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ON (σ,τ)-HIGHER DERIVATIONS IN PRIME RINGS

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Abstract. Let R be a prime ring with $char(R) \neq 2$ and σ, τ be commuting endomorphisms of R. In the present paper we show that under certain conditions on R every Jordan (σ,τ)-higher derivation on R is a (σ,τ)-higher derivation on R.

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1. Introduction

Let R be a ring with center $Z(R)$, and σ, τ be endomorphisms of R. Endomorphisms σ, τ are said to be commuting endomorphisms if $\sigma\tau = \tau\sigma$. The set of natural numbers including 0 will be denoted by $\mathbb{N}$ and $[\cdot,\cdot]$ denotes the usual commutator operator. An additive mapping d: $R \to R$ is said to be a *derivation* (resp. *Jordan derivation*) on R if $d(ab) = d(a)b + ad(b)$ (resp. $d(a^2) = d(a)a + ad(a)$) holds for all $a, b \in R$. An additive mapping d: $R \to R$ is called a (σ,τ)-*derivation* (resp. *Jordan* (σ,τ)-*derivation*) on R if $d(ab) = d(a)\tau(b)+\sigma(a)d(b)$ (resp. $d(a^2) = d(a)\tau(a) + \sigma(a)d(a)$) holds for all $a, b \in R$. Of course a $(1,1)$-derivation (resp. Jordan $(1,1)$-derivation) is a derivation (resp. Jordan derivation) on R, where 1 is the identity map on R. For an example of a (σ,τ)-derivation which is not a derivation let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$. Define σ,τ: $R \to R$ such that $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and $\tau\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$, then clearly σ,τ are endomorphisms of R. Now define a map d: $R \to R$ such that $d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Then it can be seen that d is a (σ,τ)-derivation on

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R which is not a derivation on R.

Obviously, every derivation is a Jordan derivation on R but the converse need not be true in general. However, in 1957, I.N. Herstein [11] proved that on a prime ring with $char(R) \neq 2$ every Jordan derivation is a derivation. Later on, this result was extended by several authors (see [2] and [3] where further references can be found). M. Brešar and J. Vukman [4] extended this result for (σ, τ)-derivations.

The concept of derivation was extended to higher derivation by F. Hasse and F.K. Schmidt [10] (see [1] and [9] for an historical account and applications). Let $D = \{d_n\}_{n \in \mathbb{N}}$ be a family of additive mappings d_n: $R \to R$. Following Hasse and Schmidt [10], D is said to be a *higher derivation* (resp. *Jordan higher derivation*) on R if $d_0 = I_R$ (the identity map on R) and $d_n(ab) = \sum\limits_{i+j=n} d_i(a)d_j(b)$ (resp. $d_n(a^2) = \sum\limits_{i+j=n} d_i(a)d_j(a)$) for all $a, b \in R$ and for each $n \in \mathbb{N}$.

In an attempt to generalize Herstein's result for higher derivations, C. Haetinger [8] proved that on a prime ring with $char(R) \neq 2$ every Jordan higher derivation is a higher derivation (see [6] and [7] for English versions). Now, the main purpose of this paper is to extend this result for (σ, τ)-higher derivations in rings.

2. Preliminaries and Main Results

Motivated by the existence of (σ, τ)-derivations in rings we shall introduce the notion of (σ, τ)-higher derivation in rings as follows. Let R be a ring and $D = \{f_n\}_{n \in \mathbb{N}}$ be a family of maps f_n: $R \to R$. Then D is said to be a *(σ, τ)-higher derivation* (resp. *Jordan (σ, τ)-higher derivation*) where σ, τ are endomorphisms on R if:

(i) $f_0 = I_R$;

(ii) $f_n(a + b) = f_n(a) + f_n(b)$;

(iii) $f_n(ab) = \sum\limits_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b))$

(resp. $f_n(a^2) = \sum\limits_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(a))$, for all $a, b \in R$ and for each $n \in \mathbb{N}$.

We pause to look at an example of a (σ, τ)-higher derivation on R.

Example 2.1. Let R be an algebra over field of rationals $\mathbb{Q}$ and σ, τ be endomorphisms of R. Define $d_n = \frac{\delta^n}{n!}$, for all $n \in \mathbb{N}$, where δ is a (σ, τ)-derivation on R such that $\delta\sigma = \sigma\delta$ and $\delta\tau = \tau\delta$. Consider the sequence $D = \{d_n\}_{n \in \mathbb{N}}$; we shall show that D is (σ, τ)-higher derivation. We shall use induction on n to prove the claim:

- For $n=0$, $d_0(ab)=\frac{\delta^0(ab)}{0!}=ab$.
- For $n=1$,
  $d_1(ab)=\frac{\delta^1(ab)}{1!}=\delta(ab)=\sigma(a)\delta(b)+\delta(a)\tau(b)=\sigma(a)d_1(b)+d_1(a)\tau(b)$.
- For $n=2$,

$$\begin{aligned}
d_2(ab) &= \tfrac{\delta^2(ab)}{2!}=\tfrac{\delta}{2}(\delta(ab))=\tfrac{\delta}{2}(\sigma(a)\delta(b)+\delta(a)\tau(b))=\\
&= \tfrac{1}{2}(\sigma^2(a)\delta^2(b)+\sigma(\delta(a))\delta(\tau(b))+\delta(\sigma(a))\tau(\delta(b))+\delta^2(a)\tau^2(b))=\\
&= \sigma^2(a)\tfrac{\delta^2(b))}{2!}+\delta(\sigma(a))\delta(\tau(b))+\tfrac{\delta^2(a)}{2!}\tau^2(b)=\\
&= \sigma^2(a)d_2(b)+d_1(\sigma(a))d_1(\tau(b))+d_2(a)\tau^2(b).
\end{aligned}$$

- Now suppose that $d_n=\frac{\delta^n}{n!}$ defines a (σ,τ)-higher derivation on R for each $m<n$.

Consider $d_n(ab)=\frac{\delta^n(ab)}{n!}=\frac{1}{n}\delta\left(\frac{\delta^{n-1}(ab)}{(n-1)!}\right)=\frac{1}{n}\delta(d_{n-1}(ab))$. Applying the hypothesis of induction on d_{n-1}, we have

$$\begin{aligned}
d_n(ab) &= \tfrac{\delta}{n}\sum_{j=0}^{n-1}d_j(\sigma^{n-1-j}(a))d_{n-1-j}\tau^i(b)=\tfrac{\delta}{n}\sum_{j=0}^{n-1}\tfrac{\delta^j}{j!}(\sigma^{n-1-j}(a))\tfrac{\delta^{n-1-j}}{(n-1-j)!}(\tau^j(b))=\\
&= \tfrac{1}{n}\sum_{j=0}^{n-1}\left\{\tfrac{\sigma(\delta^j(\sigma^{n-1-j}(a)))}{j!}\tfrac{\delta^{n-j}(\tau^j(b))}{(n-1-j)!}+\tfrac{\delta^{j+1}(\sigma^{n-1-j}(a))}{j!}\tfrac{\tau(\delta^{n-1-j}(\tau^j(b)))}{(n-1-j)!}\right\}=\\
&= \tfrac{1}{n}\sum_{j=0}^{n-1}\left\{d_j(\sigma^{n-j}(a))d_{n-j}(\tau^j(b))(n-j)+d_{j+1}(\sigma^{n-1-j}(a))d_{n-1-j}(\tau^{j+1}(b))(j+1)\right\}=\\
&= \sum_{j=0}^{n-1}d_j(\sigma^{n-j}(a))d_{n-j}(\tau^j(b))-\tfrac{1}{n}\sum_{j=0}^{n-2}d_j(\sigma^{n-j}(a))d_{n-j}(\tau^j(b))j-\\
&\quad -\tfrac{1}{n}d_{n-1}(\sigma(a))d_1(\tau^{n-1}(b))(n-1)+\tfrac{1}{n}\sum_{l=2}^{n}d_l(\sigma^{n-l}(a))d_{n-l}(\tau^l(b))(l-1)+\\
&\quad +\tfrac{1}{n}\sum_{l=1}^{n}d_l(\sigma^{n-l}(a))d_{n-l}(\tau^l(b)).
\end{aligned}$$

Simplifying further this equality we get,

$$\begin{aligned}
d_n(ab) &= \sum_{j=0}^{n-1}d_j(\sigma^{n-j}(a))d_{n-j}(\tau^j(b))-\tfrac{1}{n}\sum_{j=2}^{n-2}d_j(\sigma^{n-j}(a))d_{n-j}(\tau^j(b))j-\\
&\quad -\tfrac{1}{n}d_1(\sigma^{n-1}(a))d_{n-1}(\tau(b))-d_{n-1}(\sigma(a))d_1(\tau^{n-1}(b)+\tfrac{1}{n}d_{n-1}(\sigma(a))d_1(\tau^{n-1}(b))+\\
&\quad +\tfrac{1}{n}\sum_{l=2}^{n-2}d_l(\sigma^{n-l}(a))d_{n-l}(\tau^l(b))l+d_n(a)\tau^n(b)+d_{n-1}(\sigma(a))d_1(\tau^{n-1}(b))-\\
&\quad -\tfrac{1}{n}d_{n-1}(\sigma(a))d_1(\tau^{n-1}(b))-\tfrac{1}{n}\sum_{l=2}^{n}d_l(\sigma^{n-l}(a))d_{n-l}(\tau^l(b))+\\
&\quad +\tfrac{1}{n}\sum_{l=2}^{n}d_l(\sigma^{n-l}(a))d_{n-l}(\tau^l(b))+\tfrac{1}{n}d_1(\sigma^{n-1}(a))d_{n-1}(\tau(b))=\\
&= \sum_{j=0}^{n}d_j(\sigma^{n-j}(a))d_{n-j}(\tau^j(b)).
\end{aligned}$$

Thus, the family $D=\{d_n\}_{n\in\mathbb{N}}$, where $d_n=\frac{\delta^n}{n!}$, defines a (σ,τ)-higher derivation on R.

The above definitions suggest that every (σ,τ)-higher derivation on R is a Jordan (σ,τ)-higher derivation on R but the converse need not be true in general. It is also worth mentioning that in the above example if δ is assumed to be a Jordan (σ,τ)-derivation on R which is not a (σ,τ)-derivation on R, then it is equaly easy to find a Jordan (σ,τ)-higher derivation on R which is not a (σ,τ)-higher derivation on R.

In the present paper we explore the converse part of this problem and find the condition on R under which a Jordan (σ,τ)-higher derivation on R becomes (σ,τ)-higher derivation on R. In fact, the main results of the present paper are as follows:

Theorem 2.2. *Let R be a 2-torsion-free ring and σ, τ be commuting endomorphisms of R such that τ is one-one and onto. If R has a commutator which is not a right zero divisor, then every Jordan (σ,τ)-higher derivation on R is a (σ,τ)-higher derivation on R.*

Theorem 2.3. *Let R be a non-commutative prime ring with $char(R)\neq 2$ and σ,τ be commuting endomorphisms of R such that τ is one-one and onto. Then, every Jordan (σ,τ)-higher derivation on R is a (σ,τ)-higher derivations on R.*

Note that Theorem 2.2 above seems similar to [5, Theorem 1.3] for Jordan generalized higher derivations and Lie ideals.

For every fixed $n\in I\!\!N$ and for each $a,b\in R$ we denote by $\Phi_n(a,b)$ the element of R defined by

$$\Phi_n(a,b)=f_n(ab)-\sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b)).$$

It is straightforward to see that if $\Phi_n(a,b)=0$ then $D=\{f_n\}_{n\in I\!\!N}$ is a (σ,τ)-higher derivation on R.

In order to develop the proofs of the above theorems, we begin with the following known lemmas:

Lemma 2.4. ([12, Lemma 3.10]) *Let R be a prime ring with $char(R)\neq 2$ and suppose that $a,b\in R$ such that $arb+bra=0$ for all $r\in R$. Then either $a=0$ or $b=0$.*

Lemma 2.5. ([4, Lemma 4]) *Let G and H be the additive groups and let R be a 2-torsion-free ring. Let $f\colon G\times G\to R$ and $g\colon G\times G\to R$ be biadditive maps. Suppose that for each pair $a,b\in G$ either $f(a,b)=0$ or $g(a,b)^2=0$ then in this case either $f(a,b)=0$ or $g(a,b)^2=0$ for all $a,b\in G$.*

Now we prove the following:

Lemma 2.6. *Let R be ring and $D=\{f_n\}_{n\in I\!N}$ be a Jordan (σ,τ)-higher derivation, where σ,τ are commuting endomorphisms on R. Then for all $a,b,c\in R$ and each fixed $n\in I\!N$ we have;*

(i) $f_n(ab+ba)=\sum\limits_{i+j=n}(f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b))+f_i(\sigma^{n-i}(b))f_j(\tau^{n-j}(a)))$.

If R is a 2-torsion-free ring then,

(ii) $f_n(aba)=\sum\limits_{i+j+k=n}f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a))$;

(iii) $f_n(abc+cba)=\sum\limits_{i+j+k=n}(f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(c))+f_i(\sigma^{n-i}(c))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a))$,

for all $a,b,c\in R$.

Proof. (i) For $a,b\in R$, $n\in I\!N$ we have, $f_n(a^2)=\sum\limits_{i+j=n}f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(a))$.

So by linearizing the above relation on a we obtain:

$$\begin{aligned}
f_n((a+b)^2) &= \sum_{i+j=n}f_i(\sigma^{n-i}(a+b))f_j(\tau^{n-j}(a+b))=\\
&= \sum_{i+j=n}f_i(\sigma^{n-i}(a)+\sigma^{n-i}(b))f_j(\tau^{n-j}(a)+\tau^{n-j}(b))=\\
&= \sum_{i+j=n}f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(a))+\sum_{i+j=n}f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b))+\\
&\quad+\sum_{i+j=n}f_i(\sigma^{n-i}(b))f_j(\tau^{n-j}(a))+\sum_{i+j=n}f_i(\sigma^{n-i}(b))f_j(\tau^{n-j}(b)),
\end{aligned}$$

for all $a,b\in R$.

Again;

$$\begin{aligned}
f_n((a+b)^2) &= f_n(a^2+ab+ba+b^2)=f_n(a^2)+f_n(ab+ba)+f_n(b^2)=\\
&= f_n(ab+ba)+\sum_{i+j=n}f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(a))+\\
&\quad+\sum_{i+j=n}f_i(\sigma^{n-i}(b))f_j(\tau^{n-j}(b)),
\end{aligned}$$

for all $a,b\in R$.

Comparing the two expressions and reordering the indices we obtain the required result.

(ii) Using (i) and replacing b by $ab+ba$ we see that, for $\omega = a(ab+ba)+(ab+ba)a$,

$$\begin{aligned}
f_n(\omega) &= f_n(a(ab+ba)+(ab+ba)a) = \\
&= \sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(ab+ba)) \sum_{i+j=n} f_i(\sigma^{n-i}(ab+ba))f_j(\tau^{n-j}(a)) = \\
&= \sum_{i+j=n} f_i(\sigma^{n-i}(a)) \left(\sum_{r+s=j} f_r(\sigma^{j-r}\tau^{n-j}(a))f_s(\tau^{j-s}\tau^{n-j}(b)) + \right. \\
&\quad \left. + \sum_{r+s=j} f_r(\sigma^{j-r}\tau^{n-j}(b))f_s(\tau^{j-s}\tau^{n-j}(a)) \right) + \\
&\quad + \sum_{i+j=n} \left(\sum_{k+l=i} f_k(\sigma^{i-k}\sigma^{n-i}(a))f_l(\tau^{i-l}\sigma^{n-i}(b)) + \right. \\
&\quad \left. + \sum_{k+l=i} f_k(\sigma^{i-k}\sigma^{n-i}(b))f_l(\tau^{i-l}\sigma^{n-i}(a)) \right) f_j(\tau^{n-j}(a)) = \\
&= \sum_{i+j=n} f_i(\sigma^{n-i}(a)) \sum_{r+s=j} f_r(\sigma^{j-r}\tau^{n-j}(a))f_s(\tau^{j-s}\tau^{n-j}(b)) + \\
&\quad + \sum_{i+j=n} f_i(\sigma^{n-i}(a)) \sum_{r+s=j} f_r(\sigma^{j-r}\tau^{n-j}(b))f_s(\tau^{j-s}\tau^{n-j}(a)) + \\
&\quad + \sum_{i+j=n} \sum_{k+l=i} f_k(\sigma^{i-k}\sigma^{n-i}(a))f_l(\tau^{i-l}\sigma^{n-i}(b))f_j(\tau^{n-j}(a)) + \\
&\quad + \sum_{i+j=n} \sum_{k+l=i} f_k(\sigma^{i-k}\sigma^{n-i}(b))f_l(\tau^{i-l}\sigma^{n-i}(a))f_j(\tau^{n-j}(a)).
\end{aligned}$$

Using,

$$\begin{aligned}
& \sum_{i+j=n} f_i(\sigma^{n-i}(a)) \sum_{r+s=j} f_r(\sigma^{j-r}\tau^{n-j}(b))f_s(\tau^{j-s}\tau^{n-j}(a)) + \\
& + \sum_{i+j=n} \sum_{k+l=i} f_k(\sigma^{i-k}\sigma^{n-i}(a))f_l(\tau^{i-l}\sigma^{n-i}(b))f_j(\tau^{n-j}(a)) = \\
= \; & 2 \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a)),
\end{aligned}$$

we obtain,

$$\begin{aligned}
f_n(\omega) &= f_n(a(ab+ba)+(ab+ba)a) = \\
&= \sum_{i+j=n} \sum_{r+s=j} f_i(\sigma^{n-i}(a))f_r(\sigma^{j-r}\tau^{n-j}(a))f_s(\tau^{n-s}(b)) + \\
&\quad +2 \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a)) + \\
&\quad + \sum_{i+j=n} \sum_{k+l=i} f_k(\sigma^{n-k}(b))f_l(\tau^{i-l}\sigma^{n-i}(a))f_j(\tau^{n-j}(a)).
\end{aligned} \tag{1}$$

On the other hand,

$$f_n(a(ab+ba)+(ab+ba)a) = f_n((a^2b+ba^2)+2aba) = f_n(a^2b+ba^2)+2f_n(aba).$$

Now, from (i) and using the fact that $D = \{f_n\}_{n\in\mathbb{N}}$ is a Jordan (σ,τ)-higher derivation,

$$\begin{aligned}
f_n(\omega) &= f_n(a(ab+ba)+(ab+ba)a) = f_n(a^2b+ba^2)+f_n(2aba) = \\
&= \sum_{i+j=n} f_i(\sigma^{n-i}(a^2))f_j(\tau^{n-j}(b)) + \sum_{i+j=n} f_i(\sigma^{n-i}(b))f_j(\tau^{n-j}(a^2)) + 2f_n(aba) = \\
&= 2f_n(aba) + \sum_{i+j=n}\sum_{r+s=i} f_r(\sigma^{i-r}\sigma^{n-i}(a))f_s(\tau^{i-s}\sigma^{n-i}(a))f_j(\tau^{n-j}(b)) + \\
&\quad + \sum_{i+j=n} f_i(\sigma^{n-i}(b)) \sum_{k+l=j} f_k(\sigma^{j-k}\tau^{n-j}(a))f_l(\tau^{n-l}(a)) = \\
&= \sum_{r+s+j=n} f_r(\sigma^{n-r}(a))f_s(\tau^r\sigma^j(a))f_j(\tau^{n-j}(b)) \sum_{i+k+l=n} f_i(\sigma^{n-i}(b))f_k(\sigma^l\tau^{k+l}(a))f_l(\tau^{n-l}(a)) + \\
&\quad +2f_n(aba).
\end{aligned} \tag{2}$$

Comparing the above two equations (1) and (2) and reordering the indices and using the fact that $char(R) \neq 2$ we get the required result.

(iii) Linearizing the above result, we have

$$\begin{aligned}
\gamma &= f_n((a+c)b(a+c)) = \\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(a+c))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a+c)) = \\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}a)) \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(c)) + \\
&\quad + \sum_{i+j+k=n} f_i(\sigma^{n-i}(c))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a)) \sum_{i+j+k=n} f_i(\sigma^{n-i}(c))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(c)) = \\
&= f_n(aba) + \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(c)) + \\
&\quad + \sum_{i+j+k=n} f_i(\sigma^{n-i}(c))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a)) + f_n(cbc).
\end{aligned} \tag{3}$$

Again,

$$\gamma = f_n(a+c)b(a+c) = f_n(aba) + f_n(abc+cba) + f_n(cbc). \tag{4}$$

Comparing (3) and (4) and using the fact that $char(R) \neq 2$ we get the required result. □

Lemma 2.7. *Let R be a 2-torsion-free ring, and σ, τ be commuting endomorphisms of R. Let $D = \{f_n\}_{n\in\mathbb{N}}$ be a Jordan (σ,τ)-higher derivation of R. If $\Phi_m(a,b) = 0$, for each $m < n$ and for all $a, b \in R$, then*

(i) $\Phi_n(a,b)\tau^n[a,b] = 0$, *for all* $a, b \in R$;

(ii) $\Phi_n(a,b)\tau^n(r)\tau^n[b,a] + \sigma^n[b,a]\sigma^n(r)\phi_n(a,b) = 0$, *for all* $r, a, b \in R$.

Proof. (i) Take $\xi = (ab(ab)+(ab)ba)$. Then, $f_n(\xi) = f_n(ab(ab)+(ab)ba)$. Using Lemma 2.6 (iii) we have,

$$
\begin{aligned}
f_n(\xi) &= \sum_{i+j+k=n} (f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(ab)) + f_i(\sigma^{n-i}(ab))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(a)) = \\
&= \sum_{i+j=n,k=0} f_i(\sigma^{n-i}(a)f_j(\tau^{n-j}(b))\tau^n(ab) + \sum_{i+j=0,k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^n\tau^i(b))f_n(ab) + \\
&\quad + \sum_{\substack{0<i+j,k\le n-1 \\ i+j+k=n}} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b))f_k(\tau^{n-k}(ab)) = \\
&\quad + \sum_{j+k=n,i=0} \sigma^n(ab)f_j(\sigma^{n-j}(b))f_k(\tau^{n-k}(a) + \sum_{j+k=0,i=n} f_n(ab)f_j(\sigma^k\tau^n(b))f_k(\tau^{n-k}(a)) + \\
&\quad + \sum_{\substack{0<i,j+k\le n-1 \\ i+j+k=n}} f_i(\sigma^{n-i}(ab))f_j(\sigma^k\tau^i(b)f_k(\tau^{n-k}(a)) = \\
&= \sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b))\tau^n(ab) + \sigma^n(ab)f_n(ab) + \\
&\quad + \sum_{\substack{0<i+j,u+r\le n-1 \\ i+j+u+r=n}} f_i(\sigma^{n-i}(a))f_j(\sigma^{u+r}\tau^i(b))f_u(\sigma^r\tau^{i+j}(a))f_r(\tau^{n-r}(b)) + \\
&\quad + \sigma^n(ab)\sum_{j+k=n} f_j(\sigma^{n-j}(b))f_k(\tau^{n-k}(a) + f_n(ab)\tau^n(ba) + \\
&\quad + \sum_{\substack{0<l+t,j+k\le n-1 \\ l+t+j+k=n}} f_l(\sigma^{n-l}(a))f_t(\tau^l\sigma^{j+k}(b))f_j(\sigma^k\tau^{l+t}(b))f_k(\tau^{n-k}(a)).
\end{aligned} \tag{5}
$$

On the other hand,

$$
\begin{aligned}
f_n(\xi) &= f_n((ab)^2 + (ab^2a)) = \\
&= \sum_{i+j=n} f_i(\sigma^{n-i}(ab))f_j(\tau^{n-j}(ab) + \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(b^2))f_k(\tau^{n-k}(a)) = \\
&= f_n(ab)(\tau^n(ab) + \sigma^n(ab)f_n(ab) + \sum_{\substack{0<i,j\le n-1 \\ i+j=n}} f_i(\sigma^{n-i}(ab))f_j(\tau^{n-j}(ab)) + \\
&\quad + \sum_{i+p+q+k=n} f_i(\sigma^{n-i}(a))f_p(\sigma^{q+k}\tau^i(b))f_q(\tau^{i+p}\sigma^k(b))f_k(\tau^{n-k}(a)).
\end{aligned}
$$

Using, $\Phi_m(a,b) = 0$, for all $m < n$:

$$
\begin{aligned}
f_n(\xi) &= f_n(ab)\tau^n(ab) + \sigma^n(ab)f_n(ab) + \\
&\quad + \sum_{\substack{0<u+r,l+t\le n-1 \\ u+r+l+t=n}} f_u(\sigma^{n-u}(a))f_r(\tau^u\sigma^{l+t}(b))f_l(\sigma^t\tau^{u+r}(a))f_t(\tau^{n-t}(b)) + \\
&\quad + \sum_{i+p=n} f_i(\sigma^{n-i}(a))f_p(\tau^{n-p}(b))\tau^n(ba) + \sigma^n(ab)\sum_{q+k=n} f_q(\sigma^{n-q}(b))f_k(\sigma^{n-k}(a)) + \\
&\quad + \sum_{\substack{0<i+p,q+k\le n-1 \\ i+p+q+k=n}} f_i(\sigma^{n-i}(a))f_p(\sigma^{q+k}\tau^i(b))f_q(\tau^{i+p}\sigma^k(b))f_k(\tau^{n-k}(a)).
\end{aligned} \tag{6}
$$

Comparing the two equations (5) and (6) we get $\Phi_n(a,b)\tau^n[a,b] = 0$, for all $a, b \in R$.

(ii) Suppose, $\chi = abrba + barab$, where $a, b, r \in R$. Then by the Lemma 2.6 (ii), we obtain:

$$\begin{aligned}
f_n(\chi) &= f_n(a(brb)a) + f_n(b(ara)b) = \\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))f_j(\sigma^k\tau^i(brb))f_k(\tau^{n-k}(a)) + f_i(\sigma^{n-i}(b))f_j(\sigma^k\tau^i(ara))f_k(\tau^{n-k}(b)) = \\
&= \sum_{i+j+k=n} f_i(\sigma^{n-i}(a))(\sum_{l+t+u=j} f_l(\sigma^{j-l}\sigma^k\tau^i(b))f_t(\sigma^u\tau^l\sigma^k\tau^i(r))f_u(\tau^{j-u}\sigma^k\tau^i(b))f_k(\tau^{n-k}(a)) + \\
&\quad + \sum_{i+j+k=n} f_i(\sigma^{n-i}(b))(\sum_{l+t+u=j} f_l(\sigma^{j-l}\sigma^k\tau^i(a))f_t(\sigma^u\tau^l\sigma^k\tau^i(r))f_u(\tau^{j-u}\sigma^k\tau^i(a))f_k(\tau^{n-k}(b)) = \\
&= \sum_{i+l+t+u+k=n} f_i(\sigma^{n-i}(a))f_l(\sigma^{t+u+k}\tau^i(b))f_t(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+t}\sigma^k(b)f_k(\tau^{n-k}(a)) + \\
&\quad + \sum_{i+l+t+u+k=n} f_i(\sigma^{n-i}(b))f_l(\sigma^{t+u+k}\tau^i(a))f_t(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+t}\sigma^k(a))f_k(\tau^{n-k}(b)).
\end{aligned} \tag{7}$$

Again consider $f_n(\chi) = f_n((ab)r(ba) + (ba)r(ab))$. Applying Lemma 2.6 (iii);

$$f_n(\chi) = \sum_{p+q+s=n} (f_i(\sigma^{n-p}(ab))f_q(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ba)) + f_p(\sigma^{n-p}(ba))f_q(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ab)). \tag{8}$$

Equating (7) and (8) we find that;

$$\begin{aligned}
0 &= \sum_{p+q+s=n} f_p(\sigma^{n-p}(ab))f_q(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ba)) - \\
&\quad - \sum_{i+l+t+u+k=n} f_i(\sigma^{n-i}(a))f_l(\sigma^{t+u+k}\tau^i(b))f_t(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+t}\sigma^k(b)f_k(\tau^{n-k}(a)) + \\
&\quad + \sum_{p+q+s=n} f_p(\sigma^{n-p}(ba))f_q(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ab)) - \\
&\quad - \sum_{i+l+t+u+k=n} f_i(\sigma^{n-i}(b))f_l(\sigma^{t+u+k}\tau^i(a))f_t(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+t}\sigma^k(a))f_k(\tau^{n-k}(b)).
\end{aligned} \tag{9}$$

Initially calculating the first term of the right hand side of (9);

$$\begin{aligned}
&\sum_{p+q+s=n} f_p(\sigma^{n-p}(ab))f_q(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ba)) = \\
= &\sum_{p+s=n} f_p(\sigma^{n-p}(ab))\sigma^s\tau^p(r)f_s(\tau^{n-s}(ba)) + \sum_{p+s=n-1} f_p(\sigma^{n-p}(ab))f_1(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ba)) + \\
&+ \cdots + \sum_{p+s=1} f_p(\sigma^{n-p}(ab))f_{n-1}(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ba)) + \\
&+ \sum_{p+s=0} f_p(\sigma^{n-p}(ab))f_n(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ba)) = \\
= &f_n(ab)\tau^n(r)\tau^n(ba) + \sigma^n(ab)\sigma^n(r)f_n(ba) + \\
&+ \sum_{p+s=n} f_p(\sigma^{n-p}(ab))\sigma^s\tau^p(r)f_s(\tau^{n-s}(ba)) + \sum_{p+s=n-1}^{p,s\leq n-1} f_p(\sigma^{n-p}(ab))f_1(\sigma^s\tau^p(r))f_s(\tau^{n-s}(ba)) + \\
&+ \cdots + f_1(\sigma^{n-1}(ab))f_{n-1}(\tau(r))\tau^n(ba) + \sigma^n(ab)f_{n-1}(\sigma(r))f_1(\tau^{n-1}(ba)) + \sigma^n(ab)f_n(r)\tau^n(ba).
\end{aligned}$$

Using the hypothesis that $\Phi_m(a,b)=0$, for all $m<n$.

$$
\begin{aligned}
= \quad & f_n(ab)\tau^n(rba)+\sigma^n(abr)f_n(ba)+\\
& +\sum_{p+s=n}^{p,s\leq n-1}\sum_{i+j=p} f_i(\sigma^{n-i}(a))f_j(\tau^{p-j}\sigma^{n-p}(b))\sigma^s\tau^p(r)\sum_{u+k=s} f_u(\sigma^{s-u}\tau^{n-s}(b))f_k(\tau^{n-k}(a))+\\
& +\sum_{p+s=n-1}\sum_{i+j=p} f_i(\sigma^{n-i}(a))f_j(\tau^{p-j}\sigma^{n-p}(b))f_1(\sigma^s\tau^p(r))\sum_{u+k=s} f_u(\sigma^{s-u}\tau^{n-s}(b))f_k(\tau^{s-k}\tau^{n-s}(a))+\\
& +\cdots+\sum_{i+j=1} f_i(\sigma^{n-i}(a))f_j(\tau^{1-j}\sigma^{n-1}(b))f_{n-1}(\tau(r))\tau^n(ba)+\\
& +\sigma^n(ab)f_{n-1}(\sigma(r))(\sum_{u+k=1} f_u(\sigma^{1-u}\tau^{n-1}(b))f_k(\tau^{n-k}(a)))+\sigma^n(ab)f_n(r)\tau^n(ba)=\\
= \quad & f_n(ab)\tau^n(rba)+\sigma^n(abr)f_n(ba)+\\
& +\sum_{i+j+u+k=n}^{i+j,u+k\leq n-1} f_i(\sigma^{n-i}(a))f_j(\tau^i\sigma^{u+k}(b))\sigma^{u+k}\tau^{i+j}(r)f_u(\sigma^k\tau^{i+j}(b))f_k(\tau^{n-k}(a))+\\
& +\sum_{i+j+u+k=n-1} f_i(\sigma^{n-i}(a))f_j(\tau^i\sigma^{u+k+1}(b))f_1(\sigma^{u+k}\tau^{i+j}(r))f_u(\sigma^k\tau^{i+j+1}(b))f_k(\tau^{n-k}(a))+\\
& +\cdots+f_1(\sigma^{n-1}(a))\tau\sigma^{n-1}(b)f_{n-1}(\tau(r))\tau^n(ba)+\sigma^n(a)f_1(\sigma^{n-1}(b))f_{n-1}(\tau(r))\tau^n(ba)+\\
& +\sigma^n(ab)f_{n-1}(\sigma(r))f_1(\tau^{n-1}(b))\tau^n(a)+\sigma^n(ab)f_{n-1}(\sigma(r))\sigma\tau^{n-1}(b))f_1(\tau^{n-1}(a))+\\
& +\sigma^n(ab)f_n(r)\tau^n(ba).
\end{aligned}
$$

Similiarly the second term of the right hand side of (9) reduces to,

$$
\begin{aligned}
& \sum_{i+l+t+u+k=n} f_i(\sigma^{n-i}(a))f_l(\sigma^{t+u+k}\tau^i(b))f_t(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+t}\sigma^k(b)f_k(\tau^{n-k}(a))=\\
= \quad & \sum_{i+l+u+k=n} f_i(\sigma^{n-i}(a))f_l(\sigma^{u+k}\tau^i(b))\sigma^{u+k}\tau^{i+l}(r)f_u(\tau^{i+l}\sigma^k(b))f_k(\sigma^{n-k}(a))+\\
& +\sum_{i+l+u+k=n-1} f_i(\sigma^{n-i}(a))f_l(\sigma^{1+u+k}\tau^i(b))f_1(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+1}\sigma^k(b))f_k(\sigma^{n-k}(a))+\\
& +\cdots+\sum_{i+l+u+k=1} f_i(\sigma^{n-i}(a))f_l(\sigma^{n-1+u+k}\tau^i(b))f_{n-1}(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+n-1}(b))f_k(\sigma^{n-k}(a))+\\
& +\sum_{i+l+u+k=0} f_i(\sigma^{n-i}(a))f_l(\sigma^{n+u+k}\tau^i(b))f_n(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{i+l+n}(b))f_k(\sigma^{n-k}(a))=\\
= \quad & \sum_{u+k=n}\sigma^n(ab)\sigma^n(r)f_u(\sigma^{n-u}(b))f_k(\tau^{n-k}(a))+\sum_{i+l=n} f_i(\sigma^{n-i}(a))f_l(\tau^{n-l}(b))\tau^n(r))\tau^n(ba)+\\
& +\sum_{i+l+u+k=n}^{i+l,u+k\leq n-1} f_i(\sigma^{n-i}(a))f_l(\sigma^{u+k}\tau^i(b))\sigma^{u+k}\tau^{i+l}(r)f_u(\tau^{i+l}\sigma^k(b))f_k(\sigma^{n-k}(a))+\\
& +\sum_{i+l+u+k=n-1} f_i(\sigma^{n-i}(a))f_l(\sigma^{1+u+k}\tau^i(b))f_1(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{1+i+l}\sigma^k(b))f_k(\sigma^{n-k}(a))+\\
& +\cdots+f_1(\sigma^{n-1}(a))\sigma^{n-1}\tau(b)f_{n-1}(\tau(r))\tau^n(ba)+\sigma^n(a)f_1(\sigma^{n-1}(b))f_{n-1}(\tau(r))\tau^n(ba)+\\
& +\sigma^n(ab)f_{n-1}(\sigma(r))\tau^{n-1}\sigma(b)f_1(\tau^{n-1}(a))+\sigma^n(ab)f_{n-1}(\sigma(r))f_1(\tau^{n-1}\sigma(b))\tau^n(a)+\\
& +\sigma^n(a)\sigma^n(b)f_n(r)\tau^n(b)\tau^n(a)=\\
= \quad & \sum_{u+k=n}\sigma^n(abr)f_u(\sigma^{n-u}(b))f_k(\tau^{n-k}(a))+\sum_{i+l=n} f_i(\sigma^{n-i}(a))f_l(\tau^{n-l}(b))\tau^n(rba)+\\
& +\sum_{i+l+u+k=n}^{i+l,u+k\leq n-1} f_i(\sigma^{n-i}(a))f_l(\sigma^{u+k}\tau^i(b))\sigma^{u+k}\tau^{i+l}(r)f_u(\tau^{i+l}\sigma^k(b))f_k(\sigma^{n-k}(a))+\\
& +\sum_{i+l+u+k=n-1} f_i(\sigma^{n-i}(a))f_l(\sigma^{1+u+k}\tau^i(b))f_1(\sigma^{u+k}\tau^{i+l}(r))f_u(\tau^{1+i+l}\sigma^k(b))f_k(\sigma^{n-k}(a))+\\
& +\cdots+f_1(\sigma^{n-1}(a))\sigma^{n-1}\tau(b)f_{n-1}(\tau(r))\tau^n(ba)+\sigma^n(a)f_1(\sigma^{n-1}(b))f_{n-1}(\tau(r))\tau^n(ba)+\\
& +\sigma^n(ab)f_{n-1}(\sigma(r))\tau^{n-1}\sigma(b)f_1(\tau^{n-1}(a))+\sigma^n(ab)f_{n-1}(\sigma(r))f_1(\tau^{n-1}\sigma(b))\tau^n(a)+\\
& +\sigma^n(ab)f_n(r)\tau^n(ba).
\end{aligned}
$$

Now, substracting the two terms and using the hypothesis that $\sigma\tau = \tau\sigma$ their difference yields;

$$\begin{aligned}
& f_n(ab)\tau^n(rba) - \sigma^n(abr)\sum_{u+k=n} f_u(\sigma^{n-u}(b))f_k(\tau^{n-k}(a))+ \\
& +\sigma^n(abr)f_n(ba) - \sum_{i+l=n} f_i(\sigma^{n-i}(a))f_l(\tau^{n-l}(b))\tau^n(rba) = \\
= \; & \sigma^n(abr)(f_n(ba) - \sum_{u+k=n} f_u(\sigma^{n-u}(b))f_k(\tau^{n-k}(a)))+ \\
& +(f_n(ab) - \sum_{i+l=n} f_i(\sigma^{n-i}(a))f_l(\tau^{n-l}(b)))\tau^n(rba) = \\
= \; & \sigma^n(abr)\Phi_n(b,a) + \Phi_n(a,b)\tau^n(rba).
\end{aligned}$$

Similarly, the difference of the last two terms of equation (9) yields $\sigma^n(bar)\Phi_n(a,b) + \Phi_n(b,a)\tau^n(rab)$. Thus, (9) becomes

$$\begin{aligned}
0 \; = \; & \sigma^n(abr)\Phi_n(b,a) + \Phi_n(a,b)\tau^n(rba) + \sigma^n(bar)\Phi_n(a,b) + \Phi_n(b,a)\tau^n(rab) = \\
= \; & \Phi_n(a,b)\tau^n(r)\tau^n[b,a] + \sigma^n[b,a]\sigma^n(r)\Phi_n(a,b).
\end{aligned}$$

□

In view of Lemma 2.6 (i), it is easy to see that the function Φ defined in the beginning of this section is antisymmetric. For any $a, b \in R$, $n \in I\!N$ we have,
$f_n(ab)+f_n(ba) = f_n(ab+ba) = \sum_{i+j=n} (f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b))+f_i(\sigma^{n-i}(b))f_j(\tau^{n-j}(a)))$
or, $f_n(ab)-\sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b)) = -(f_n(ba)-\sum_{i+j=n} f_i(\sigma^{n-i}(b))f_j(\tau^{n-j}(a))$
or, $\Phi_n(a,b) = -\Phi_n(b,a)$.

It can also be seen that the function Φ is additive in both the arguments, i.e., for $a, b, c \in R$, $n \in I\!N$ consider,

$$\begin{aligned}
& \Phi_n(a, b+c) = f_n(a(b+c)) - \sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b+c)) = \\
= \; & f_n(ab) - \sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(b)) + f_n(ac) - \sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\tau^{n-j}(c)) = \\
= \; & \Phi_n(a,b) + \Phi_n(a,c).
\end{aligned}$$

Analogously, it can also be be shown that $\Phi_n(a+b,c) = \Phi_n(a,c) + \Phi_n(b,c)$.

Proof of Theorem 2.2. Let $x, y \in R$ be the fixed elements of R such that $c[x,y] = 0 \implies c = 0$ for every $c \in R$.

We'll prove the result by induction on n. We know that for $n = 0$, $\Phi_0(a,b) = 0$. Hence proceeding by induction we can assume that $\Phi_m(a,b) = 0$ for all $m < n$. Using Lemma 2.7 (i) we have

$$\Phi_n(a,b)[\tau^n(a), \tau^n(b)] = 0, \text{for all } a, b \in R. \tag{10}$$

In particular,

$$\Phi_n(x,y) = 0. \tag{11}$$

Replacing, a by $a + x$, in (10) we get

$$\Phi_n(x, b)[\tau^n(a), \tau^n(b)] + \Phi_n(a, b)[\tau^n(x), \tau^n(b)] = 0, \text{for all } a, b \in R. \tag{12}$$

Replace b by $b + y$ in (12). Then

$$\begin{aligned} 0 \quad = \quad & \Phi_n(a, b)[\tau^n(x), \tau^n(y)] + \Phi_n(a, y)[\tau^n(x), \tau^n(b)] + \\ & +\Phi_n(a, y)[\tau^n(x), \tau^n(y)] + \Phi_n(x, b)[\tau^n(a), \tau^n(y)], \text{ for all } a, b \in R. \end{aligned} \tag{13}$$

Replacing a by x in (13) and using (11) we obtain

$$\Phi_n(x, b)[\tau^n(x), \tau^n(y)] = 0, \text{ for all } b \in R. \tag{14}$$

Again replace b by y in (12) and use (11) to get $\Phi_n(a, y)[\tau^n(x), \tau^n(y)] = 0$, for every $a \in R$. Hence,

$$\Phi_n(a, y) = 0, \text{ for all } a \in R. \tag{15}$$

Combining (13), (14)and (15) we have that $\Phi_n(a, b)[\tau^n(x), \tau^n(y)] = 0$, and so $\Phi_n(a, b) = 0$, for all $a, b \in R$. □

Some special cases of the above theorem are themselves of great interest and we list them as corollaries:

Corollary 2.8. *Let R be a 2-torsion-free ring. If R has a commutator which is not a right zero divisor of R then every Jordan higher derivation on R is a higher derivation on R.*

Corollary 2.9. *Let R be a 2-torsion-free ring and σ, τ be the commuting endomorphisms of R such that τ is one-one and onto. If R has a commutator which is not a zero divisor then every Jordan (σ, τ)- derivation on R is a $(\sigma, \tau)-$derivation on R.*

Proof of Theorem 2.3. Given that R is non-commutative. Now we'll proceed by induction on n. We know that for $n = 0$, $\Phi_0(a, b) = 0$. Hence, we may assume that $\Phi_m(a, b) = 0$ for all $m < n$.
Using Lemma 2.7 (ii) we have

$$\Phi_n(a, b)\tau^n(r)\tau^n[a, b] + \sigma^n[a, b]\sigma^n(r)\Phi_n(a, b) = 0, \text{ for all } a, b, r \in R.$$

Now, multiplying the above equation by $\tau^n[a, b]$ from the right and using Lemma 2.7 (i), we have

$$\Phi_n(a, b)\tau^n(r)\tau^n[a, b]\tau^n[a, b] = 0, \text{ for all } a, b, r \in R.$$

Since τ is invertible, the above equation gives

$$\tau^{-n}(\Phi_n(a,b))r([a,b])^2 = 0, \text{ for all } a,b,r \in R.$$

Now, by primeness of R for each fixed $a,b \in R$, either $\Phi_n(a,b) = 0$ or $[a,b]^2 = 0$. Using Lemma 2.5 either $\Phi_n(a,b) = 0$ or $[a,b]^2 = 0$, for all $a,b \in R$. Suppose that $[a,b]^2 = 0$, for all $a,b \in R$. Now let $t \in R$ such that $t^2 = 0$. Replacing b by t in the latter identity and using the fact that $t^2 = 0$, we find that $(at)^2 + (ta)^2 - ta^2t = 0$. This implies that $(ta)^2t = 0$ i.e., $(ta)^3 = 0$ for all $a \in R$. Thus tR is a nonzero nil right ideal satisfying $z^3 = 0$ for all $z \in tR$. By Lemma 1.1 of [12] R has a nonzero nilpotent ideal. But since, R is prime we find that $tR = \{0\}$ and hence, $t = 0$. Thus $[a,b]^2 = 0$ for all $a,b \in R$ shows that $[a,b] = 0$ for all $a,b \in R$. Hence R is commutative, a contradiction. Therefore, $\Phi_n(a,b) = 0$ for all $a,b \in R$. □

In the hypothesis of the above theorem, if the underlying ring is arbitrary prime, then for $\sigma = \tau$ we can prove the following:

Theorem 2.10. *Let R be a prime ring with char$(R) \neq 2$ and σ be an automorphism on R. Then every Jordan (σ,σ)-higher derivation on R is a (σ,σ)-higher derivation on R.*

Proof. Let us define $\Phi_n(a,b) = f_n(ab) - \sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\sigma^{n-j}(b))$. For $n = 0$, $\Phi_0(a,b) = 0$ and also for $n = 1$, $\Phi_1(a,b) = 0$. Proceeding by induction let us assume that $\Phi_m(a,b) = 0$, for each $m < n$. When $\sigma = \tau$ Lemma 2.7 (ii) reduces to $\Phi_n(a,b)\sigma^n(r)\sigma^n[b,a] + \sigma^n[b,a]\sigma^n(r)\Phi_n(a,b) = 0$, for all $a,b \in R$. This implies that $\sigma^{-n}(\Phi_n(a,b))r[b,a] + [b,a]r\sigma^{-n}(\Phi_n(a,b)) = 0$. Using Lemma 2.4, we find that for each fixed pair $a,b \in R$ either $\Phi_n(a,b) = 0$ or $[b,a] = 0$. Now for each fixed $a \in R$, we put $A_1 = \{b \in R \mid \Phi_n(a,b) = 0\}$ and $A_2 = \{b \in R \mid [b,a] = 0\}$. Clearly A_1 and A_2 are the additive subgroups of R whose union is R. By Braurer's trick, we have either $R = A_1$ or $R = A_2$. Again using the similar procedure we can see that either $R = \{a \in R \mid R = A_1\}$ or $R = \{a \in R \mid R = A_2\}$, that is, either $\Phi_n(a,b) = 0$ for all $a,b \in R$ or R is commutative. If R is commutative then from Lemma 2.6 (i) we can easily obtain that $f_n(ab) = \sum_{i+j=n} f_i(\sigma^{n-i}(a))f_j(\sigma^{n-j}(b))$ for all $a,b \in R$, that is, $\Phi_n(a,b) = 0$, for all $a,b \in R$. Thus, in both the cases $\Phi_n(a,b) = 0$, for all $a,b \in R$. This completes the proof of our theorem. □

An immediate consequence of the above theorem is the following corollary which is a famous result due to Herstein;

Corollary 2.11. ([11, Theorem 3.1]) *Let R be a prime ring with char$(R) \neq 2$. Then every Jordan derivation on R is a derivation on R.*

The above theorem also reduces to the main theorem of [8];

Corollary 2.12. ([8, Theorem 2.1.10]) *Let R be a prime ring with char$(R) \neq 2$. Then every Jordan higher derivation on R is a higher derivation on R.*

In conclusion it is tempting to conjecture as follows:

Conjecture. *Let R be a 2-torsion-free prime (semiprime) ring and let σ, τ be commuting endomorphisms of R such that τ is one-one and onto. Then every Jordan (σ, τ)-higher derivation of R is a (σ, τ)-higher derivation of R.*

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