International Electronic Journal of Algebra
Volume 6 (2009) 31-37

NI RINGS WHICH ARE WEAKLY π-REGULAR

C. Selvaraj and S. Petchimuthu

Received: 10 April 2008; Revised 17 June 2009
Communicated by A. Çiğdem Özcan

Abstract. In this paper, we prove that if a ring R with identity is NI and satisfies (CZ2), then R is right (left) weakly π-regular if and only if $R/\mathcal{N}^*(R)$ is right (left) weakly π-regular, if and only if every strongly prime ideal of R is maximal.

Mathematics Subject Classification (2000): 16D50, 16E20
Key words: strongly prime, weakly π -regular, completely prime.

1. Introduction

Throughout this paper, R denotes an associative ring with identity. We use $\mathcal{P}(R)$, $\mathcal{N}^*(R)$ and $\mathcal{N}(R)$ to denote the prime radical of R, the unique maximal nil ideal and the set of all nilpotent elements of R, respectively. Recall that a ring R is called reduced if it has no non zero nilpotent elements and called nil semisimple if it has no non zero nil ideals. An ideal P is said to be prime (semiprime) if for any $a, b \in R, aRb \subseteq P (aRa \subseteq P)$ implies that either $a \in P$ or $b \in P (a \in P)$. An ideal P is called strongly prime [5] if P is prime and R/P is nil semisimple. All strongly prime ideals are taken to be proper. We say that an ideal P of a ring R is minimal strongly prime if P is minimal among strongly prime ideals of R. We use $Spec(R)$ and $(m)Spec(R)$ to denote the set of all strongly prime ideals of R and the set of all minimal strongly prime ideals of R, respectively. Observe that for a ring R,

$$\begin{aligned} \mathcal{N}^*(R) &= \cap\{a \in R \,|\, (a) \text{ is nil ideal of } R\} \\ &= \cap\{P \,|\, P \text{ is a minimal strongly prime ideal of } R\} \\ &= \cap\{P \,|\, P \text{ is a strongly prime ideal of } R\}. \end{aligned}$$

An ideal P is called completely prime if for any $a, b \in R, ab \in P$ implies $a \in P$ or $b \in P$. Note that every completely prime ideals are strongly prime, every strongly prime ideals are prime, but the converse need not be holds.

A ring R is called 2-primal if $\mathcal{P}(R) = \mathcal{N}(R)$ [1]. We refer to [1, 2, 3] for more detail of 2-primal rings. A ring R is called NI if $\mathcal{N}^*(R) = \mathcal{N}(R)$. Clearly, every 2-primal rings are NI, but the converse need not true by [5, Example 1.2]. Note that a ring R is reduced if and only if R is semiprime and 2-primal, if and only if R is nil semisimple and NI. Hong and Kwak [4] characterized a ring R whose unique maximal nil ideal $\mathcal{N}^*(R)$ coincides with the set of all its nilpotent elements $\mathcal{N}(R)$ (i.e., an NI ring in our sense). Recently Hwang, Jeon and Lee [5] studied the basic structure of NI rings. In this paper, we show that if a ring R with identity is NI and satisfies (CZ2), then R is right (left) weakly π-regular if and only if $R/\mathcal{N}^*(R)$ is right (left) weakly π-regular, if and only if every strongly prime ideal of R is maximal.

2. Preliminaries

Definition 2.1. [4] An ideal I of a ring R is said to have Insertion Factors Property (IFP) if for any $a, b \in I, ab \in I$ implies that $aRb \subseteq I$. A ring R is said to have IFP if the ideal (0) has IFP.

Definition 2.2. The Jacobson radical of a ring R is denoted by $\mathcal{J}(R)$, and is defined by

$$\mathcal{J}(R) = \cap\{maximal\ left\ ideals\ of\ R\}.$$

Definition 2.3. A ring R is called local if the set of all non invertible elements of R is an ideal of R.

Definition 2.4. [3] Let R be a ring with $x, y \in R$ and n a positive integer. We say that R satisfies the

(i) $(CZ1)$ condition if whenever $(xy)^n = 0$ then $x^m y^m = 0$ for some positive integer m.
(ii) $(CZ2)$ condition if whenever $(xy)^n = 0$ then $x^m R y^m = 0$ for some positive integer m.

Observe that any local ring with nil Jacobson radical satisfies condition $(CZ2)$ [3, Definition 1.3].

Definition 2.5. A ring R is right (left) weakly π-regular if for any $a \in R$, there exists a natural number $n = n(a)$ depending on a such that $a^n \in a^n R a^n R (a^n \in R a^n R a^n)$. A ring R is called weakly π-regular if it is both right and left weakly π-regular.

Definition 2.6. [4] For a ring R and $P \in Spec(R)$ we get

$$\begin{aligned} O(P) &= \{a \in R \mid aRb = 0 \text{ for some } b \in R \setminus P\}, \\ O_P &= \{a \in R \mid ab = 0 \text{ for some } b \in R \setminus P\}, \\ \overline{O(P)} &= \{a \in R \mid a^m \in O(P) \text{ for some positive integer } m\}, \\ \overline{O}_P &= \{a \in R \mid a^m \in O_P \text{ for some positive integer } m\}, \\ N(P) &= \{a \in R \mid aRb \subseteq \mathcal{N}^*(R) \text{ for some } b \in R \setminus P\}. \end{aligned}$$

3. NI rings which are weakly π-regular

In this section, we show that if R is an NI ring which satisfies the condition (CZ2), then (i) every strongly prime ideals are maximal if and only if R is right (left) weakly π-regular; (ii) $P \in (m)Spec(R)$ if and only if $P = \overline{O(P)}$ and give an example which illustrates the condition (CZ2) is not superfluous in (ii).

The proof of the following theorem is given in [4, Corollary 13]. But we prove it in a different way.

Theorem 3.1. *For a ring R, the following are equivalent.*

(i) *R is an NI ring;*

(ii) *Every minimal strongly prime ideal of R is completely prime.*

Proof. (ii) $\Rightarrow$ (i) Let $x^n = 0$ for some positive integer n. Then $x^n \in P$ for all completely prime ideal P of R. Hence $x \in P$ for all P. Since every minimal strongly prime ideal of R is completely prime, $x \in \bigcap_{P \in (m)Spec(R)} P = \mathcal{N}^*(R)$. Therefore R is an NI ring.

(i) $\Rightarrow$ (ii) Let P be a minimal strongly prime ideal of R. Let $a, b \in R$ such that $ab \in P$ and $b \notin P$. We show that $a \in P$.

Case (i) Suppose that $(ab)^k = 0$ for some k. Then $(ab)^k \in \mathcal{N}^*(R)$. Since $\mathcal{N}^*(R)$ is completely semiprime ($i.e.$, $a^2 \in \mathcal{N}^*(R)$ implies $a \in \mathcal{N}^*(R)$ for $a \in R$) , $a^k b^k \in \mathcal{N}^*(R)$. Since $b \notin P$, there exist $z_1, z_2, z_3, \cdots, z_{k-1} \in R$ such that $bz_1bz_2 \cdots z_{k-1}b \notin P$ and since $\mathcal{N}^*(R)$ has IFP, $a^kR(\ bz_1bz_2 \cdots z_{k-1}b) \in \mathcal{N}^*(R)$. Again using completely semiprimeness of $\mathcal{N}^*(R)$, we have $aRbz_1bz_2 \cdots z_{k-1}b \in \mathcal{N}^*(R)$. Hence $a \in N(P) \subseteq P$.

Case (ii) Suppose $(ab)^k \neq 0$ for all $k > 0$. Let $S = \{(ab)^n \mid n \geq 1\}, L = R \setminus P$ and $T = \{r \in R \mid r \neq 0, r = (ab)^{t_0}x_1(ab)^{t_1}x_2 \cdots (ab)^{t_n}$ where $t_i \geq 1, i = 1, 2, \cdots, n-1, t_i \geq 0, i = 0, n$ and $x_i \in L$ for all $i\}$. Clearly, $S \neq \{0\}$ and $L \subseteq T$. Let $M = S \cup T$. We shall prove that M is a multiplicative monoid in $R \setminus \{0\}$. Let $x, y \in M$. If $x, y \in S$, then $xy \in S \subseteq M$. If $x \in S$ and $y \in T$, then let $x = (ab)^q$ for some $q > 0$ and $y = (ab)^{t_0}x_1(ab)^{t_1} \cdots x_n(ab)^{t_n}$. Suppose $xy = 0$. Take $m =$

$q + t_0 + t_1 + \cdots + t_n$ and $w = by_1x_1y_2x_2 \cdots y_nx_n \notin P$ for some $y_1, y_2, \cdots, y_n \in R$ because $b, x_1, x_2, \cdots, x_n \notin P$. Since $xy = 0$ and $\mathcal{N}^*(R)$ has IFP, we can easily prove that $(aw)^m \in \mathcal{N}^*(R)$. By case (i), $a \in P$. Therefore assume that $xy \neq 0$. Then clearly from the definition of $T, xy \in T \subseteq M$. Similarly, we can show that if $x, y \in T$ then $xy \in T \subseteq M$. Thus we have M is a multiplicative monoid in $R \setminus \{0\}$. By Zorn's lemma, there is an ideal Q of R which is maximal with respect to the property that $Q \cap M = \phi$. By [5, Lemma 2.2], Q is a strongly prime ideal of R. Since $Q \cap M = \phi, ab \notin Q$ and $Q \subseteq P$. Since P is a minimal strongly prime, $Q = P$. Therefore $ab \notin P$, which is a contradiction and consequently $a \in P$. □

The following theorem shows that in the case of NI ring which satisfies (CZ2), the condition "$R/\mathcal{N}^*(R)$ is right weakly π-regular" in [4, Proposition 18] can be replaced by the condition "R is right weakly π-regular".

Theorem 3.2. *Let R be an NI ring satisfying (CZ2). Then the following are equivalent.*

(1) *R is right (left) weakly π-regular;*
(2) *$R/\mathcal{N}^*(R)$ is right (left) weakly π-regular;*
(3) *Every strongly prime ideal of R is maximal.*

Proof. It is enough to prove that (3) ⇒ (1), because (1) ⇒ (2) is clear and (2) ⇒(3) is proved in [4, Proposition 18]. Suppose R is not right weakly π-regular. Then there exists an element $a \in R$ such that a is not right weakly π-regular. So we have $a^k \notin a^kRa^kR$ for every positive integer k. Hence $a^k \neq 0$ for all $k > 0$ and $a \notin aRaR$. Then RaR is contained in a maximal ideal which is also a strongly prime ideal. Let T be the union of all strongly prime ideals which contain a. Let $S = R \backslash T$. Since every strongly prime ideal is maximal, every strongly prime ideal is minimal. Since every minimal strongly prime ideal is completely prime by Theorem 3.1, S is a multiplicatively closed set. Let $F = \{a^{t_0}b_1a^{t_1}...b_na^{t_n} \neq 0 \mid b_i \in S$ and $t_i \in \{0\} \cup \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers$\}$. Let $L = \{a, a^2, ...\}$. Let $M = F \cup L$. Clearly, $S \subseteq F \subseteq M$. We shall claim that M is a multiplicative monoid in $R \setminus \{0\}$. Let $x, y \in M$. Assume $x \in F$ and $y \in L$. Suppose that $xy = 0$. Take $x = a^{t_0}b_1a^{t_1}...b_na^{t_n}$ and $y = a^r$. Choose $m = t_0 + t_1 + ... + t_n + r$ and $b = b_1y_1b_2y_2...y_{n-1}b_n \notin P$ for some $y_1, y_2, ..., y_{n-1} \in R$. Then $xy = 0$ implies that $(ab)^m \in \mathcal{N}^*(R)$ and hence $(ab)^k = 0$ for some k. Since R satisfies (CZ2), $a^qRb^q = 0$ for some $q > 0$. Observe that a strongly prime ideal cannot contain both a^q and b^q, otherwise a strongly prime ideal would contain both of them which contradicts the definition of S and T. Hence $Ra^qR + Rb^qR = R$. So $a^qR = a^qRa^qR + a^qRb^qR$.

Since $a^qRb^q = 0$, $a^q \in a^qRa^qR$. This shows that a is right weakly π-regular, a contradiction. Hence $0 \neq xy \in M$. Similarly, we can prove that if $x, y \in F$ then $0 \neq xy \in M$. Thus M is a multiplicative monoid in $R\backslash\{0\}$. By Zorn's Lemma, there is an ideal Q which is maximal with respect to the property that $Q \cap M = \phi$. By [5, Lemma 2.2], Q is a strongly prime ideal of R. Since $a \notin Q, Q + RaR = R$. Hence $1 = b + c$ for some $b \in Q$ and $c \in RaR$. This gives $b \notin T$. So that $b \in S \subseteq F \subseteq M$, which implies $Q \cap M \neq \phi$, a contradiction and consequently R is right weakly π-regular. The proof of the left case is similar. □

Corollary 3.3. *Let R be a 2-primal ring satisfying condition $(CZ2)$. Then the following are equivalent.*

(1) *R is right weakly π-regular;*
(2) *$R/\mathcal{N}^*(R)$ is right weakly π-regular;*
(3) *$R/\mathcal{P}(R)$ is right weakly π-regular;*
(4) *Every prime ideal of R is maximal;*
(5) *Every strongly prime ideal of R is maximal.*

Proof. (1) ⇒ (3) is clear. (3) ⇒ (4) and (4) ⇒ (5) are proved in [4, Corollary 19]. (1) ⇒ (2) and (2) ⇒ (5) follow from Theorem 3.2. □

Theorem 3.4. *Let R be an NI ring and P a strongly prime ideal of R.*

(1) *If R satisfies $(CZ1)$, then P is a minimal strongly prime ideal of R if and only if $P = \overline{O}_P$.*
(2) *If R satisfies $(CZ2)$, then P is a minimal strongly prime ideal of R if and only if $P = \overline{O(P)}$.*

Proof. (1) Let P be a minimal strongly prime ideal of R. Then by Theorem 3.1, P is completely prime and so $S = R \setminus P$ is a multiplicatively closed set. If we suppose that $a^k = 0$ for some $k > 0$, then there is nothing to prove. Assume that $a^k \neq 0$ for all $k > 0$. Construct M as in the proof of Theorem 3.2. Let $x, y \in M$. Then either $xy = 0$ or $xy \neq 0$. By the similar method to that of Theorem 3.2, we obtain either $(ad)^k = 0$ for some $k, d \in R \setminus P$ or $Q \cap M = \phi$ for some strongly prime ideal Q. Suppose the latter is true. Then $Q \subseteq P$. Since P is minimal strongly prime, $Q = P$. So that $a \in Q$ and hence $Q \cap M \neq \phi$, a contradiction. Thus

$$(ad)^k = 0 \text{ for some } k > 0 \tag{1}$$

Since R satisfies (CZ1), $a^qd^q = 0$ for some $q > 0$. Hence $a^q \in O_P$, because $d^q \in S$ and consequently $a \in \overline{O}_P$. Hence $P \subseteq \overline{O}_P$. Let $x \in \overline{O}_P$. Then there exist a positive integer n and $s \in R\backslash P$ such that $x^ns = 0$ and so $x^ns \in \mathcal{N}^*(R)$. Since R is NI ,

$\mathcal{N}^*(R)$ is completely semiprime and therefore we obtain $xs \in \mathcal{N}^*(R)$. Since $\mathcal{N}^*(R)$ has IFP, $xRs \subseteq \mathcal{N}^*(R) \subseteq P$. Since P is strongly prime, $x \in P$. Therefore $\overline{O}_P \subseteq P$. Thus $\overline{O}_P = P$.

Conversely, assume that $\overline{O}_P = P$. We have to show that P is a minimal strongly prime ideal of R. Suppose that there is a strongly prime ideal Q of R such that $Q \subseteq P$. Then $P = \overline{O}_P \subseteq \overline{O}_Q \subseteq Q$. So that $P = Q$. Therefore P is a minimal strongly prime ideal of R.

(2) Let P be a minimal strongly prime ideal of R and let $a \in P$. From equation (1) of part (1), we have $(ad)^k = 0$ for some $k > 0$ and $d \in R \setminus P$. Since R satisfies (CZ2), $a^q R d^q = 0$ for some $q > 0$. Since P is minimal strongly prime, P is completely prime by Theorem 3.1. Hence $d^q \in R \setminus P$ and so that $a^q \in O(P)$ and consequently $a \in \overline{O(P)}$. Let $x \in \overline{O(P)}$. Then $x^n Rs = 0$ for some $n > 0$ and $s \in R \setminus P$. Hence $x^n Rs \subseteq \mathcal{N}^*(R)$. From the completely semiprimeness of $\mathcal{N}^*(R)$ and strongly primeness of P, we obtain $x \in P$. Therefore $\overline{O(P)} \subseteq P$. Thus $P = \overline{O(P)}$.

The converse is similar to the converse of part (1). □

Corollary 3.5. *Let R be a ring which satisfies the condition $(CZ1)$. Then R is NI if and only if $P = \overline{O}_P$ for every minimal strongly prime ideal P of R.*

Proof. Suppose that $P = \overline{O}_P$ for every minimal strongly prime ideal P of R. Then $\mathcal{N}^*(R) = \bigcap_{P \in (m)Spec(R)} \overline{O}_P$. Let $x^n = 0$ for some $n > 0$. Then $x^n \in O_P$ for all $P \in (m)Spec(R)$ and consequently $x \in \mathcal{N}^*(R)$. Thus R is NI. The converse follows from Theorem 3.4. □

The following example shows that the conditions (CZ1) and (CZ2) are not superfluous in Theorem 3.4.

Example 3.6. *There is an NI ring in which $\overline{O}_P \neq P$ and $\overline{O(P)} \neq P$ for some minimal strongly prime ideal P of R:*

Let S be a domain that is not right Ore. So there are two non zero elements a and b in S such that $aS \cap bS = 0$. Consider the ring $R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$. Since S is NI, the ring R is also NI by [5, Proposition 4.1]. It can be easily checked that the ideal $P = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$ is a minimal strongly prime ideal of R.

Clearly, $x = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in P$. But we claim that $x \notin \overline{O}_P$. Assume to the contrary that $x \in \overline{O}_P$. Then there is a positive integer n and an element $y \in R \setminus P$ such that $x^n y = 0$, say $y = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$, where $\alpha, \beta, \gamma \in S$. Since $y \in R \setminus P$, $\gamma \neq 0$.

Now

$$x^n y = \begin{pmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} = 0.$$

It follows that $a^n\beta + a^{n-1}b\gamma = 0$ *and so* $a\beta + b\gamma = 0$. *Thus* $a\beta = b(-\gamma) \in aS \cap bS = 0$. *So* $b\gamma = 0$. *Since* $\gamma \neq 0$ *and* S *is a domain,* $b = 0$, *which is a contradiction and consequently* $x \notin \overline{O}_P$. *Thus* $P \neq \overline{O}_P$.

We note that if $\overline{O(P)} = P$ *then* $\overline{O}_P = P$, *for any strongly prime ideal* P *of* R. *Therefore* $\overline{O(P)} \neq P$.

Acknowledgment: The authors wish to express their indebtedness and gratitude to the referee for the helpful suggestions and valuable comments.

References

[1] G.F. Birkenmeier, H.E. Heatherly and E.K. Lee, Completely prime ideals and associated radicals, Proc. Biennial Ohio State-Denison conference 1992, edited by S.K. Jain and S.T. Rizvi, World Scientific, New Jersey (1993), 102-129.

[2] G.F. Birkenmeier, J.Y. Kim and J.K. Park, *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc., 122 (1994), 53-58.

[3] G.F. Birkenmeier, J.Y. Kim and J.K. Park, *A characterization of minimal prime ideals*, Glasgow Math J., 40 (1998) 223-236.

[4] C.Y. Hong and T.K. Kwak, *On minimal strongly prime ideals*, Comm. Algebra, 28 (2000), 4867-4878.

[5] S.U. Hwang, Y.C. Jeon and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra, 302 (2006),186-199.

C. Selvaraj and **S. Petchimuthu**
Department of Mathematics
Periyar University
Salem-636 011, India.
email: selvavlr@yahoo.com (C. Selvaraj)
spmuthuss@yahoo.co.in (S. Petchimuthu)