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WEAKLY LASKERIAN, WEAKLY COFINITE MODULES AND GENERALIZED LOCAL COHOMOLOGY

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Abstract. Let R be a commutative Noetheran ring, I an ideal of R and M be a finitely generated projective R-module. Let N be an R module and t a non-negative integer such that $\mathrm{Ext}^t_R(M/IM, N)$ is weakly Laskerian. Then for any weakly Laskerian submodule U of the first non I-weakly cofinite module $H^t_I(M,N)$, the R-module $\mathrm{Hom}_R(M/IM, H^t_I(M,N)/U)$ is weakly Laskerian. As a consequence the set of associated primes of $H^t_I(M,N)/U$ is finite.

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1. Introduction

The notion of generalized local cohomology was first introduced by J. Herzog [10] in his Habilitationsschrift. These modules have attracted the interest of others as well, see for example [5,6,11,17]. Let I be an ideal of R and let M, N be two R-modules. The i-th generalized local cohomology module of M and N with respect to I is defined by

$$H^i_I(M,N) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/I^nM, N).$$

With $M = R$ we obtain the ordinary local cohomology $H^i_I(N) = H^i_I(R,N)$ introduced by A. Grothendieck.

One of the famous conjecture in commutative and homological algebra which raised by Huneke [12] is: If N is a finitely generated R-module, then the set of associated primes of $H^i_I(N)$ is finite for all ideals I of R and all $i \geq 0$. Singh [16] and Katzman [13] showed that this conjecture is not true in general. However, there is considerable papers which show that this question has affirmative answer in many cases, see for example [1,14].

In [1], the authors proved that for a finitely generated R-module N, the first non-finitely generated local cohomology $H^t_I(N)$ has only finitely many associated prime ideals.

In this paper, among other things, by concerning the notion of weakly Laskerian modules, we prove that the weakly Laskerian property of the modules $H^i_I(M,N)$ for $i \leq t$ inherits the same property for the modules $\mathrm{Ext}^i_R(M/IM,N)$ for all $i \leq t$. Also we improve and generalize the above mentioned result of [1] to a large class of modules. More precisely we prove the following theorem (Theorem 1.1). This result has been proved in the special case of ordinary local cohomology, by Divaani-Aazar and Mafi in [7], using the spectral sequence argument. Our method of proof is completely different from the proof of [1] and that of [7].

Theorem 1.1. *Let M be a finitely generated projective R-module and N an arbitrary R-module. Let t be a non-negative integer such that $Ext^t_R(M/IM,N)$ is weakly Laskerian and that $H^i_I(M,N)$ is I-weakly cofinite for all $i < t$. Then for any weakly Laskerian submodule U of $H^t_I(M,N)$, the R-module $Hom_R(M/IM, H^t_I(M,N)/U)$ is weakly Laskerian. In particular the set of associated primes of $H^t_I(M,N)/U$ is finite.*

2. Main Results

Throughout, R is a commutative Noetherian ring, I is a ideal of R and M,N are two R-modules with M finitely generated. First we recall some known results on generalized local cohomology which we need in the sequel.

Remark 2.1. i) As usual, $\Gamma_I(.)$ is the functor from the category of R-modules to itself which assigns to each R-module X in this category the module $\Gamma_I(X) = \{x \in X | I^t x = 0 \quad \text{for some} \quad t \geq 0\}$. Then the functor $\Gamma_I(.)$ is covariant, R-linear and left exact. Let $E^\bullet$ be an injective resolution of N. Then from [5] one has

$$H^i_I(M,N) \cong H^i(\Gamma_I(\mathrm{Hom}_R(M,E^\bullet)))$$

$$\cong H^i(\mathrm{Hom}_R(M,\Gamma_I(E^\bullet))).$$

We observe that the above isomorphisms imply that all $H^i_I(M,N)$ are I-torsion modules (an R-module X is said to be I-torsion if $\Gamma_I(X) = X$).

ii) If N is an I torsion module or $I \subseteq (0:_R M)$, then $H^i_I(M,N) = \mathrm{Ext}^i_R(M,N)$ for each $i \geq 0$ (see [5]).

iii) If $f : R \to R'$ is a flat homomorphism of commutative Noetherain rings, then

$$H^i_I(M,N) \otimes_R R' \cong H^i_{IR'}(M \otimes_R R', N \otimes_R R'),$$

in particular for each prime ideal $\mathfrak{p}$ of R

$$H^i_I(M,N)_{\mathfrak{p}} \cong H^i_{I_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}).$$

This gives that $\mathrm{Supp}_R(H^i_I(M,N)) \subseteq V(I)$, where $V(I)$ is the set of all prime ideals containing I.

iv) From the definition of generalized local cohomology it follows that for any short exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules, there is a long exact sequence

$$0 \to H^0_I(M,X) \to H^0_I(M,Y) \to H^0_I(M,Z) \to H^1_I(M,X) \to \cdots ,$$

of generalized local cohomology modules.

Definition 2.2. i) An R-module N is said to be *Laskerian* if any submodule of N is an intersection of a finite number of primary submodules.
ii) An R-module N is said to be *weakly Laskerian* if the set of associated primes of any quotient of N is finite [7].

Example 2.3. i) Any Laskerian R-module is weakly Laskerian. Also any Noetherian and any Artinian R-module is weakly Laskerian.

ii) Recall that for an R-module N, the Goldie dimension of N, denoted by $\mathrm{Gdim}N$, is defined as the cardinal of the set of indecomposable submodules of the injective hull $E(N)$ of N, which appear in a decomposition of $E(N)$ as a direct sum of indecomposable submodules. In other words $\mathrm{Gdim}N = \sum_{\mathfrak{p}\in \mathrm{Ass}_R N} \mu^0(\mathfrak{p}, N)$, where $\mu^0(\mathfrak{p}, N)$ is the 0-th Bass number of N relative to $\mathfrak{p}$. Now the I-relative Goldie dimension of N, which is introduced in [9], denoted by $\mathrm{Gdim}_I N$, is defined by $\mathrm{Gdim}_I N = \sum_{\mathfrak{p}\in V(I)} \mu^0(\mathfrak{p}, N)$. This yields that an I-torsion module all of whose quotients have finite I-relative Goldie dimension is weakly Laskerian.

Lemma 2.4. *i) The class of weakly Laskerian modules is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of all R-modules. In particular any finite direct sum of weakly Laskerian modules is weakly Laskerian.*

ii) Let M, N be two R-modules. If M is finitely generated and N is weakly Laskerian, then $\mathrm{Ext}^i_R(M,N)$ and $\mathrm{Tor}^R_i(M,N)$ are weakly Laskerian for all $i \geq 0$.

Proof. See the proof of [7, Lemma 2.3]. □

Following Zöschinger [19], an R-module N is said to be *minimax* if N has a finitely generated submodules T such that N/T is Artinian. Now by Lemma 2.4 i) and Example 2.3 i), it is clear that any minimax module is weakly Laskerian.

Proposition 2.5. *Let M, N be two R-modules. If M is finitely generated and N is a weakly Laskerian module with $\mathrm{Ass}_R N \subseteq V(I)$. Then $H^i_I(M,N)$ is weakly Laskerian for all $i \geq 0$.*

Proof. We note that by [3, 2.1.12], $\mathrm{Ass}_R(N/\Gamma_I(N)) \subseteq \mathrm{Ass}_R N \setminus \mathrm{Ass}_R\Gamma_I(N)$. So the assumption on N gives that $\Gamma_I(N) = N$. Hence by Remark 2.1 ii) we have $H^i_I(M,N) = \mathrm{Ext}^i_R(M,N)$. Now the result follows by Lemma 2.4 ii). □

Definition 2.6. (see [8, Definition 2.4]) The R-module N is said to be *I-weakly cofinite* if $\mathrm{Supp}_R N \subseteq V(I)$ and $\mathrm{Ext}^i_R(R/I,N)$ is weakly Laskerian for all $i \geq 0$.

Proposition 2.7. *If $\mathrm{Hom}_R(R/I,N)$ is weakly Laskerian and $\mathrm{Supp}_R N \subseteq V(I)$, then the set $\mathrm{Ass}_R N$ is finite. In particular if N is I-weakly cofinite then $\mathrm{Ass}_R N$ is finite.*

Proof. Since $\mathrm{Hom}_R(R/I,N)$ is weakly Laskerian and $\mathrm{Supp}_R N \subseteq V(I)$, it follows from [2, Exersise 1.2.27], that $\mathrm{Ass}_R N = \mathrm{Ass}_R N \cap V(I) = \mathrm{Ass}(\mathrm{Hom}_R(R/I,N))$ is finite. □

Remark 2.8. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If two of the modules in the sequence are I-weakly cofinite, then so is the third one. Consequently if $f : M \to N$ is a homomorphism between two I-weakly cofinite modules and one of three modules $\mathrm{Ker}f, \mathrm{Im}f$ and $\mathrm{Coker}f$ is I-weakly cofinite, then all three of them are I-weakly cofinite.

Proposition 2.9. *Let M be a finitely generated projective R-module, N be an R-module such that $H^i_I(M,N)$ is I-weakly cofinite for all i (respectively for all $i \leq t \in \mathbb{N}$), then $\mathrm{Ext}^i_R(M/IM,N)$ is a Laskerian R-module for all i (repectively for all $i \leq t$).*

Proof. We prove by induction on i. Let $i = 0$ and put $\bar{M} = M/IM$. We have

$$\begin{aligned} \mathrm{Hom}_R(\bar{M},N) &\cong \mathrm{Hom}_R(\bar{M},\Gamma_I(N)) \\ &\cong \mathrm{Hom}_R(R/I,\mathrm{Hom}_R(M,\Gamma_I(N))) \cong \mathrm{Hom}_R(R/I,H^0_I(M,N)), \end{aligned}$$

so that the third isomorphism is by Remark 2.1 i). Thus the result follows in this case.

So let $i > 0$ and set $\bar{N} = N/\Gamma_I(N)$. The short exact sequence $0 \to \Gamma_I(N) \to N \to \bar{N} \to 0$, gives the long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(\bar{M},\Gamma_I(N)) \to \mathrm{Ext}^i_R(\bar{M},N) \to \mathrm{Ext}^i_R(\bar{M},\bar{N}) \to \cdots,$$

of Ext-modules for all $i \geq 0$ and by Remark 2.1 ii) the isomorphism

$$H^i_I(M,N) \cong H^i_I(M,\bar{N}),$$

of generalized local cohomology modules for all $i \geq 1$. Since

$$\mathrm{Ext}^i_R(\bar{M},\Gamma_I(N)) \cong \mathrm{Ext}^i_R(R/I,\mathrm{Hom}_R(M,\Gamma_I(N))) \cong \mathrm{Ext}^i_R(M,H^0_I(M,N))$$

by [15, 9.21], and the later module is weakly Laskerian, so in view of Lemma 2.4 i) we may assume that $\Gamma_I(N) = 0$ (note that $H_I^0(M, \bar{N}) = 0$).

Let $E(N)$ be the injective hull of N and put $L = E(N)/N$. Then $\Gamma_I(E(N)) = 0$ and so by Remark 2.1 i) $H_I^0(M, E(N)) = 0$. Also using the Hom Vanishing Lemma of [4, p.11] we see that $\mathrm{Hom}_R(\bar{M}, E(N)) = 0$. Therefore using the exact sequence $0 \to N \to E(N) \to L \to 0$ we get $H_I^{i+1}(M, N) \cong H_I^i(M, L)$ and $\mathrm{Ext}_R^{i+1}(\bar{M}, N) \cong \mathrm{Ext}_R^i(\bar{M}, L)$ for all $i \geq 0$. Now the result follows by induction. □

Theorem 2.10. *Let M be a finitely generated projective R-module and N an arbitrary R-module. Let t be a non-negative integer such that $Ext_R^t(M/IM, N)$ is weakly Laskerian. Assume that $H_I^i(M, N)$ is I-weakly cofinite for all $i < t$. Then for any weakly Laskerian submodule U of $H_I^t(M, N)$, the R-module $Hom_R(M/IM, H_I^t(M, N)/U)$ is weakly Laskerian. In particular the set of associated primes of $H_I^t(M, N)/U$ is finite.*

Proof. We consider the exact sequence

$$0 \to U \to H_I^t(M, N) \to H_I^t(M, N)/U \to 0,$$

to obtain the exact sequence

$$\mathrm{Hom}_R(M/IM, H_I^t(M, N)) \to \mathrm{Hom}_R(M/IM, H_I^t(M, N)/U) \to \mathrm{Ext}_R^1(M/IM, U).$$

So by Lemma 2.4 it is sufficient to show that $\mathrm{Hom}_R(M/IM, H_I^t(M, N))$ is weakly Laskerian. To do this, we use induction on $t \geq 0$. For $t = 0$, we have

$$\mathrm{Hom}_R(M/IM, H_I^0(M, N)) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(M/IM, N)),$$

which is weakly Laskerian by our assumption and Lemma 2.4 ii). So let $t > 0$ and the case $t-1$ is settled. The exact sequence $0 \to \Gamma_I(N) \to N \to \bar{N} := N/\Gamma_I(N) \to 0$, induces the exact sequence

$$\mathrm{Ext}_R^t(M/IM, N) \to \mathrm{Ext}_R^t(M/IM, \bar{N}) \to \mathrm{Ext}_R^{t+1}(M/IM, \Gamma_I(N)),$$

of Ext-modules and the isomorphisms

$$H_I^i(M, N) \cong H_I^i(M, \bar{N}),$$

for all $i > 0$. Now by [15, 9.21], we have

$$\mathrm{Ext}_R^{t+1}(M/IM, \Gamma_I(N)) \cong \mathrm{Ext}_R^{t+1}(R/I, H_I^0(M, N)),$$

which is weakly Laskerian by our assumption. Also by the previous isomorphism $H_I^i(M, \bar{N})$ is I- weakly cofinite for $i < t$ (note that $H_I^0(M, \bar{N}) = 0$). So using Lemma 2.4 i), once again we can assume that $\Gamma_I(N) = 0$. Let $E(N)$ be the injective hull of N and put $L = E(N)/N$. Then as the previous proposition we get $\mathrm{Ext}_R^{i+1}(M/IM, N) \cong \mathrm{Ext}_R^i(M/IM, L)$ and $H_I^{i+1}(M, N) \cong H_I^i(M, L)$ which gives

$$\mathrm{Hom}_R(M/IM, H_I^{i+1}(M, N)) \cong \mathrm{Hom}_R(M/IM, H_I^i(M, L))$$

for all $i \geq 0$. The assertion now follows by induction.

The final part of theorem follows by Proposition 2.7. □

The following corollary now almost immediately yields.

Corollary 2.11. (cf. [1, Proposition 2.2] and [7, Corollary 2.7]) *Let M be a finitely generated projective R-module. Let t be a non-negative integer such that N and $H_I^i(M, N)$ are weakly Laskerian for all $i < t$. Let U be a weakly Laskerian submodule of $H_I^t(M, N)$. Then $Hom_R(M/IM, H_I^t(M, N)/U)$ is weakly Laskerian. Consequently the set of associated primes of $H_I^t(M, N)/U$ is finite.*

Proof. This follows by Theorem 2.10 and Proposition 2.7. □

We conclude the paper with the following corollary which extend the main result of [18].

Corollary 2.12. *Let M be a finitely generated projective R-module and N an R-module. Let t be a non-negative integer such that $Ext_R^t(M/IM, N)$ is weakly Laskerian. Assume that $H_I^i(M, N)$ are I-cofinite R-module for all $i < t$. Let U be a weakly Laskerian submodule of $H_I^t(M, N)$. Then $Hom_R(M/IM, H_I^t(M, N)/U)$ is weakly Laskerian. In particular the set of associated primes of $H_I^t(M, N)/U$ is finite.*

Question. Do Proposition 2.9 and Theorem 2.10 hold true for finitely generated R-module M of finite projective dimension? Are they true for any finitely generated R-module M?

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