : aR \to R$ be any right R-homomorphism. Clearly, $r(a) \subseteq r(f(a))$. So $f(a) \in lr(f(a)) \subseteq lr(a) = l(1R \cap r(a)) = l(1) + Ra = Ra$ by (4)

because $a1 \notin Z_r$, i.e., $f(a) \in Ra$. Write $f(a) = ca$ for some $c \in R$. Then we can define $g : R_R \to R_R$ by $g(r) = cr, r \in R$. Obviously, $g|_{aR} = f$. Consequently, R is right Jcp-injective. □

A ring R is called right p-injective if and only if $lr(a) = Ra$ for all $a \in R$ [8, Lemma 1.1]. A ring R is called right NPP [15] if for any $a \in N(R)$, aR is projective. Clearly, right PP rings are right NPP. [15, Theorem 2.10] shows that right NPP rings are right nonsingular. A ring R is called von Neumann regular if $a \in aRa$ for all $a \in R$. Clearly, R is von Neumann regular if and only if R is right *pp* right p-injective. A ring R is called ZI if $ab = 0$ implies that $aRb = 0$ for all $a, b \in R$. For example, a reduced ring (that is $a^2 = 0$ implies that $a = 0$ for all $a \in R$) is ZI. Clearly, R is a regular ZI ring if and only if R is a strongly regular ring (that is, $a \in a^2R$ for all $a \in R$).

Corollary 2.2. *(1) If R satisfies one of the following conditions, then R is right p-injective if and only if R is right Jcp-injective.*

(a) R is right nonsingular.

(b) R is right NPP.

(c) R is right PP.

(2) R is von Neumann regular if and only if R is right pp and right Jcp-injective.

Corollary 2.3. *Let R be ZI right Jcp-injective. Then the following conditions are equivalent.*

(1) R is semiprime.

(2) R is strongly regular.

(3) R is von Neumann regular.

(4) $J(R) = 0$.

(5) R is reduced.

(6) R is right pp.

Proof. Assume (1). Let $a \in R$ and write $T = aR \cap r(a)$. Then $T^2 = 0$ by hypothesis and so $T = 0$. This shows that R is a right nonsingular ring and $r(a^2) = r(a)$. So $Ra = Ra^2$ by hypothesis and $R = l(0) = l(aR \cap r(a)) = Ra \oplus l(a)$ because R is a right Jcp-injective ring. Hence R is von Neumann regular and reduced. Certainly, R is also a right *pp* ring with $J(R) = 0$. □

Example 2.4. Let V be a two-dimensional vector space over a field F, the trivial extension $R = T(F, V) = F \oplus V$ is a commutative, local, artinian ring with $J^2 = 0$ and $J = Z_r$. But R is not a p-injective ring [13]. On the other hand, if $x \in R$

with $x \notin Z_r$, then x is invertible. So $lr(x) = R = Rx$. This implies that R is a right Jcp-injective. Hence there exists a right Jcp-injective ring which is not right p-injective.

Recall that a ring R is right $C2$ if every right ideal T which is isomorphic to a summand of R_R is a summand [13]. In [13], it is shown that right p-injective rings are right $C2$. We can generalize the result to Jcp-injective rings. An element $a \in R$ is called right regular if $r(a) = 0$. [7, Theorem 1] is improved in the next theorem.

Theorem 2.5. *Let R be right Jcp-injective. Then:*

(1) Any right regular element of R is left invertible.

(2) $Z_r \subseteq J(R)$.

(3) Every left or right R-module is divisible.

(4) If P is a reduced principal right ideal of R, then $P = eR$, where $e = e^2 \in R$ and $(1-e)R$ is an ideal of R.

(5) R is right $C2$.

(6) If $aR|R, bR|R$ with $aR \cap bR = 0$, then $(aR \oplus bR)|R$.

(7) The following conditions are equivalent for a $a \notin Z_r$:

(a) aR is projective.

(b) $aR|R$.

(c) aR is a Jcp-injective module.

Proof. (1) Let $c \in R$ such that $r(c) = 0$. Then $c \notin Z_r$ and so by Theorem 2.1, $R = lr(c) = Rc$, which proves (1).

(2) If $z \in Z_r$ and $a \in R$, then $r(1 - az) = 0$ implies that $v(1 - az) = 1$ for some $v \in R$ by (1). This proves that $z \in J(R)$.

(3) If c is a non-zero-divisor in R, then $dc = 1$ for some $d \in R$ by (1). Now $l(c) = 0$ implies that $cd = 1$ and for any right R-module M, $M = Mdc \subseteq Mc \subseteq M$ implies that $M = Mc$. Similarly, any left R-module is divisible.

(4) Let P be a non-zero reduced principal right ideal. Then $P = cR$ for some $c \in R$. Since $c^2 \notin Z_r$, $lr(c^2) = Rc^2$. Hence $r(c) = r(c^2)$ shows that $Rc = lr(c) = lr(c^2) = Rc^2$. Therefore $c = bc^2$ for some $b \in R$, which implies that $c = cbc$ because P is reduced, whence P is generated by the idempotent $e = cb$. Also, for any $a \in R$, $(ea - eae)^2 = 0$ implies $ea = eae$, whence $eR(1-e) = 0$. Therefore $R(1-e) \subseteq (1-e)R$ which establishes the last part of (4).

(5) Assume that $aR \cong eR$, where $a, e^2 = e \in R$. Then $a \notin Z_r$, so $Ra = lr(a)$ by Theorem 2.1. Since $aR \cong eR$, by [11, Theorem 1.2], there exists a $f^2 = f \in R$

such that $af = a$ and $r(a) = r(f)$. Hence $Ra = lr(a) = lr(f) = Rf$. Consequently, $aR|R$.

(6) Follows from (5).

(7) By (5), we have $(a) \Leftrightarrow (b)$. Obviously, $(c) \Rightarrow (b)$ always holds.

$(b) \Rightarrow (c)$ We only need to show that $l_{aR}r_R(b) = aRb$ for each $b \notin Z_r$. In fact, if $ac \in l_{aR}r_R(b)$, then $r_R(b) \subseteq r_R(ac)$, so $ac \in l_Rr_R(ac) \subseteq l_Rr_R(b) = Rb$. Because $aR = eR$ for some $e^2 = e \in R$, $ac = eac \in eRb = aRb$. Hence $l_{aR}r_R(b) \subseteq aRb$ which shows that $l_{aR}r_R(b) = aRb$. □

Example 2.6. Faith and Menal [4] give an example of a right noetherian ring R in which every right ideal is an annihilator, but which is not right artinian. Thus R is left *Jcp*-injective. But R is not right $C2$, hence it is not right *Jcp*-injective. Therefore there exists a left *Jcp*-injective ring which is not right *Jcp*-injective.

A ring R is called right FGF if every finitely generated right R-module can be embedded in a free module. It is an open question whether every right FGF ring is quasi-Frobenius. The conjecture is known to be true if the ring is right $C2$ [13]. Hence we derive that if every right FGF ring R is a right *Jcp*-injective, then R is quasi-Frobenius.

A ring R is called right *Johns* if it is right noetherian and every right ideal is an annihilator. If the matrix ring $M_n(R)$ is right *Johns* for every $n \geq 1$, then R is called strongly right *Johns*. [13, Theorem 4.6] shows that R is quasi-Frobenius if and only if R is strongly right *Johns* and right $C2$. Hence we have that R is quasi-Frobenius if and only if R is strongly right *Johns* and right *Jcp*-injective.

Recall that a ring R is directly finite if $uv = 1$ in R implies that $vu = 1$. For example, semilocal rings are directly finite. Obviously, R is directly finite if and only if every epimorphism $R_R \to R_R$ is an isomorphism. It is known that (1) if each monomorphism $R_R \to R_R$ is an isomorphism, then R is a directly finite; (2) R is directly finite if and only if $R/J(R)$ is directly finite.

Recall that a module M_R is $GC2$ if $N \subseteq M$ with $N_R \cong M$ implies that $N|M$. A ring R is right $GC2$ if R_R is $GC2$. Clearly, a right $C2$ ring is right $GC2$.

Yiqiang Zhou shows that if M_R is $GC2$ and M_R is finite dimensional, then $End(M_R)$ is a semilocal ring.

Corollary 2.7. *Let R be a right Jcp-injective ring. Then:*

(1) If R_R is of finite Goldie dimensional, then R is semilocal.

(2) If $J(R)$ is nilpotent, then R is right noetherian if and only if R is right artinian.

(3) The following conditions are equivalent.

(a) $R/J(R)$ is directly finite.

(b) Every monomorphism $R_R \to R_R$ is an isomorphism.

(c) R is directly finite.

If every complement right ideal of R is not singular, then the conditions above are also equivalent to

(d) R/Z_r is directly finite.

Proof. (1) Since R is right $C2$, R is right $GC2$. Hence $R \cong End(R_R)$ is semilocal, because R_R is finite dimensional.

(2) If R is right noetherian, then R is semilocal by (1). Hence R is semiprimary because $J(R)$ is nilpotent. Consequently, R is right artinian.

(3) $(b) \Rightarrow (c) \Leftrightarrow (a)$ and $(d) \Rightarrow (c)$ are trivial.

$(c) \Rightarrow (b)$ Assume that $f : R_R \to R_R$ is a monomorphism. Since R is right $GC2$, $Im(f) = eR$ for some $e^2 = e$ because $Im(f) \cong R_R$. Write $f(1) = a$, then $aR = f(R) = eR$. Hence $a = ea = aba$ for some $b \in R$ with $e = ab$. Thus $ba = 1$ because $r(a) = 0$. By (c), $ab = 1$, so $f(R) = aR = eR = abR = R$. This implies that f is an epimorphism.

$(c) \Rightarrow (d)$ Let $a, b \in R$ such that $1 - ab \in Z_r$. Since R is right Jcp-injective, $1 - ab \in J(R)$ by Theorem 2.7(2). Let $ab = 1 + x$ for some $x \in J(R)$. So $ab(1+x)^{-1} = 1$. Since R is directly finite, $b(1+x)^{-1}a = 1$. If $x \notin Z_r$, then there exists a nonzero right ideal I of R which is maximal with respect to the property that "$I \cap r(x) = 0$". By hypothesis, there exists $b \in I$ such that $b \notin Z_r$. Hence $xb \notin Z_r$. Let $f : xbR \to R$ be defined by $f(xbr) = br$ for all $r \in R$. Then f is a well defined right R-homomorphism. Since R is right Jcp-injective, $f = c\cdot$, $c \in R$. Hence $b = f(xb) = cxb$ and so $(1 - cx)b = 0$. Since $cx \in J(R)$, $b = 0$ which is a contradiction. Hence $x \in Z_r$, which implies that $1 - ba \in Z_r$, so R/Z_r is directly finite. □

In fact, from the proof of Corollary 2.7(3), we know that every monomorphism $R_R \to R_R$ is an isomorphism if and only if R is directly finite and right $GC2$.

Call a ring R abelian if every idempotent element of R is central. As examples of abelian rings we have ZI rings and reduced rings. Clearly, abelian rings are directly finite. Call a ring I-finite if it contains no infinite set of orthogonal idempotents. I-finite rings are also directly finite.

In [11], it is proved that if R is right p-injective, then $J(R) = Z_r$. We do not know whether the result holds for right Jcp-injective. But, from the proof of Corollary 2.7(3), we can obtain the following corollary.

Corollary 2.8. *Let R be a right Jcp-injective ring. Then:*

(1) If R satisfies one of the following conditions, then every monomorphism $R_R \to R_R$ is an isomorphism.

(a) R is abelian.

(b) R is I-finite.

(c) R is semilocal.

(2) If each non-zero complement right ideal of R is not contained in Z_r, then

(a) $J(R) = Z_r$.

(b) for each $a \in R \setminus J(R)$, there exists a $c \in R$ such that the inclusion $r(a) \subset r(a - aca)$ is proper.

Call a ring R right mininjective [12] if for each right minimal element $k \in R$, $Rk = lr(k)$. Right p-injective rings are right mininjective. But we don't know temporarily whether the result holds for a right Jcp-injective ring. Call a ring R right principally small injective if for any $a \in J(R)$, every R-homomorphism from aR to R_R can be extended to one from R_R into R_R. Clearly, R is right principally small injective if and only if $Ra = lr(a)$ for all $a \in J(R)$.

Call a ring R right SPP if for any $a \notin Z_r$, aR is projective.

Call a ring R right PS [10] if each minimal right ideal of R is projective as a right R-module. Clearly, the following conditions are equivalent for a ring R: (1) R is right PS. (2) $Z_r \cap Soc(R_R) = 0$. (3) Every homomorphic image of a mininjective right R-module is mininjective. Examples of right PS rings contain right pp and right universally mininjective [12]. Clearly, R is right universally mininjective if and only if R is right PS and right mininjective.

Call a ring R right $MC2$ [13] if each projective minimal right ideal is a summand in R_R. As examples, we have right mininjective and right $C2$ rings. In [16], we show that (1) R is right $MC2$ if and only if $Z_r \cap Soc(R_R) = J(R) \cap Soc(R_R)$. (2) R is right universally mininjective if and only if R is right PS and right $MC2$. (3) If R is a right $MC2$ I- finite ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent. Since right $C2$ rings are right $MC2$, by Theorem 2.5, we have: if R is a right Jcp-injective I-finite ring, then $R \cong R_1 \times R_2$, where R_1 is semisimple and every simple right ideal of R_2 is nilpotent.

Theorem 2.9. *(1) R is right p-injective if and only if R is right Jcp-injective and right principally small injective.*

(2) R is right SPP if and only if every homomorphic image of a right Jcp-injective R-module is Jcp-injective.

(3) The following conditions are equivalent for a ring R.

(a) R is right pp.

(b) R is right NPP and right SPP.

(c) R is right nonsingular and right SPP.

(4) Let R be a right Jcp-injective and right PS ring. Then R is right universally mininjective, and so R is right mininjective.

(5) Let R be right Jcp-injective and S_r be essential in R_R. Then the following conditions are equivalent.

(a) R is right PS.

(b) R is right universally mininjective.

(c) R is semiprimitive.

(d) R is right nonsingular.

(e) R is semiprime.

In this case, R is right p-injective.

(6) Let R be right Jcp-injective and semiperfect with S_r essential in R_R. Then the following conditions are equivalent.

(a) R is right PS;

(b) R is right pp;

(c) R is semisimple.

(7) The following conditions are equivalent for a ring R.

(a) R is von Neumann regular.

(b) R is right nonsingular right SPP and right $C2$.

(c) R is right nonsingular right SPP and right Jcp-injective.

Proof. (1) (The "only if" part) Let R be a right Jcp-injective and principally small injective ring. By Theorem 2.5(2), $Z_r \subseteq J(R)$. Let $a \in R$. If $a \notin Z_r$, then $lr(a) = Ra$ by Theorem 2.1. If $a \notin Z_r$, then $a \in J(R)$. We claim that $Ra = lr(a)$. $Ra \subseteq lr(a)$ is clear. Let $x \in lr(a)$, then $r(a) \subseteq r(x)$. Let $f : aR \to R$ be defined by $f(ar) = xr$. Then f is a well defined right R-homomorphism. Since R is right principally small injective, there exists a right R-homomorphism $g : R \to R$ such that $f(a) = g(a)$. Hence $x = f(a) = g(a) = g(1)a \in Ra$ and so $lr(a) \subseteq Ra$.

(2) Similar to [15, Theorem 2.10(1)].

(3) Follows from the definitions and [15, Theorem 2.10(2)].

(4) Since R is right Jcp-injective, R is right $C2$ by Theorem 2.5(5). Hence R is right $MC2$, and so R is right universally mininjective because R is right PS. Hence R is right mininjective.

(5) By (4), we have $(a) \Rightarrow (b)$. Assume that (b) holds, then $S_r \cap J(R) = 0$. Hence $J(R) = 0$ because S_r is essential in R_R. So $(b) \Rightarrow (c) \Rightarrow (e) \Rightarrow (a)$ hold. Since $Z_r \subseteq J(R)$, $(c) \Rightarrow (d) \Rightarrow (a)$ hold.

In this case, by Corollary 1.2, R is right p-injective.

(6) $(c) \Rightarrow (b) \Rightarrow (a)$ are trivial. Assume (a). Then R is right p-injective by (5). So R is a right GPF ring by [11], and then $S_r = l(J(R))$ by [11, Corollary 2.2]. But by (5), $J(R) = 0$, so $S_r = R$. Hence R is a semisimple ring.

(7) Follows from Corollary 2.2. □

Clearly, the ring R in Example 2.4 is not PS. Otherwise, R is a semisimple ring. In fact, since R is artinian, R is a semiperfect ring, and S_r is essential in R_R. Hence R is a semisimple ring by Theorem 2.9(6). But $J(R) = Z_r \neq 0$, which is a contradiction. Hence R is not a PS ring, and so is not universally mininjective. On the other hand, we claim that R is not mininjective. In fact, if $V = vF \oplus wF$, then $(0, vR)$ is a minimal right ideal of R, and let $\theta : V \to V$ be a linear transformation with $\theta(v) = w$. Then $(0, x) \longmapsto (0, \theta(x))$ is an R- linear map from $(0, v)R \to R$ which cannot be extended to $R \to R$ because $w \notin vF$. So R is not a mininjective ring. Hence there exists a Jcp-injective ring which is not mininjective.

Example 2.10. The trivial extension $R = T(Z, Z_{2^\infty})$ is a commutative ring with $Z_r = J \neq 0$ which is not right $C2$, so is not right Jcp-injective. On the other hand, R has a simple essential socle, so R is a mininjective ring. So there exists a mininjective ring which is not Jcp-injective.

A ring R is called right quasi-regular if $a \in aRa$ for all $a \notin Z_r$. Clearly, R is von Neumann regular if and only if R is right nonsingular and right quasi-regular.

Theorem 2.11. *(1) Let R be a right Jcp-injective and a right SPP ring. Then*

(a) $Z_l \subseteq J(R) = Z_r$;

(b) for each $a \notin Z_r$, $a = aba$ for some $b \in R$. So $R/J(R)$ is von Neumann regular.

(2) If R is right SPP and Abelian, then $N(R) \subseteq Z_r$.

(3) Let R be a right SPP ring. Then the following conditions are equivalent.

(a) R is reduced.

(b) R is abelian right nonsingular.

(c) R is abelian right NPP.

(4) The following conditions are equivalent for a ring R.

(a) R is right quasi-regular.

(b) R is right Jcp-injective and right SPP.

(c) Every right R-module is Jcp-injective.

(d) Every cyclic right R-module is Jcp-injective.

Proof. (1) First, we show that $J(R) \subseteq Z_r$. Otherwise, there exists a $a \in J(R)$ such that $b \notin Z_r$. So $r(a) = r(e)$ for some $e^2 = e \in R$ because R is right SPP. Since R is right Jcp-injective, $Re = lr(e) = lr(a) = Ra \subseteq J(R)$. This is a contradiction. Similarly, we can show that $Z_l \subseteq Z_r$. By Theorem 2.5, we have $Z_l \subseteq J(R) = Z_r$. Next, let $a \notin Z_r$. Then $Ra = Re$, so $a = ae \in aRa$. Hence $R/J(R)$ is von Neumann regular.

(2) If $N(R) \nsubseteq Z_r$, then there exists $a \in N(R)$ such that $a \notin Z_r$. So $r(a) = gR$, $g^2 = g \in R$. Let $a^n = 0$ and $a^{n-1} \neq 0$. Hence $a^{n-1} \in r(a) = gR$, so $a^{n-1} = ga^{n-1}$, Since R is abelian, $a^{n-1} = a^{n-1}g = 0$, which is a contradiction. So $N(R) \subseteq Z_r$.

(3) Follows from (2).

(4) $(b) \Rightarrow (a)$ follows from (1). $(c) \Rightarrow (d)$ is trivial.

$(a) \Rightarrow (c)$ Let M be any right R-module, $a \in R$ with $a \notin Z_r$ and $f : aR \to M$ any right R-homomorphism. Since R is right quasi-regular, $a = aba$ for some $b \in R \setminus Z_r$. Let $ab = e$ and $f(e) = m$, where $m \in M$. Then $h : R \to M$ defined by $g(r) = mr, r \in R$ is a right R-homomorphism and $g(ar) = mar = f(e)ar = f(ab)ar = f(aba)r = f(a)r = f(ar)$, so M is Jcp-injective.

$(d) \Rightarrow (a)$. Let $a \notin Z_r$. By (d), aR is Jcp-injective, so the identity map $aR \to aR$ can be extended to one of R into R. Hence $a = aba$ for some $b \in R$. □

Theorem 2.12. *Let e be an idempotent of R such that $ReR = R$ and let $S = eRe$. Then:*

(1) If R is right Jcp-injective, then so is S.

(2) $eZ_r(R)e \subseteq Z_r(S)$. Hence if S is right nonsingular, then so is R.

(3) If R is right nonsingular, then so is S.

(4) If R is right SPP, then for any $a \in S \setminus Z_r(S)$, there exist an idempotent g of R such that $r_S(a) = egeS$.

(5) If R is right principally small injective, then so is S.

(6) If R is right quasi-regular, then so is S.

(7) If R is von Neumann regular, then so is S.

(8) If R is right p-injective, then so is S.

Proof. (1) Let $x \in S \setminus Z_r(S)$. Then $x \notin Z_r$. Otherwise, there exists an essential right ideal I of R such that $xI = 0$. Since $eR \cap I \neq 0$, $eI \neq 0$. Since $R = ReR$, $eI = eIR = eIReR = eIeR$, $eIe \neq 0$. We claim that eIe is an essential right ideal of S. Let K be any nonzero right ideal of S. Then $KR \cap I \neq 0$. Let $0 \neq y \in KR \cap I$

and let $1 = \sum_{i=1}^{n} a_i e b_i$, $a_i, b_i \in R$. Then $y = y1 = \sum_{i=1}^{n} y a_i e b_i$, so there exists some $i_0 \in \{1, 2, \cdots, n\}$ such that $y a_{i_0} e \neq 0$. Since $e y a_{i_0} e = y a_{i_0} e \in K R a_{i_0} e \cap I \subseteq K \cap I = K \cap eIe$, eIe is an essential right ideal of S. Since $xeIe = xIe = 0$, $x \in Z_r(S)$. This is a contradiction. Hence $x \notin Z_r$. Since R is right Jcp-injective, $l_R r_R(x) = Rx$. Now let $z \in l_S r_S(x)$. Then $r_S(x) \subseteq r_S(z)$. Let $a \in r_R(x)$. Then $xa = 0$, so $0 = xaa_i e = xeaa_i e$, $i = 1, 2, \ldots, n$. Hence $eaa_i e \in r_S(x) \subseteq r_S(z)$, so $zaa_i e = zeaa_i e = 0$. Thus $za = \sum_{i=1}^{n} zaa_i e b_i = 0$, so $a \in r_R(z)$. This shows that $r_R(x) \subseteq r_R(z)$. So $z \in l_R r_R(z) \subseteq l_R r_R(x) = Rx$. Hence $z = eze \in eRex = Sx$, which implies that $l_S r_S(x) \subseteq Sx$ and so $l_S r_S(x) = Sx$. Hence S is right Jcp-injective.

(2) By (1), we can easily see that $eZ_r(R)e = Z_r \cap eRe \subseteq Z_r(S)$. If $Z_r(S) = 0$, then $Z_r = RZ_r(R)R = ReRZ_r(R)ReR = ReZ_r(R)eR = 0$.

(3) If $Z_r(S) \neq 0$, then there exists $0 \neq x \in Z_r(S)$. Since $Z_r = 0$, there exists a nonzero right ideal I of R such that $r_R(x) \cap I = 0$. If $xI \neq 0$, then $eIe \neq 0$, so $eIe \cap r_S(x) \neq 0$. Let $0 \neq y \in eIe \cap r_S(x)$ and let $y = ez, z \in I$. So $xz = xez = xy = 0$ and then $0 \neq z \in r_R(x) \cap I$, which is a contradiction. Hence $xI = 0$, so $I \subseteq r_R(x) \cap I = 0$, which is also a contradiction. Thus $Z_r(S) = 0$.

(4) Let $a \notin Z_r(S)$. By (2), $a \notin Z_r$. Hence $r_R(a) = gR, g^2 = g \in R$ by hypothesis. So $r_S(a) = egeS$.

(5) It is trivial.

(6) Let $a \in S$ with $a \notin Z_r(S)$. By (2), $a \notin Z_r$. So $a = aba$ for some $b \in R$. Hence $a = aebea = a(ebe)a$, where $ebe \in S$ and so S is right quasi-regular.

(7) Follows from (3) and (6).

(8) Follows from (1), (5) and Theorem 2.9(1). □

The following theorem is a generalization of [11, Theorem 1.1].

Theorem 2.13. *Let R be a right Jcp-injective ring, and let $a, b \in R$ with $b \notin Z_r$. Then:*

(1) If bR embeds in aR, then Rb is an image of Ra.

(2) If aR is an image of bR, then Ra embeds in Rb.

(3) If $bR \cong aR$, then $Ra \cong Rb$.

(4) If K is a simple projective right ideal of R, then RK is the homogenous component of S_r containing K.

(5) If A, B are right ideals of R with $A \cap (B + Z_r) = 0$ and A is an ideal of R, then $Hom_R(A_R, B_R) = 0$.

Proof. (1) Let $\sigma : bR \to aR$ be a monomorphism. Since R is right Jcp-injective and $b \notin Z_r$, we can let $\sigma = v\cdot, v \in R$. Then $vb = au, u \in R$, so define $\varphi : Ra \to Rb$ by $\varphi(ra) = rau = rvb$, then φ is a well defined left R-homomorphism. Since $vb \notin Z_r$ and $r(vb) = r(b)$, $Rb = lr(b) = lr(vb) = Rvb$. Hence, clearly, φ is epic.

(2) Let $\sigma : bR \to aR$ be epic, and let v, u and φ be as in (1). Write $a = \sigma(bs) = vbs, s \in R$. Then $\varphi(ra) = 0$ gives $0 = rau = rvb$, whence $ra = rvbs = 0$. Hence φ is monic.

(3) Follows from the proof of (1) and (2).

(4) If $K = kR$ where $k \in R$ and $\sigma : K \to S$ is an R- isomorphism, where $S \subseteq R$, then $r(k) = r(\sigma(k))$. So $Rk = lr(k) = lr(\sigma(k)) = R\sigma(k)$ because $k \notin Z_r$. Hence $S = \sigma(k)R \subseteq RkR = RK$, so the $K-$ component is in RK. The other inclusion always holds.

(5) If there exists a $0 \neq f \in Hom_R(A_R, B_R)$, then there exists $0 \neq a \in A$ such that $f(a) \neq 0$. Then $f(a) = va$ where $v \in R$, because $A \cap Z_r = 0$. Since A is an ideal, $va \in A$. Hence $f(a) \in A \cap B = 0$, which is a contradiction. Hence $Hom_R(A_R, B_R) = 0$. □

3. Finiteness conditions

In [3], a right (left) annihilator M of a ring R is called maximal if for any right (left) annihilator N with $M \subseteq N$, either $N = M$ or $N = R$. In this case, $M = r(a)$ $(l(a))$ for some $0 \neq a \in R$.

Using the idea of [3], we start with the following theorem.

Theorem 3.1. *Let R be a semiprime right Jcp-injective ring whose complement right ideals are non-small. Then every maximal right (left) annihilator of R is a maximal right (left) ideal of R generated by an idempotent.*

Proof. Let L be a maximal right annihilator, then by [3, Theorem 3.1], there exists $0 \neq a \in R$ such that

(1) $L = r(a)$; (2) $r(a) = r(y)$ for every $0 \neq y \in Ra$; (3) $Z_r \cap Ra = 0$.

Hence, there exists a non-zero complement right ideal I of $r(a)$ such that $r(a) \oplus I$ is essential in R_R. Then, by hypothesis, there exists $0 \neq b \in I$ such that $b \notin Z_r$ and $ab \neq 0$. Since $r(ab) = r(b)$, $ab \notin Z_r$. Hence $Rb = lr(b) = lr(ab) = Rab$. Write $b = cab$, $c \in R$. Then $b \in r(a - aca)$. But $b \notin r(a)$ and hence $a = aca$. Let $d = ca$, by [3, Theorem 3.1], $L = r(a) = r(d) = eR$, where $e = 1 - d$, $d^2 = d$.

Now, we claim that L is a maximal right ideal of R. Similar to [3, Theorem 3.1], we only need to show that dRd is a division ring. In fact, we can assume that $a = d$, if $0 \neq x \in dRd$, then $r(x) = r(d)$ by (2). Hence, $x \notin Z_r$ by (3), so,

$Rx = lr(x) = lr(d) = Rd$. Write $d = ux$ where $u \in R$. Then $d = d^2 = dux = dudx$ because $x = dx$. That is, dRd is indeed a division ring. □

It is well known that if R is semiprime, then $S_r = S_l$. Hence from Theorem 3.1, we have the following result.

Corollary 3.2. *Let R be a right Jcp-injective ring whose complement right ideals are non-small. Then the following hold.*

(1) If R is semiprime, then R contains a maximal right (left) annihilator if and only if $S_r = S_l \neq 0$.

(2) If R is prime and contains a maximal right (left) annihilator, then R is left and right primitive and left and right nonsingular ring. So R is right p-injective.

Proof. (1) It is trivial.

(2) Since R is prime, R is left and right PS. Hence $Z_r \cap S_r = Z_l \cap S_l = 0$, so $Z_l = Z_r = 0$ because $0 \neq S_r = S_l$ is an essential left and right ideal. Hence R is left and right nonsingular. Let kR be a minimal right ideal of R. Then $R/r(k)$ is a faithful simple right R-module and hence R is right primitive. Similarly, R is left primitive. □

Let R be a ring and let S be an ideal of R such that R/S satisfies ACC on right annihilators. If $Y_1, Y_2, \cdots$ are subsets of $l(S)$, then [11, Lemma 2.1] shows that there exists $n \geq 1$ such that $r(Y_{n+1}Y_n \cdots Y_2Y_1) = r(Y_n \cdots Y_2Y_1)$. The following theorem is similar to [11, Theorem 2.2].

Theorem 3.3. *Let R be a right Jcp-injective ring whose complement right ideals are non-small. If R/S_r satisfies ACC on right annihilators, then*

(1) $J(R)$ is nilpotent;

(2) R is semiprime if and only if R is semiprimitive.

Proof. (1) First, if I is a complement right ideal of R, then $I \subsetneq J(R)$. Since R is right Jcp-injective, $Z_r \subseteq J(R)$. So $I \subsetneq Z_r$, by the proof of $(c) \Rightarrow (d)$ of Corollary 2.7(3), $J(R) = Z_r$. Hence $J(R)S_r = Z_rS_r = 0$. Let $a_1, a_2, \cdots$ be given in $J(R) \subseteq l(S_r)$. We have $r(a_na_{n-1} \cdots a_1) = r(a_{n+1}a_na_{n-1} \cdots a_1)$ for some $n \geq 1$. This implies that $a_na_{n-1} \cdots a_1R \cap r(a_{n+1}) = 0$, so $a_na_{n-1} \cdots a_1 = 0$ because $a_{n+1} \in J(R) = Z_r$. Hence $J(R)$ is left $T-$nilpotent, and so $(J(R) + S_r)/S_r$ is left $T-$ nilpotent. But R/S_r has ACC on right annihilators. Hence $(J(R) + S_r)/S_r$ is nilpotent. Then there exists an integer m such that $J^m \subseteq S_r$, and so $J^{2m} \subseteq JS_r = 0$. □

[1, Remark 1] conjecture that every flat right R-module is finitely projective if and only if the ascending chain $r(a_1) \subsetneqq r(a_2a_1) \subsetneqq r(a_3a_2a_1) \subsetneqq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \ldots$ of R. But [17] has answered this conjecture in the negative. However, the following Theorem gives an affirmative answer to this conjecture for right Jcp-injective ring. On the other hand, it is also a generalization of [3, Theorem 3.4].

Theorem 3.4. *Let R be a right Jcp-injective ring whose complement right ideals are non-small. Then the following are equivalent.*

(1) R is right perfect.

(2) Every flat right R-module is finitely projective.

(3) Every flat right R-module is singly projective.

(4) The ascending chain $r(a_1) \subsetneqq r(a_2a_1) \subsetneqq r(a_3a_2a_1) \subsetneqq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \ldots$ of R.

It is well known that Z_r is nilpotent for any ring R with ACC on right annihilators.

In [3], Chen and Ding show that if $Z_r \neq 0$ and the ascending chain $r(a_1) \subsetneqq r(a_2a_1) \subsetneqq r(a_3a_2a_1) \subsetneqq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \ldots$ of R, then there exists a $0 \neq b \in Z_r$ such that $r(b) = r(y)$ for every $0 \neq y \in Rb$. Hence we have the following corollary.

Corollary 3.5. *Let R be a right Jcp-injective ring whose complement right ideals are non-small. Then the following hold.*

(1) If R satisfies ACC on right annihilators, then R is semiprimary.

(2) If R is semiprime and the ascending chain $r(a_1) \subsetneqq r(a_2a_1) \subsetneqq r(a_3a_2a_1) \subsetneqq \cdots$ terminates for every infinite sequence $a_1, a_2, a_3, \ldots$ of R, then R is a semisimple artinian ring.

Proof. (1) By Theorem 3.4, $R/J(R)$ is semisimple artinian. Hence R is semiprimary because $J(R) = Z_r$ is nilpotent.

(2) If $Z_r \neq 0$, then there exists a $0 \neq b \in Z_r$ such that $r(b) = r(y)$ for every $0 \neq y \in Rb$. Since R is semiprime, there exists a $0 \neq c \in R$ such that $bcb \neq 0$. Hence $r(b) = r(bcb)$. Consequently, bc is not nilpotent. But $Z_r = J(R)$ is right $T-$nilpotent, which is a contradiction. Hence $J(R) = Z_r = 0$ and by Theorem 3.4, R is semisimple artinian. □

Call a left ideal L of a ring R left weakly essential, if for all $0 \neq a \in R$ with $a \notin Z_r$, $Ra \cap L \neq 0$. Clearly, an essential left ideal is left weakly essential.

A ring R is called right Kasch if every simple right modules can be embedded in R, or equivalently if $l(M) \neq 0$ for every maximal right ideal M of R.

The following theorem is similar to [11, Theorem 2.4].

Theorem 3.6. *Let R be a right Jcp-injective ring. Then:*

(1) If R is right Kasch, then $l(J)$ is left weakly essential.

(2) If R is semiperfect, then R is a right Kasch ring if and only if S_r is left weakly essential.

Proof. (1) Let $0 \neq b \in R$, $b \notin Z_r$ and choose M maximal in bR. Let $\sigma : bR/M \rightarrow R_R$ is a monomorphism. If $\gamma : bR \rightarrow R$ is defined by $\gamma(x) = \sigma(x + M)$, then $\gamma = a\cdot$ where $a \in R$ because $b \notin Z_r$. So $ab = \gamma(b) = \sigma(b + M) \neq 0$. But $abJ = \sigma(bJ) = 0$ because $(bR/M)J = 0$. Therefore $0 \neq ab \in Rb \cap l(J)$, as required.

(2) If R is right Kasch, then $l(J)$ is left weakly essential by (1). Since R is a semiperfect ring, $S_r = l(J)$. Hence S_r is left weakly essential.

Conversely, we assume that S_r is left weakly essential. Let M be a maximal right ideal of R. Then there exists a $e^2 = e \in R$ such that $1 - e \in M$ and $eR \cap M \subseteq J$ because R is a semiperfect ring. Hence $S_r \subseteq l(J) \subseteq l(eR \cap M)$, and so $l(eR \cap M)$ is left weakly essential by hypothesis. Since $e \notin Z_r$, $0 \neq Re \cap l(eR \cap M) = l((1 - e)R) \cap l(eR \cap M) = l((1 - e)R + (eR \cap M)) = l(M)$. This implies that R is right Kasch. □

Call a right R-module M right *nil*-injective [15] if for each nilpotent element $a \in R$, the right R-homomorphism from aR to M can be extended to one from R into M. If R is *nil*-injective as a right R-module, then we call R a right *nil*-injective ring.

Theorem 3.7. *Let R be a right Jcp-injective ring. Assume that every simple singular right R-module is right nil-injective. Then the following hold.*

(1) R is right p-injective, and so R is semiprimitive right nonsingular.

(2) If R is right SPP, then R is von Neumann regular.

(3) $Z_l = 0$.

Proof. First, we show that $Z_r = 0$. If not, then there exists $0 \neq b \in Z_r$ such that $b^2 = 0$. We claim that $Z_r + r(b) = R$. Otherwise, there exists a maximal essential right ideal M of R such that $Z_r + r(b) \subseteq M$, then R/M is a right *nil*-injective R-module. Define $f : bR \rightarrow R/M$ by $f(br) = r + M$ for all $r \in R$. Clearly, f is a well defined right R-module homomorphism. Hence there exists a $a \in R$ such that

$1-ab \in M$. So $1 \in M$ because $ab \in Z_r \subseteq M$, which is a contradiction. Hence $Z_r + r(b) = R$. Since R is right Jcp-injective, $Z_r \subseteq J(R)$ by Theorem 2.5. So $J(R)+r(b) = R$. This implies that $r(b) = R$ and so $b = 0$, which is a contradiction. Hence $Z_r = 0$. By Corollary 2.2, R is right p-injective. By [11, Theorem 2.1], $J(R) = 0$.

(2) This is an immediate consequence of Theorem 2.9(7).

(3) If $Z_l \neq 0$, then there exists $0 \neq b \in Z_l$ such that $b^2 = 0$. We show that $Z_l+r(b) = R$. Otherwise, there exists a maximal right ideal M such that $Z_l+r(b) \subseteq M$. If M is not an essential right ideal of R, then $M = r(e)$, where $e^2 = e \in R$. If $be \neq 0$, then $beR \cong eR$ as right R-module. By Theorem 2.5(5), $beR = gR$, where $g^2 = g \in R$, so $g \in Z_l$ because $beR \subseteq Z_l$. This is a contradiction. So $be = 0$. Then $e \in r(b) \subseteq M = r(e)$, which is impossible. Thus M is an essential right ideal of R, so R/M is nil-injective R-module. Similar to the proof of (1), there exists $a \in R$ such that $1 - ab \in M$. So $1 \in M$ because $ab \in Z_l \subseteq M$, which is a contradiction. Hence $Z_l + r(b) = R$. Let $1 = x + y, x \in Z_l, y \in r(b)$. Then $b = bx$ and so $b(1 - x) = 0$. Since $x \in Z_l$ and $l(x) \cap l(1 - x) = 0$, $l(1 - x) = 0$. This shows that $b = 0$, which is a contradiction. So $Z_l = 0$. □

4. Weakly injectivity

Let $E(M)$ be an injective hull of M_R. M is called right weakly injective if for any finite generated submodule $N_R \subseteq E(M)$, there exists $X_R \cong M$ and $N_R \subseteq X_R \subseteq E(M)$. Clearly, right injective rings are right weakly injective, but the converse is not true in general.

Lemma 4.1. *Let R be a right Jcp-injective ring. If R_R is essential in X_R, where $X_R \cong R_R$, then $X = R$.*

Proof. Let $f : R_R \to X_R$ be the isomorphism and $f(1) = b \in X$. Then $bR = Im(f) = X$. Since $1 \in R \subseteq X$, let $1 = bu, u \in R$. Hence $R_R = 1R = buR$ and $r(u) = 0$. Since R is right Jcp-injective, by Theorem 2.5(1), there exists $d \in R$ such that $du = 1$. Let $e = ud$. Then $e^2 = e$ and $uR = eR$. Hence $R = buR = beR$. It is clear that $X = bR = b(eR \oplus (1-e)R) = beR + b(1-e)R$. If $x \in beR \cap b(1-e)R$, then there exist $r_1, r_2 \in R$ such that $x = ber_1 = b(1 - e)r_2$, so $f^{-1}(x) = er_1 = (1 - e)r_2$. Hence $f^{-1}(x) = 0$ and then $x = 0$, so $X = bR = beR \oplus b(1 - e)R = R \oplus b(1 - e)R$. Since R_R is essential in X_R, $b(1 - e)R = 0$, and so $X = beR = R$.

□

The following theorem is a generalization of [11, Theorem 1.3].

Theorem 4.2. *A ring R is right self-injective if and only if R is right Jcp-injective and R_R is weakly injective.*

Proof. We only need to show that $E(R_R) \subseteq R$. For each $a \in E(R_R)$, since $R + aR \subseteq E(R_R)$, there exists $X \subseteq E(R_R)$ such that $R + aR \subseteq X$ and $X_R \cong R_R$. Since R is right Jcp-injective, $X = R$ by Lemma 4.1. Hence $R = E(R_R)$.

□

Call a ring R right coflat if for each finitely generated right ideal I of R, every homomorphism from I_R to R can be extended to the one of R into R. Call R right FP-injective if for each finitely presented right ideal I of R, every homomorphism from I_R to R can be extended to the one of R to R. Right FP-injective rings are right coflat rings and right self-injective rings are right FP-injective rings.

Corollary 4.3. *The following conditions are equivalent for a right weakly injective ring R.*

(1) *R is right self-injective.*

(2) *R is right p-injective.*

(3) *R is right coflat.*

(4) *R is right FP-injective.*

(5) *R is right Jcp-injective.*

Call a ring R right *np*-injective, if for any non-nilpotent element c of R and any right R-homomorphism $g : cR \to R$, there exists $m \in R$ such $g(ca) = mca$ for all $a \in R$. An important source of right *np*- injective rings is given by Yue Chi Ming [8], which is a generalization of right *p*- injective ring.

Call a ring R right weakly *np*-injective, if for any non-nilpotent element c of R, there exists a positive integer number n such that for any right R-homomorphism $g : c^nR \to R$, there exists $m \in R$ such $g(c^na) = mc^na$ for all $a \in R$. Evidently, right weakly *np*-injective rings are the generalization of right *np*-injective and right YJ-injective.

Call a ring R right *Gnp*-injective, if for any non-nilpotent element c of R, $lr(c) = Rc \oplus X_c$, where X_c is a left ideal of R. Obviously, right *Gnp*-injective rings are the generalization of right *np*-injective and right AP-injective.

Call a ring R right weakly *Gnp*-injective, if for any non-nilpotent element c of R, there exists a positive integer number n such that $lr(c^n) = Rc^n \oplus X_c$, where X_c is a left ideal of R. Obviously, right weakly *Gnp*-injective rings are the generalization of right weakly *np*-injective and right AGP-injective [14].

Lemma 4.4. *Let R be a right weakly Gnp-injective ring. If R_R is essential in X_R, where $X_R \cong R_R$, then $X = R$.*

Proof. Let $f : R_R \to X_R$ be the isomorphism and $f(1) = b \in X$. Then $bR = Im(f) = X$. Since $1 \in R \subseteq X$, let $1 = bu, u \in R$. Hence $R_R = 1R = buR$ and $r(u) = 0$. Since R is right weakly *Gnp*-injective, there exists an $n \geq 1$ such that $lr(u^n) = Ru^n \oplus X_u$ where X_u is a left ideal of R because u is a non-nilpotent element. Hence $R = l(0) = lr(u^n) = Ru^n \oplus X_u$ because $r(u) = 0$. Write $Ru^n = Re, e^2 = e \in R$, then $u^n = u^n e = u^n du^n$ where $e = du^n$. So $1 - du^n \in r(u^n) = r(u) = 0$, this implies that $vu = 1$, where $v = du^{n-1}$, Then, similar to the proof of Lemma 4.1, we can complete the proof. □

Similar to Theorem 4.2, we have the following theorem.

Theorem 4.5. *A ring R is right self-injective if and only if R is right weakly Gnp-injective and R_R is weakly injective.*

Corollary 4.6. *The following are equivalent for a right weakly injective ring R.*

(1) *R is right self-injective.*
(2) *R is right YJ-injective.*
(3) *R is right AP-injective.*
(4) *R is right AGP-injective.*
(5) *R is right np-injective.*
(6) *R is right Gnp-injective.*
(7) *R is right weakly np-injective.*
(8) *R is right weakly Gnp-injective.*

5. On a Theorem of Camillo

Camillo [2], Nicholson and Yousif [11] and Chen and Ding [3] have studied p-injective rings and YJ-injective rings. In this section, we extend their works.

An element $u \in R$ is said to be right uniform if $u \neq 0$ and uR is a uniform right ideal of R.

Theorem 5.1. *Let R be a right Jcp-injective ring. If $u \in R$ is right uniform with $u \notin Z_r$, then $M_u := \{x \in R \mid uR \cap r(x) \neq 0\}$ is a maximal left ideal containing $l(u)$.*

Proof. Since uR is uniform, M_u is a left ideal. Clearly, $l(u) \subseteq M_u \neq R$. If $a \notin M_u$, then $au \neq 0$ because $uR \cap r(a) = 0$. Since $u \notin Z_r$ and $r(u) = r(au)$, $au \notin Z_r$. Hence $Ru = lr(u) = lr(au) = Rau$ because R is right *Jcp*-injective, write $u = cau$

where $c \in R$. So $1 - ca \in l(u) \subseteq M_u$. Hence $R = Ra + M_u$, which implies that M_u is maximal. □

Corollary 5.2. *If R is a right Jcp-injective right uniform ring, then R is local and $Z_l \subseteq J(R) = Z_r$*

Proof. By hypothesis, $Z_r = \{x \in R \,|\, r(x)$ is essential in $R_R\} = \{x \in R \,|\, r(x) \neq 0\} = \{x \in R \,|\, 1R \cap r(x) \neq 0\} = M_1 \supseteq J(R)$ because $1 \notin Z_r$. Hence $J(R) = Z_r = M_1$ is a maximal left ideal. So R is local. □

Corollary 5.3. *Let R be right Jcp-injective and left Kasch. Assume that every nonzero right ideal contains a uniform right ideal, which is not contained in Z_r. Then every maximal left ideal M has the form $M = M_u$ for some right uniform element u.*

Proof. Let M be a maximal left ideal. Then $r(M) \neq 0$ because R is left Kasch. By hypothesis, there exists a uniform right ideal uR such that $uR \subseteq r(M)$ and $u \notin Z_r$. So $M = lr(M) \subseteq l(u) \subseteq M_u$. Hence $M = M_u$. □

Similar to [11, Lemma 3.1 and Theorem 3.1], we can obtain the following theorems.

Theorem 5.4. *Let R be right Jcp-injective, and assume that $Rb_1 \oplus Rb_2 \oplus \cdots \oplus Rb_n \subseteq R$ is a direct sum with $(Rb_1 \oplus Rb_2 \oplus \cdots \oplus Rb_n) \cap Z_r = 0$. Then:*

(1) Any R-linear map $\alpha : b_1R + b_2R + \cdots + b_nR \to R$ extends to $\alpha : R \to R$.

(2) Write $S = b_1R + b_2R + \cdots + b_kR$ and $T = b_{k+1}R + b_{k+2}R + \cdots + b_nR$, $1 \leq k < n$, then $l(S \cap T) = l(S) + l(T)$.

Theorem 5.5. *If R is right Jcp-injective and $\oplus_{i \geq 1} B_i$ is a direct sum of ideals of R with $(\oplus_{i\geq 1} B_i) \cap Z_r = 0$, then $A \cap (\oplus_{i \geq 1} B_i) = \oplus_{i \geq 1}(A \cap B_i)$ for any ideal A of R.*

Theorem 5.6. *Let R be right Jcp-injective and let $W = u_1R \oplus \cdots \oplus u_nR$ be a direct sum of uniform right ideals u_iR of R with $W \cap Z_r = 0$. If $M \subseteq R$ is a maximal left ideal that is not of the form M_u for any right uniform element u, then there exists $m \in M$ such that $r(1-m) \cap W$ is essential in M.*

Since division rings are von Neumann regular, every module over division rings is p-injective. Hence every right module over division rings is right Jcp-injective. We now characterize division rings in terms of the following notion: R is called a right F-ring if, for any maximal right ideal M of R and any $b \in M$, R/bM is a flat right R-module. Division rings are right F-rings.

Theorem 5.7. *The following are equivalent for a semiprime right uniform ring R.*

(1) R is a division ring;

(2) R is a right p-injective right F-ring;

(3) R is a right YJ-injective right F-ring;

(4) R is a right Jcp-injective right F-ring.

Proof. It is obvious that (1) implies (2), which, in turn, implies (3) and (4).

Assume (4). Since R is a right uniform ring and right *Jcp*-injective, by Corollary 5.2, R is a local ring with $Z_r = J(R)$. Since R is a right F- ring, $J(R)^2 = 0$ and so $J(R) = 0$ because R is a semiprime ring. Hence R is a division ring. □

R is called a right CAM-ring if, for any maximal essential right ideal M of R (if it exists) and for any right subideal I of M which is either a complement right subideal of M or a right annihilator ideal in R, I is an ideal of M.

Right CAM-rings generalize semisimple artinian. [8] shows that semiprime right CAM-ring R is either semisimple artinian or reduced. If R is also right *Jcp*-injective, then R is either semisimple artinian or strongly regular ring. We yield the following theorem.

Theorem 5.8. *The following are equivalent for a ring R.*

(1) R is either a semisimple artinian or a strongly regular ring.

(2) R is a semiprime right CAM-ring whose singular simple right modules are flat.

(3) R is a semiprime right Jcp-injective, right $CAM-$ring.

(4) R is a semiprime right CAM-ring, $MERT$ ring whose singular simple right R-modules are Jcp-injective.

Proof. (1) ⇒ (i) where $i = 2, 3, 4$ are obvious.

(2) ⇒ (1) Assume (2). If R is not a semisimple artinian ring, then R is reduced. Let $0 \neq a \in R$. If $aR \oplus r(a) \neq R$, then $aR \oplus r(a) \subseteq M$ for some maximal right ideal M of R. If M is not an essential right ideal of R, then $M = eR$, where $e^2 = e \in R$. Because R is reduced, $ae = ea = 0$ and $e \in r(a) \subseteq M = r(e)$, a contradiction. Hence M is an essential right ideal of R and so R/M is a singular simple right R-module. By (2), R/M is flat, then there exists $m \in M$ such that $a = ma$. But then $a = am$, because R is reduced. Now we obtain $1 - m \in r(a)$, and so $1 \in M$, a contradiction. Hence $aR \oplus r(a) = R$ and then R is a strongly regular ring.

(3) ⇒ (1) If R is not a semisimple artinian ring, then R is reduced. By Corollary 1.5, R is a regular ring, But R is an abelian ring, so R is a strongly regular ring.

(4) ⇒ (1) We can directly assume that R is reduced. So R is a right nonsingular ring. Let $0 \neq a \in R$. If $aR \oplus r(a) \neq R$, then $aR \oplus r(a) \subseteq M$ for some maximal essential right ideal M of R. Hence R/M is a singular simple right R-module. By hypothesis, R/M is right Jcp-injective. Then there exists a $c \in R$ such that $1 - ca \in M$. But then $1 \in M$, because R is a $MERT$ ring and M is an ideal. It is a contradiction. Hence $aR \oplus r(a) = R$ and then R is a strongly regular ring. □

A ring R is called right CM if, for any maximal essential right ideal M of R, every complement right subideal is an ideal of M. [8, Proposition 3] shows that simple projective right module over right CM ring is injective.

A ring R is right finitely embedded if, $Soc(R_R)$ is finite generated and right essential in R_R. Note that a right finitely embedded right PS ring need not be semiprime. We conclude the paper with a few characteristic properties of semisimple artinian rings.

Theorem 5.9. *The following are equivalent for a ring R.*

(1) R is a semisimple artinian ring.

(2) R is a right CM, right finitely embedded and right PS ring.

(3) R is a semiprime, right Jcp-injective and left or right Goldie ring.

Proof. Clearly, (1) ⇒ (2) and (3).

(2) ⇒ (1) Since R is a right PS right finitely embedded ring, $Soc(R_R)$ is a semisimple projective right R-module. Because R is a right CM ring, $Soc(R_R)$ is an injective right R-module. Hence $Soc(R_R) = eR$, where $e^2 = e \in R$. But then $Soc(R_R) = R$, because $Soc(R_R)$ is essential in R_R. Hence R is semisimple artinian.

(3) ⇒ (1) Assume (3). Then R has a left (or right) fraction ring Q, and Q is a semisimple artinian ring. If Q is a left fraction ring, then for every $x \in Q$, $x = a^{-1}b$, where $a, b \in R$ and $l(a) = r(a) = 0$, so $a \notin Z_r$. Since R is a right Jcp-injective ring, there exists $c \in R$ such that $ca = 1$ and then $ac = 1$. Hence $a^{-1} \in R$ and so $x \in R$. Thus $R = Q$ is a semisimple artinian ring. □

References

[1] G. Azumaya, *Finite splitness and finite projectivity*, J. Algebra, 106 (1987),114-134.

[2] V. Camillo, *Commutative ring whose principal ideals are annihilators*, Port. Math., 46 (1)(1989), 33-37.

[3] J.L. Chen and N.Q. Ding, *On General principally injective rings*, Comm. Algebra, 27 (1999), 2097-2116.

[4] C. Faith and P. Menal, *A counter example to a conjecture of Johns*, Proc. Amer. Math. Soc., 116 (1992), 21-26.

[5] J.Y. Kim, N.K. Kim and S.B. Nam, *Some comments on simple singular GP-injective modules*, Kyungpook Math. J., 41 (2001), 23-27.

[6] N.K. Kim, S.B. Nam and J.Y. Kim, *On simple singular GP-injective modules*, Comm. Algebra, 27(5) (1999), 2087-2096.

[7] R. Yue Chi Ming, *On annihilator ideals*, Math. J. Okayama Univ., 19 (1976), 51-53.

[8] R. Yue Chi Ming, *On Quasi-Frobeniusean and artinian rings*, Publications De L'institut Math ématique, 33(47) (1983),239-245.

[9] S.B. Nam, N.K. Kim and J.Y. Kim, *On simple GP-injective modules*, Comm. Algebra, 23(14) (1995), 5437-5444.

[10] W.K. Nicholson and J.F. Watters, *Rings with projective socle*, Proc. Amer. Math. Soc., 102 (1988), 443-450.

[11] W.K. Nicholson and M.F. Yousif, *Principally injective rings*, J. Algebra, 174 (1995), 77-93.

[12] W.K. Nicholson and M.F. Yousif, *Mininjective rings*, J. Algebra, 187 (1997), 548-578.

[13] W.K. Nicholson and M.F. Yousif, *Weakly continuous and C2-rings*, Comm. Algebra, 29(6) (2001), 2429-2466.

[14] S. Page and Y. Zhou, *Generalization of principally injective ring*, J. Algebra, 206 (1998), 706-721.

[15] J.C. Wei and J.H. Chen, *Nil-injective rings*, Int. Electron. J. Algebra, 2 (2007), 1-21.

[16] J.C. Wei, *The rings characterized by minimal left ideal*, Acta Math. Sin.(Engl. Ser.), 21(3) (2005), 473-482.

[17] S.L. Zhu, *On rings over which every flat left module is finitely projective*, J. Algebra, 139 (1991), 19-27.

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