International Electronic Journal of Algebra
Volume 8 (2010) 18-29

VALUATION, DISCRETE VALUATION AND DEDEKIND MODULES

J. Moghaderi and R. Nekooei

Received: 18 April 2009; Revised: 01 May 2010
Communicated by Surender K. Jain

Abstract. The purpose of this paper is to introduce *valuation* and *discrete valuation modules* over an integral domain. Some basic results and characterizations are obtained and these results are used to characterize*Dedekind multiplication modules* with *discrete multiplication valuation modules*.

Mathematics Subject Classification (2000): 13C13
Keywords: multiplication module, Dedekind module, prime submodule, valuation, discrete valuation, fractional submodule

1. Introduction

Throughout this paper, R denotes an integral domain, with quotient field K, $T = R - \{0\}$ and M is a unitary R-module. A submodule N of M is called *prime (primary)* if $N \neq M$ and for arbitrary $r \in R$ and $m \in M$, $rm \in N$ implies $m \in N$ or $r \in (N : M)$ ($r^n \in (N : M)$, for some $n \in \mathbb{N}$), where $(N : M) = \{r \in R | rM \subseteq N\}$. It is clear that when N is a prime submodule, $(N : M)$ is a prime ideal of R. The *radical of* N, given by $radN$, is the intersection of all prime submodules of M containing N (see [7,9,10]). If there is no prime submodule containing N, then we put $radN = M$. An R-module M is called a *multiplication R-module*, if for each submodule N of M, there exists an ideal I of R such that $N = IM$. (For more information about multiplication modules, see [1,4,14,16].) An integral domain R is called a *valuation ring*, if for each $x \in K - \{0\}$, $x \in R$ or $x^{-1} \in R$ (see [5,6,12]). In the first section of this paper, we generalize the notion of valuation to a torsionfree R-module and obtain results which characterize it. Then we prove some interesting results for multiplication valuation modules. In the second section, we introduce *fractional submodules, discrete valuation modules* and obtain some basic results. Finally in the third section, we obtain relations between *Dedekind* modules and discrete valuation modules and give some characterizations for Dedekind multiplication modules.

This research has been supported by Mahani Mathematical Research Center.

2. Valuation Modules

Let R be an integral domain with quotient field K and M a torsionfree R-module. For $y = \frac{r}{s} \in K$ and $x \in M$, then following [13], we say that $yx \in M$ if there exists $m \in M$ such that $rx = sm$. It is clear that this is a well-defined operation.

Lemma 2.1. *Let R be an integral domain with quotient field K and M a torsionfree R-module. Then the following conditions are equivalent:*
i) For all $y \in K$ and all $x \in M$, $yx \in M$ or $y^{-1}M \subseteq M$.
ii) For all $y \in K$, $yM \subseteq M$ or $y^{-1}M \subseteq M$.

Definition 2.2. *Let R be an integral domain with quotient field K. A torsionfree R-module M is called valuation R-module (VM) if one of the conditions of Lemma 2.1 holds.*

Example 2.3. *i) Let R be a domain. R is a valuation ring if and only if R is a valuation R-module.*
ii) Any vector space is a valuation module.
iii) Let $R = \mathbb{Z}$ and p be a prime integer number. If $M = \{p^n \frac{a}{b} | a, b, n \in \mathbb{Z},\ b \neq 0,\ n \geq 0,\ (p, a) = (p, b) = (a, b) = 1\}$, then M is a valuation module.
iv) Let M be a valuation R-module, then any K-subvector space of M_T, which contains M is a valuation module.
v) $\mathbb{Z}$ is not a valuation $\mathbb{Z}$-module.

Following [2], an R-module M is said to be *integrally closed* whenever $y^n m_n + \cdots + y m_1 + m_0 = 0$, for some $n \in \mathbb{N}$, $y \in K$ and $m_i \in M$, then $y m_n \in M$. By [5, Proposition 5.18], any valuation ring is integrally closed. As the following shows, valuation modules also have this property.

Lemma 2.4. *Any valuation module is integrally closed.*

Proof. Let M be a valuation R-module and $y^n m_n + \cdots + y m_1 + m_0 = 0$, for some $n \in \mathbb{N}$, $y \in K$ and $m_i \in M$. Since M is a VM, if $y m_n \notin M$ then $y^{-1}M \subseteq M$. So $y^{-1} m_i \in M$ for all i, $0 \leq i \leq n-1$ and hence $y^{-t} m_i \in M$, for all $t \in \mathbb{N}$ and all i, $0 \leq i \leq n-1$. Therefore $y m_n = -m_{n-1} - y^{-1} m_{n-2} - \cdots - y^{1-n} m_0 \in M$ and M is integrally closed. □

A subset N of an R-module M is called *R-stable*, if $RN \subseteq N$, i.e. for all $r \in R$ and $x \in N$, $rx \in N$.

Proposition 2.5. *Let K be the quotient field of a domain R and M a torsionfree R-module. Let S be the set, ordered by inclusion, of all non-empty subsets of M.*

Then the following conditions are equivalent:
i) M is a valuation module.
ii) $S' = \{(N : M)|N \in S\}$ is totally ordered.
iii) For $U = \{rM|r \in R\}$ the subset of S, U' is totally ordered.

Proof. i)⇒ii) Let $N, L \in S$ be such that there exist $r \in (N : M)\backslash(L : M)$ and $s \in (L : M)\backslash(N : M)$. So $rM \subseteq N$, $sM \subseteq L$ and there exist $\alpha, \beta \in M$ such that $s\alpha \notin N$, $r\beta \notin L$. Since M is a VM for $y = \frac{s}{r} \in K$ and $\alpha \in M$, if $y\alpha \in M$, there exists $m \in M$ such that $s\alpha = rm \in rM \subseteq N$, which is a contradiction. If $y^{-1}M \subseteq M$ then $y^{-1}\beta \in M$ and so there exists $n \in M$ such that $r\beta = sn \in sM \subseteq L$, which is again a contradiction. Therefore S' is totally ordered.

ii)⇒iii) This is clear.

iii)⇒i) Let $y = \frac{s}{r} \in K$. Since $rM, sM \in U$, $(sM : M) \subseteq (rM : M)$ or $(rM : M) \subseteq (sM : M)$. So $sM \subseteq rM$ or $rM \subseteq sM$. Therefore $yM \subseteq M$ or $y^{-1}M \subseteq M$ and M is a VM. □

Corollary 2.6. *Let R be a domain and M a torsionfree R-module. Then M is a valuation module if and only if for any submodules N, L of M, $(N : M) \subseteq (L : M)$ or $(L : M) \subseteq (N : M)$.*

Corollary 2.7. *Let K be the quotient field of a domain R and M a faithful multiplication R-module. Let S be the set, ordered by inclusion, of all R-stable non-empty subsets of M. Then the following conditions are equivalent:*
i) M is a valuation module.
ii) S is totally ordered.
iii) $U = \{rM|r \in R\}$ the subset of S is totally ordered. Moreover, in this case S is the set of all submodules of M.

Proof. Since M is multiplication, the equivalence is easily obtained from Proposition 2.5. For the last part, let $N \in S$. It is enough to show that for any $\alpha, \beta \in N$, $\alpha - \beta \in N$. By (ii), $R\alpha \subseteq R\beta$ or $R\beta \subseteq R\alpha$. Let $R\alpha \subseteq R\beta$, there exists $r \in R$ such that $\alpha = r\beta$. So $\alpha - \beta = (r - 1)\beta \in N$. □

Corollary 2.8. *Let R be a domain and M a faithful multiplication R-module. Then M is a valuation module if and only if for any two submodules N, L of M, $N \subseteq L$ or $L \subseteq N$. If M is also a valuation R-module, then*
i) M has a unique maximal submodule.
ii) for a proper submodule N of M, $radN$ is a prime submodule of M.
iii) for a proper submodule N of M, if $radN = N$, then N is a prime submodule.

Remark. $\mathbb{R}^2$ is a valuation $\mathbb{R}$-module, but not a multiplication $\mathbb{R}$-module. Note that $\mathbb{R} \oplus (0) \nsubseteq (0) \oplus \mathbb{R}$ and $(0) \oplus \mathbb{R} \nsubseteq \mathbb{R} \oplus (0)$.

Note that $\mathbb{R}$ does not have non-zero maximal submodules as an $\mathbb{R}$-module. Any vector space is a VM, but an infinite dimensional vector space has infinite number of maximal submodules. So it is not necessary that each valuation module has a (unique) maximal submodule.

Theorem 2.9. *Let M be a valuation R-module. Then*
i) For any submodule N of M, such that $\frac{M}{N}$ is a torsionfree R-module, $\frac{M}{N}$ is a VM.
ii) If M is finitely generated, then for each $p \in Spec(R)$, M_p is a valuation R_p-module.
iii) If M' is a torsionfree R-module and $\varphi : M \to M'$ is an epimorphism, then M' is a valuation module too.

Proof. i) Let $\frac{L_1}{N}$, $\frac{L_2}{N}$ be two submodules of $\frac{M}{N}$. So L_1, L_2 are submodules of M, containing N. Since M is a VM, by Corollary 2.6, $(L_1 : M) \subseteq (L_2 : M)$ or $(L_2 : M) \subseteq (L_1 : M)$. Let $(L_1 : M) \subseteq (L_2 : M)$. It is clear that $(\frac{L_1}{N} : \frac{M}{N}) \subseteq (\frac{L_2}{N} : \frac{M}{N})$ and so by Corollary 2.6, $\frac{M}{N}$ is a VM.
ii) Let $p \in Spec(R)$. Since R is a domain and M is torsionfree, it is easy to see that R_p is a domain and M_p is a torsionfree R_p-module. Let N_p, L_p be two submodules of M_p, corresponding to submodules N and L of M. Since M is a VM, by Corollary 2.6, $(N : M) \subseteq (L : M)$ or $(L : M) \subseteq (N : M)$. Let $(N : M) \subseteq (L : M)$. Since M is finitely generated, so $(N_p : M_p)_{R_p} \subseteq (L_p : M_p)_{R_p}$. Hence M_p is a valuation R_p-module.
iii) By part (i). □

Prüfer modules has been defined by Naoum and Al-Alwan in [13, page 407]. The R-module M is *uniserial*, if its submodules are totally ordered by inclusion or equivalently given $a, b \in M$, either $aR \subseteq bR$ or $bR \subseteq aR$. It is clear that if M is a torsionfree uniserial R-module, then M is a valuation R-module. Now, let M be a torsionfree module over a Prüfer domain R, then M is a Prüfer module if and only if for every maximal ideal P of R, the R_P-module M_P is uniserial (see [11, Theorem 2.4]).

The following two lemmas give the relations between valuation rings and valuation modules.

Lemma 2.10. *Let R be a valuation ring and M a torsionfree R-module. Then M is a valuation R-module.*

Lemma 2.11. *If M is a multiplication valuation R-module, then M is finitely generated and R is a valuation ring.*

Proof. Let I, J be ideals of R, then IM, JM are submodules of M and since M is a VM, by Corollary 2.8, $IM \subseteq JM$ or $JM \subseteq IM$. Let $IM \subseteq JM$. Now by [1, Corollary 3.3, Lemma 4.1] M is finitely generated, and so $I \subseteq J$. So by [5, Proposition 5.2], R is a valuation ring. □

Let M be a multiplication module. If M is a Dedekind module then by [2, Theorem 3.12], R is a Dedekind domain. Also by [2, Corollary 3.15], M is Noetherian. Hence by [2, Corollary 3.7], every multiplication Dedekind R-module M is isomorphic to an ideal of R.

Lemma 2.12. *Let R be a valuation domain. Then every finitely generated torsion-free R-module is free.*

Proof. [6, §3.6, Lemma 1]. □

Corollary 2.13. *Let M be a multiplication valuation module over an integral domain R. Then M is isomorphic to R.*

Proof. By Lemma 2.11, R is a valuation ring. Since M is finitely generated and torsionfree, by Lemma 2.12, M is free and so isomorphic to R. □

Corollary 2.14. *Let M be a multiplication valuation R-module. Then any finitely generated submodule of M is cyclic.*

Let M be a multiplication R-module, $N = IM$ and $L = JM$ for some ideals I and J of R. Following [4], the product of N and L is denoted by $N.L$ or NL and is defined by IJM. We consider $N^t = I^tM$, for any $t \in \mathbb{R}$. By [4, Lemma 3.6], if M is finitely generated faithful multiplication, then $ann(\frac{M}{N})ann(\frac{M}{L}) = ann(\frac{M}{NL})$ or $(N : M)(L : M) = (NL : M)$.

Theorem 2.15. *Let M be a multiplication valuation R-module, $N = IM$ a proper submodule of M, for ideal I of R and $L = \bigcap_{n=1}^{\infty} N^n = \bigcap_{n=1}^{\infty} I^nM$. Then*
i) L is a prime submodule of M.
ii) If, for some positive integer t, $N^t = N^{t+1}$, then N is an idempotent prime submodule.
iii) If U is a submodule of M with $N \subseteq radU$, then U contains a power of N.
iv) L contains every prime submodule of M which is properly contained in N.
v) Every prime submodule of M which is properly contained in N, is contained in every power of N.

Theorem 2.16. *Let R be a domain, P a prime submodule of a multiplication valuation R-module M and $P = pM$, where $p = (P : M) \in Spec(R)$. We have*
i) If Q is p-primary and $x \in M - P$, then $Q = I(x)$, where $I = \{y \in K | yx \in Q\}$.
ii) If $x \in M - P$, then $P = p(x)$.
iii) If $P \neq P^2$, then the only p-primary submodules of M are powers of P.
Furthermore, let P be a maximal submodule. Then
iv) If Q_1, Q_2 are p-primary, then Q_1Q_2 is a p-primary submodule.
v) The intersection of all p-primary submodules of M is a prime submodule and there are no prime submodules of M properly between it and P.

Following [4], an element u of an R-module M is said to be unit provided that u is not contained in any maximal submodule of M. By [4, Theorem 3.19], in a multiplication R-module M, $u \in M$ is unit if and only if $M = Ru$.

Theorem 2.17. *Let R be a local ring (not necessarily an integral domain) with unique principal maximal ideal $I = (p)$ and M a multiplication R-module such that $\bigcap_{n=1}^{\infty}(p^n)M = (0)$. Then the only proper submodules of M are (0) and $(p^m)M$, for some $m \geq 1$. Furthermore, if M is faithful, then either p is nilpotent or M is a valuation module.*

Proof. $N = IM$ is the unique maximal submodule of M. Let L be a proper submodule of M, so $L \subseteq N$. If for all $n \in \mathbb{N}$, $L \subseteq N^n = I^nM$, then $L \subseteq \bigcap_{n=1}^{\infty} I^nM = \bigcap_{n=1}^{\infty}(p^n)M = (0)$. Otherwise there exists $n \in \mathbb{N}$ such that $L \subseteq (p^n)M$, but $L \nsubseteq (p^{n+1})M$. Let $a \in L \backslash (p^{n+1})M$. Since $L \subseteq (p^n)M$, there exists $\alpha \in M$ such that $a = p^n\alpha$ and $\alpha \notin N$. But N is the unique maximal submodule of M. Hence α is a unit and $M = R\alpha$. So $(p^n)M \subseteq L$ and therefore $L = (p^n)M$.

Now assume that M is faithful and p is not nilpotent. Since M is a multiplication module and for all nonzero submodules L_1, L_2 of M, $L_1 = (p^t)M$ and $L_2 = (p^s)M$, for some $t, s \in \mathbb{N}$, we have $L_1 \subseteq L_2$ or $L_2 \subseteq L_1$. Hence by Corollary 2.8, it is enough to show that R is a domain. Let for $a, b \in R$, $ab = 0$. It follows that $aM = (0)$ or $aM = (p^n)M$, for $n \in \mathbb{N}$ and similarly $bM = (0)$ or $bM = (p^m)M$, for $m \in \mathbb{N}$. If $aM = (p^n)M$ and $bM = (p^m)M$ then $0 = (ab)M = (a)M(b)M = (p^n)M(p^m)M = (p^{n+m})M$. Since M is torsionfree, so $p^{n+m} = 0$, which is a contradiction. Hence R is a domain and therefore M is a VM. □

Theorem 2.18. *Any finitely generated valuation module over a domain R, is unique (up to isomorphism) and isomorphic to a finite direct sum of the integral closure of R in its field of fractions.*

Proof. Let M be a finitely generated valuation R-module. Consider the subring $S = \{y \in K : yM \subseteq M\}$ of K, the field of fractions of R. Then $R \subseteq S \subseteq K$ and M is a finitely generated S-module. It is easy to see that S is a valuation ring. By usual determinant argument, every element of S is integral over R. Thus $S \subseteq T$ where T denotes the integral closure of R in K. On the other hand, since S is a valuation ring, it is integrally closed, and so $S = T$. Moreover, since M is finitely generated and torsionfree over the valuation ring S, M is free as S-module with finite rank. This gives that M is unique (up to isomorphism) and isomorphic to a finite direct sum of the integral closure of R in its field of fractions. □

There are plenty of valuation modules which are not finitely generated. For example, every valuation ring between R and K is a valuation module over R (as the one given in Example 2.3. (iii)).

Let $\Theta(M) = \{y \in K : yM \subseteq M\}$. Then $\Theta(M)$ is a subring of K with $R \subseteq \Theta(M)$ and M is a $\Theta(M)$-module (see [15]). Let M be a valuation R-module and $y \in K$, so $yM \subseteq M$ or $y^{-1}M \subseteq M$. Therefore $\Theta(M)$ is a valuation ring. Now let S be an overring of R. If S is a valuation ring, then it is clear that S is a valuation R-module. Let M be a finitely generated R-module, so M is a finitely generated $\Theta(M)$-module. If M is a valuation R-module, then $\Theta(M)$ is a valuation ring and hence $\Theta(M)$ is integrally closed. Now suppose that $y \in K$ with $yM \subseteq M$, then by using the standard determinant argument, we obtain that y is integral over R. Therefore $\Theta(M) \subseteq \overline{R}$ and so $\Theta(M)$ is integrally closed. Hence by Lemma 2.12, we have the following theorem.

Theorem 2.19. *Let M be a finitely generated module over an integrally closed ring R. If M is a valuation module, then M is a free R-module and R is a valuation ring.*

3. Fractional Submodules and Discrete Valuation Modules

A *fractional ideal* of R is an R-submodule U of K such that $aU \subseteq R$, for some $a \in R$, $a \neq 0$ (see [5,12]). In this section we generalize this notion to a module and define discrete valuation modules. Furthermore, we prove some basic results and obtain relations between fractional submodules and discrete valuation modules.

Definition 3.1. Let R be an integral domain and M a torsionfree R-module. An R-submodule N of M_T is called a *fractional submodule* of M if there exists $r \in T = R - \{0\}$ such that $rN \subseteq M$.

Example 3.2. *i) Let $M = R$. Then any fractional ideal of R is a fractional submodule of M.*

ii) Let $\alpha \in M_T - \{0\}$. Then $N = R\alpha$ is a fractional submodule, called a cyclic fractional submodule.

iii) Any R-submodule of M is a fractional submodule, called an integral submodule.

iv) Let N be an R-submodule of M and $\alpha_1, \ldots, \alpha_n \in M_T - \{0\}$. Then $L = N + R\alpha_1 + \cdots + R\alpha_n$ is a fractional submodule of M.

v) Let N and L be two fractional submodules of M. Then $N + L$ and $N \cap L$ are also fractional submodules.

Lemma 3.3. *Let N be a finitely generated R-submodule of M_T. Then N is a fractional submodule and conversely, if M is a Noetherian R-module, then every fractional submodule of M is finitely generated.*

Proposition 3.4. *Let R be an integral domain and M a torsionfree R-module. For the following statements we have (i)$\Rightarrow$ (ii)$\Rightarrow$ (iii)$\Rightarrow$ (iv)$\Rightarrow$ (v) and if M is multiplication, then (v)$\Rightarrow$ (i).*
i) The set of cyclic fractional submodules of M is linearly ordered by inclusion.
ii) The set of fractional submodules of M is linearly ordered by inclusion.
iii) The set of cyclic integral submodules of M is linearly ordered by inclusion.
iv) The set of integral submodules of M is linearly ordered by inclusion.
v) M is a valuation R-module.

Proof. The proof that (i)$\Rightarrow$ (ii)$\Rightarrow$ (iii)$\Rightarrow$ (iv) is clear.

iv)$\Rightarrow$ v) Let $y = \frac{r}{t} \in K - \{0\}$, then rM, tM are integral submodules of M and so by (iv), $rM \subseteq tM$ or $tM \subseteq rM$. Therefore $yM \subseteq M$ or $y^{-1}M \subseteq M$.

v)$\Rightarrow$ i) Let $\alpha = \frac{x}{t}, \beta = \frac{y}{s} \in M_T - \{0\}$. Put $N = R\alpha$, $L = R\beta$. Then N, L are cyclic fractional submodules of M. Since Rsx, Rty are submodules of M, so by Corollary 2.8, $Rsx \subseteq Rty$ or $Rty \subseteq Rsx$. Therefore $N \subseteq L$ or $L \subseteq N$. □

Let N be a fractional submodule of M. Consider $N' = [M : N] = \{y \in K | yN \subseteq M\}$. It is clear that N' is an R-submodule of K and $N'N \subseteq M$. Similar to the definition in [13], a fractional R-submodule N of M is called *invertible*, if $N'N = M$. In particular, if $M = R$, then any invertible fractional ideal of R is an invertible fractional submodule.

By [5], a valuation ring is a *discrete valuation ring* if and only if it is Noetherian.

Definition 3.5. A Noetherian valuation module is called a *discrete valuation module* (DVM).

Proposition 3.6. *Let R be a local domain with unique principal maximal ideal $I = (p) \neq 0$, and M a faithful multiplication R-module such that $\bigcap_{n=1}^{\infty}(p^n)M = (0)$. Then M is a DVM.*

Proof. It is clear that R is a DVR, and hence by Corollary 2.13, M is a DVM. □

It is easy to see that if M is a DVM, then for each $p \in Spec(R)$, M_p is a DV R_p-module.

Proposition 3.7. *Let M be a multiplication valuation R-module. Then M is a DVM if and only if every prime submodule of M is cyclic.*

Theorem 3.8. *Let R be a domain, and $dimR = 1$. Let M be a Noetherian, faithful multiplication R-module and $L = JM$, for $J \in \max(R)$. Consider the following:*
i) M is a DVM.
ii) Every non-zero proper submodule of M is a power of L.
iii) Every primary submodule of M is a power of its radical.
Then (i) ⇔ (ii) and if R is local then (ii) ⇔ (iii).

Proof. i)⇒ ii) Let M be a DVM and N be a nonzero proper submodule of M. Since M is Noetherian and $dimR = 1$, N is a J-primary. Since M is a DVM, it is easy to see that R is a Noetherian local ring. So by the Nakayama Lemma, $J^2 \neq J$ and so $L^2 \neq L$. Now by Theorem 2.16(iii), there exists $n \in \mathbb{N}$ such that $N = L^n$.

ii)⇒ i) If every non-zero proper submodule of M is a power of L then, by Corollary 2.8, M is a DVM.

Now let R be local.

ii)⇒ iii) Let Q be p-primary. If $Q = 0$ then $radQ = Q = 0$. Now let $Q \neq 0$. Since R is local, so $J = p$ and there exists $n \in \mathbb{N}$ such that $Q = L^n = (JM)^n = (radQ)^n$.

iii)⇒ ii) Let N be a non-zero proper submodule of M. Since M is Noetherian, R is local and $dimR = 1$, so N is J-primary. Hence there exists $n \in \mathbb{N}$ such that $N = (radN)^n = (\sqrt{(N : M)}M)^n = (JM)^n = L^n$. □

Proposition 3.9. *Let R be a local domain with $dimR = 1$. Let M be a Noetherian, faithful multiplication R-module. If every non-zero fractional submodule of M is invertible, then M is a DVM.*

Proof. Let $L = JM$, for $J \in Max(R)$. By Theorem 3.8, it is enough to show that every non-zero proper submodule of M is a power of L. Let $S = \{0 \neq N < M \mid N \neq L^n, \text{ for all } n \in \mathbb{N}\}$. If $S \neq \emptyset$, as M is Noetherian, then S has a maximal element N. Hence $N \subset L$ and $L^{'}N \subseteq M$. If $L^{'}N = M$ then $N = L$, which is a contradiction. So $L^{'}N \subset M$. On the other hand, $N \subseteq L^{'}N$. If $N \subset L^{'}N$ then $L^{'}N \notin S$ and so $L^{'}N = L^t$, for some $t \in \mathbb{N}$. Therefore $N = L^{t+1}$, which is a contradiction. If $N = L^{'}N$ then $LN = N$ and $IJM = IM$, where I is an ideal of R such that $N = IM$. Now by the Nakayama Lemma $N = IM = 0$, which is again a contradiction. Therefore $S = \emptyset$. □

4. Dedekind Modules

Following [13], a non-zero R-module M is called a *Dedekind module* (DM), if each non-zero submodule of M is invertible. (For more information, see [2].)

By [2, Corollary 3.15], a multiplication R-module M is a Dedekind R-module if and only if M is Noetherian, integrally closed and every nonzero prime submodule of M is maximal. In what follows we give some characterizations for DM with fractional submodules and DVM.

Theorem 4.1. *Let R be a domain and M a torsionfree R-module. Then M is a DM if and only if every non-zero fractional submodule of M is invertible.*

Proof. Let M be a DM and N be a non-zero fractional submodule of M. There exists $r \in T$ such that $rN \subseteq M$. Since M is torsionfree, $rN \neq 0$ and so is invertible. Hence $L(rN) = M$, where $L = [M : rN]$. Therefore $(rL)N = M$ and it is easy to see that $rL = [M : N]$. The converse is clear by the definition of DM. □

Theorem 4.2. *Let R be a domain and M a Noetherian faithful multiplication R-module such that every non-zero prime submodule of M is maximal. Then the following conditions are equivalent:*
i) M is a DM.
ii) M_p is a DVM, for any $p \in Spec(R) - \{0\}$.
iii) Every primary submodule of M is a power of a prime submodule.

Proof. i)⇔ ii) By [3, Theorem 19] and Corollary 2.13, M is a DM if and only if R is a Dedekind domain if and only if R_p is a DVR for every $p \in Spec(R)$ if and only if M_p is a DVM for every $p \in Spec(R)$.

ii)⇒ iii) Let Q be p-primary submodule of M, where $Q = qM$, $\sqrt{q} = p$. If $Q = 0$, then $radQ = Q = 0$. Let $Q \neq 0$. So $p \neq 0$ and $Q_p = qM_p$ is a non-zero proper submodule of DVM, M_p. By Theorem 3.8, there exists $n \in \mathbb{N}$ such that $Q_p = p^n M_p$. Hence $q = p^n$ and therefore $Q = qM = (pM)^n$, where $pM \in Spec(M)$.

iii)⇒ ii) Let $p \in Spec(R) - \{0\}$. By Theorem 3.8, it is enough to show that every non-zero proper submodule of M_p is a power of $L = pM_p$. Let Q_p be a non-zero proper submodule of M_p. So $Q = qM$ is a non-zero proper submodule of M, where

$q = (Q : M)$. Since M_p is Noetherian and R_p is local with $dimR_p = 1$, so $Q_p = qM_p$ is pR_p-primary. Hence $(q_p \cap R)M$ is p-primary and there exist $n \in \mathbb{N}$ such that $(q_p \cap R)M = p^n M$. Therefore $Q_p = p^n M_p = L^n$. □

Theorem 4.3. *Let R be a local domain, with unique principal maximal ideal (p) and $dimR = 1$. Let M be a faithful multiplication R-module. Then M is a DM if and only if M is a DVM.*

Proof. Let M be a DVM. Since M is multiplication and $dimR = 1$, by [2, Corollary 3.15] M is a DM. Conversely, let M be a DM. It is enough to show that M is a VM. Since M is faithful multiplication, R is Noetherian and so $\bigcap_{n=1}^{\infty}(p^n) = (0)$. Therefore $\bigcap_{n=1}^{\infty}(p^n)M = (0)$ and by Theorem 2.17, M is a VM. Hence M is a DVM. □

Acknowledgment. The authors would like to thank the referee for his/her useful suggestions that improved the presentation of this paper.We would like to also thank Dr. S. Karimzadeh for her useful hints in the proofs of some of the propositions.

References

[1] Z. Abd El-bast and P.F. Smith, *Multiplication modules*, Comm. Algebra, 16(4) (1988), 755–779.

[2] M. Alkan, B. Sarac and Y. Tiras, *Dedekind modules*, Comm. Algebra, 33(5)(2005), 1617–1626.

[3] M. Alkan and Y. Tiras, *On Invertible and Dense Submodules*, Comm. Algebra, 32(10)(2004), 3911–3919.

[4] R. Ameri, *On The Prime Submodules of Multiplication Modules*, IJMMS, 27(2003), 1715–1724.

[5] M.F. Atiyah and I.G.MacDonald, Introduction to Commutative Algebra, Addison-Wesley,(1969).

[6] N. Bourbaki, Commutative Algebra, Addison-Wesley,(1972).

[7] S. Hedayat and R. Nekooei, *Characterization of prime submodules of a finitely generated free module over a PID*, Houston J. Math., 31(1)(2005), 75–85.

[8] S. Hedayat and R. Nekooei, *Prime and radical submodules of free modules over a PID*, Houston J. Math., 32(2)(2006), 355–367.

[9] S.Hedayat and R. Nekooei, *Primary Decomposition of submodules of a finitely generated module over a PID*, Houston J. Math., 32(2)(2006), 369–377.

[10] S. Karimzadeh and R. Nekooei, *On Dimension of Modules*, Turkish J. Math, 31(2007), 95–109.

[11] S. Karimzadeh and R. Nekooei, *Some remarks on Dedekind modules*, Accepted in Algebra Colloquium.

[12] M.D. Larsen and P.J. McCarthy, Multiplicative theory of ideals, Academic Press, London,(1971).

[13] A.G. Naoum and F.H. Al-Alwan, *Dedekind modules*, Comm. Algebra, 24(2)(1996), 397–412.

[14] R. Nekooei, *On finitely generated multiplication modules*, Czechoslovak Math. J., 55(130)(2005), 503–510.

[15] B.Sarac, P. F. Smith and Y. Tiras, *On Dedekind module*, Comm. Algebra, 35 (2007) 1532–1538.

[16] P.F. Smith, *Some remarks on multiplication modules*, Arch. Math. 50(1988), 223–235.

J. Moghaderi
Department of Mathematics
Hormozgan University
Iran
e-mail: j.moghaderi@yahoo.com

R. Nekooei
Department of Mathematics
Shahid Bahonar University of Kerman
Iran
e-mail: rnekooei@mail.uk.ac.ir