PLANARITY OF INTERSECTION GRAPHS OF IDEALS OF RINGS

Sayyed Heidar Jafari and Nader Jafari Rad

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Abstract. In this paper we characterize planar intersection graphs of ideals of a commutative ring with 1.

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1. Introduction

For graph theory terminology in general we follow [6]. Specifically, let \( G = (V, E) \) be a graph with vertex set \( V \) of order \( n \) and edge set \( E \). We denote the degree of a vertex \( v \) in \( G \) by \( d_G(v) \), which is the number of edges incident to \( v \). A graph \( G \) is complete if there is an edge between every pair of the vertices. A subset \( X \) of the vertices of a graph \( G \) is called independent if there is no edge with two endpoints in \( X \). A graph \( G \) is called bipartite if its vertex set can be partitioned into two subsets \( X \) and \( Y \) such that every edge of \( G \) has one endpoint in \( X \) and other endpoint in \( Y \). A complete bipartite graph is a bipartite graph in which any vertex of a partite set is adjacent to all vertices in another partite set. A graph \( G \) is said to be star if \( G \) contains one vertex in which all other vertices are joined to this vertex and \( G \) has no other edges. The complement \( \overline{G} \) of \( G \) is the graph with vertex set \( V(\overline{G}) = V(G) \), and \( E(\overline{G}) = \{uv : uv \notin E(G)\} \). The complement of a complete graph is the null graph.

Let \( F = \{S_i : i \in I\} \) be an arbitrary family of sets. The intersection graph \( G(F) \) is the one-dimensional skeleton of the nerve of \( F \), i.e., \( G(F) \) is the graph whose vertices are \( S_i, i \in I \) and in which the vertices \( S_i \) and \( S_j \) (\( i, j \in I \)) are adjacent if and only if \( S_i \neq S_j \) and \( S_i \cap S_j \neq \emptyset \). It is shown that every simple graph is an intersection graph, ([5]).

It is interesting to study the intersection graphs \( G(F) \) when the members of \( F \) have an algebraic structure. Bosak [2] in 1964 studied graphs of semigroups. Then Csáky and Pollk [4] in 1969 studied the intersection graphs of subgroups of a finite...

Chakrabarty et al. [3] studied intersection graphs of ideals of rings. The intersection graph of ideals of a ring $R$, denoted $\Gamma(R)$, is the undirected simple graph (without loops and multiple edges) whose vertices are in a one-to-one correspondence with all nontrivial left ideals of $R$ and two distinct vertices are joined by an edge if and only if the corresponding left ideals of $R$ have a nontrivial (nonzero) intersection. Clearly the set of vertices is empty for left simple rings. In this case we refer $\Gamma(R)$ as the empty graph.

Chakrabarty et al. [3] studied planarity of intersection graphs of the ring $\mathbb{Z}_n$. In this paper we will characterize all commutative rings with 1 which $\Gamma(R)$ is planar.

We denote by $K_n$ the complete graph on $n$ vertices, and by $K_{m,n}$ the complete bipartite graph which one partite set is of cardinality $m$ and another partite set is of cardinality $n$.

2. Results

We will repeatedly use Kuratowski's theorem, which states that a graph is planar if and only if it does not contain a subdivision of $K_5$ or $K_{3,3}$ (see [6, Theorem 6.2.2]).

Let $R$ be a commutative ring with 1. We begin with the following lemma.

Lemma 2.1. If $\Gamma(R)$ is planar, then any chain of ideals of $R$ has length at most five.

Proof. Let $I_1 \subset I_2 \subset ... \subset I_5$ be a chain of nontrivial proper ideals of $R$. Then $I_1, I_2, ..., I_5$ induce a $K_5$ as an induced subgraph in $\Gamma(R)$. This completes the proof. \qed

Corollary 2.2. If $\Gamma(R)$ is planar, then $R$ is both Noetherian and Artinian.

Lemma 2.3. If $\Gamma(R)$ is null and $R$ contains at least two proper nontrivial distinct ideals, then $R \cong R_1 \times R_2$, where $R_1, R_2$ are fields.

Proposition 2.4. $\Gamma(R_1 \times R_2)$ is planar if and only if one of $\Gamma(R_1), \Gamma(R_2)$ is empty, and another is empty or null with at most two vertices.

Proof. ($\Rightarrow$) Let $I_1, I_2$ be two nontrivial ideals of $R_2$ with $I_1 \subseteq I_2$. Then $0 \times I_1, 0 \times I_2, 0 \times R_2, R_1 \times I_1, R_1 \times I_2$ form a $K_5$, a contradiction. So $\Gamma(R_1), \Gamma(R_2)$ are null or empty. We show that $\Gamma(R_1)$ or $\Gamma(R_2)$ is empty. Suppose that both $\Gamma(R_1)$ and $\Gamma(R_2)$
are null. Let $I < R_1, J < R_2$, (nontrivial). Then $0 \times R_2, 0 \times J, I \times R_2, I \times J, R_1 \times J$ form a $K_5$, a contradiction. Assume that $\Gamma(R_1)$ is empty. Suppose that $\Gamma(R_2)$ is null. By Lemma 2.3, $\Gamma(R_2)$ has at most two vertices. 

$(\Leftarrow)$ Is straightforward. \hfill $\Box$

**Corollary 2.5.** $\Gamma(R_1 \times R_2 \times R_3)$ is planar if and only if $R_i$ is a field for $i = 1, 2, 3$.

**Proof.** Notice that if $R_3$ is not a field and $I \leq R_3$, then $R_2 \times 0, R_2 \times I$ is an edge in $\Gamma(R_2 \times R_3)$, and by Proposition 2.4, $\Gamma(R_1 \times R_2 \times R_3)$ is not planar. \hfill $\Box$

Let $Max(R)$ be the set of all maximal ideals of $R$.

**Lemma 2.6.** If $\Gamma(R)$ is planar, then $|Max(R)| \leq 3$.

**Proof.** Let $\Gamma(R)$ is planar. Suppose that $|Max(R)| \geq 4$. Let $M_1, M_2, M_3$ be three distinct maximal ideals of $R$. Let $I = M_1 \cap M_2 \cap M_3$. Since $|Max(R)| \geq 4$, we have $I \neq 0$. Then $M_1, M_2, M_3, M_1 \cap M_2, M_1 \cap M_3, M_2 \cap M_3$ form a $K_6$, as an induced subgraph, a contradiction. \hfill $\Box$

We divide the rest of the paper into two subsection according to $|Max(R)|$.

2.1. $|Max(R)| \neq 1$. Let $J(R)$ be the Jacobson radical of $R$. We first consider the case $|Max(R)| = 3$.

**Corollary 2.7.** If $|Max(R)| = 3$ and $\Gamma(R)$ is planar, then $J(R) = 0$.

**Corollary 2.8.** If $|Max(R)| = 3$ and $\Gamma(R)$ is planar, then $R \cong R_1 \times R_2$.

**Proof.** Let $Max(R) = \{M_1, M_2, M_3\}$. By Corollary 2.7, $M_1 \cap (M_2 \cap M_3) = 0$. On the other hand $M_1 + (M_2 \cap M_3) = R$. So the result follows. \hfill $\Box$

**Theorem 2.9.** If $|Max(R)| = 3$, then $\Gamma(R)$ is planar if and only if $R = R_1 \times R_2 \times R_3$, where $R_i$ is a field for $i = 1, 2, 3$.

**Proof.** Follows from Corollary 2.8, Lemma 2.3 and Proposition 2.4. \hfill $\Box$

We next assume that $|Max(R)| = 2$.

**Lemma 2.10.** (Nakayama, [1]) Let $M$ be a finitely generated $R$-modulate. If $J(R)M = M$, then $M = 0$.

**Lemma 2.11.** If $Max(R) = \{M_1, M_2\}$ and $\Gamma(R)$ is planar, then $R \cong M_1^3 \times M_2^3$. 


We first show that \( M_3^1 \cap M_3^2 = 0 \). Suppose that \( M_3^1 \cap M_3^2 \neq 0 \). By Corollary 2.2, \( M_1, M_2 \) are finitely generated \( R \)-modules. By Nakayama’s lemma \( M_1, M_2, (M_1 \cap M_2), (M_1 \cap M_2)^2, (M_1 \cap M_2)^3 \) are all mutually distinct. Then \( M_1, M_2, (M_1 \cap M_2), (M_1 \cap M_2)^2, (M_1 \cap M_2)^3 \) form a \( K_5 \) as an induced subgraph, a contradiction. So \( M_3^1 \cap M_3^2 = 0 \). On the other hand \( M_3^1 + M_3^2 = R \). This completes the result.

Theorem 2.12. If \( |\text{Max}(R)| = 2 \), then \( \Gamma(R) \) is planar if and only if one of \( \Gamma(R_1), \Gamma(R_2) \) is empty, and another is empty or null with one vertex.

Proof. Notice that by Lemma 2.11, \( R \cong R_1 \times R_2 \). Now the result follows by Proposition 2.4.

2.2. \( |\text{Max}(R)| = 1 \). In this subsection \( R \) is a local ring. Let \( M \) be the unique maximal ideal of \( R \). The following lemmas are easily verified.

Lemma 2.13. If \( \Gamma(R) \) is planar, then \( M^5 = 0 \).

Lemma 2.14. Let \( I \trianglerighteq R \). Then \( \frac{I}{IM} \) is a vector space over \( \frac{R}{M} \). Further, any subspace of \( \frac{I}{IM} \) is in the form \( \frac{J}{IM} \), where \( J \trianglerighteq R \) and \( IM \subseteq J \subseteq I \).

Lemma 2.15. Let \( I \trianglerighteq R \). If \( \dim(\frac{I}{IM}) \geq 3 \), then \( \Gamma(R) \) is not planar.

Proof. Let \( u_1, u_2, u_3 \) be three linear independent vectors in \( \frac{I}{IM} \). Let \( W = \langle u_1, u_2, u_3 \rangle \). Since \( \dim(\frac{W}{(u_1)}) = 2 \), \( \frac{W}{(u_1)} \) contains exactly \( |\frac{R}{M}| + 1 \) subspaces of dimension 1. This implies that \( W \) contains at least 3 subspaces \( W_1, W_2, W_3 \) of dimension 2 containing \( u_1 \). On the other hand \( W_4 = \langle u_2, u_3 \rangle \) is another subspace of \( W \) of dimension 2. We obtain that \( W_1, W_2, W_3, W_4 \) are for subspaces of dimension 2 such that \( W_i \cap W_j \neq 0 \) for \( i, j \in \{1, 2, 3, 4\} \). Suppose that \( W_i = \frac{J}{IM} \) for \( i = 1, 2, 3, 4 \). Now \( J_1, J_2, J_3, J_4, M \) form a \( K_5 \).

Corollary 2.16. Let \( M^2 = 0 \). Then \( \Gamma(R) \) is planar if and only if \( \dim(M) = 1 \) or 2 as a vector space over \( \frac{R}{M} \).

Proof. Follows by Lemma 2.15 with putting \( I = M \).

Corollary 2.17. Let \( M^2 = 0 \). Then \( \Gamma(R) \) is planar if and only if \( \Gamma(R) \) is either an star or \( K_4 \).

Lemma 2.18. Let \( M^2 \neq 0 \). If \( \Gamma(R) \) is planar, then \( \dim(\frac{M}{M^2}) = 1 \) and \( \frac{M}{M^2} \cong \frac{M^2}{M^2} \) as an isomorphism of \( R \)-modules.
By Lemma 2.15, \( \dim(\frac{M}{x}) \leq 2 \). Suppose that \( \dim(\frac{M}{x}) = 2 \). It follows that \( \frac{M}{x} \) contains at least three subspaces \( W_1, W_2, W_3 \) of dimension 1. Let \( W_i = \frac{x}{x} \) for \( i = 1, 2, 3 \). Then \( J_1, J_2, J_3, M, M^2 \) form a \( K_5 \), a contradiction. Thus \( \dim(\frac{M}{x}) = 1 \). As a consequent, \( M = \langle a \rangle \) for some \( a \in R \). We define the map \( \phi : \frac{M}{x} \rightarrow \frac{M^2}{x} \) by \( \phi(ra + M^2) = ra^2 + M^3 \). Since \( \frac{M^2}{x} \) is a simple \( R \)-module, it is straightforward to see that \( \phi \) is an \( R \)-isomorphism.

**Corollary 2.19.** Let \( M^2 \neq 0 \) and \( M^3 = 0 \). Then \( \Gamma(R) \) is planar if and only if \( \Gamma(R) = K_2 \).

**Proof.** Let \( \Gamma(R) \) be planar. By Lemma 2.18, \( M = Ra \) where \( a \in R \). Let \( I \) be a minimal ideal of \( R \). We show that \( I = M^2 \). Since \( I \) is a simple \( R \)-module, we obtain \( I \cong \frac{R}{M^2} \). Then \( I = \langle x \rangle \), where \( x \in R \). If \( x \in M \setminus M^2 \), then \( x = ra \), where \( r \in R \setminus M \). So \( r \) is invertible and \( \langle x \rangle = \langle a \rangle = M \), a contradiction. We deduce that \( x \in M^2 \), and so \( I \subseteq M^2 \). Since \( M^2 \) is simple, we obtain \( I = M^2 \). Thus \( M^2 \) is the unique minimal ideal of \( R \), and \( \Gamma(R) = K_2 \). The converse is obvious.

**Lemma 2.20.** Let \( M^3 \neq 0 \) and \( M^4 = 0 \). If \( \Gamma(R) \) is planar, then \( \dim(\frac{M}{M^3}) = 1 \) and \( \frac{M}{M^3} \cong \frac{M^2}{M^3} \cong \frac{M^4}{M^7} \).

**Corollary 2.21.** Let \( M^3 \neq 0 \) and \( M^4 = 0 \). Then \( \Gamma(R) \) is planar if and only if \( \Gamma(R) = K_3 \) or \( K_4 \).

**Proof.** By Lemma 2.20, \( M = Ra \) where \( a \in R \). Let \( I \) be a minimal ideal of \( R \). We show that \( I = M^3 \). Since \( I \) is a simple \( R \)-module, we obtain \( I \cong \frac{R}{M^3} \). Then \( I = \langle x \rangle \), where \( x \in R \). If \( x \in M \setminus M^2 \), then \( x = ra \), where \( r \in R \setminus M \). So \( r \) is invertible and \( \langle x \rangle = \langle a \rangle = M \), a contradiction. If \( x \in M^2 \setminus M^3 \), then \( x = ra^2 \), where \( r \in R \setminus M \). As before we can see that \( \langle x \rangle = \langle a^2 \rangle = M^2 \), a contradiction. We deduce that \( x \in M^3 \), and so \( I \subseteq M^3 \). Since \( M^3 \) is simple, we obtain \( I = M^3 \). Thus \( M^3 \) is the unique minimal ideal of \( R \), and \( \Gamma(R) \) is complete. Now the result follows.

**Lemma 2.22.** Let \( M^4 \neq 0 \) and \( M^5 = 0 \). If \( \Gamma(R) \) is planar, then \( \dim(\frac{M}{M^4}) = 1 \) and \( \frac{M}{M^4} \cong \frac{M^2}{M^4} \cong \frac{M^3}{M^4} \cong \frac{M^5}{M^8} \).

**Corollary 2.23.** Let \( M^4 \neq 0 \) and \( M^5 = 0 \). Then \( \Gamma(R) \) is planar if and only if \( \Gamma(R) = K_4 \).

**Proof.** By Lemma 2.22, \( M = Ra \) where \( a \in R \). Let \( I \) be a minimal ideal of \( R \). Similar to the proof of Corollary 2.21, we obtain \( I = M^4 \). Thus \( M^4 \) is the unique minimal ideal of \( R \), and \( \Gamma(R) \) is complete. Now the result follows.
As a consequent of Corollaries 2.17, 2.19, 2.21 and 2.23 we obtain the following.

**Theorem 2.24.** If $|\text{Max}(R)| = 1$, then $\Gamma(R)$ is planar if and only if $\Gamma(R)$ is an star, $K_1$, $K_3$, or $K_4$.

**References**


Sayyed Heidar Jafari and Nader Jafari Rad

Department of Mathematics

Shahrood University of Technology

Shahrood, Iran

e-mails: shjafari55@gmail.com (Sayyed Heidar Jafari)

n.jafarirad@gmail.com (Nader Jafari Rad)