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DECOMPOSITION OF TENSOR PRODUCTS OF MODULAR IRREDUCIBLE REPRESENTATIONS FOR SL_3
(*WITH AN APPENDIX BY C.M. RINGEL)*

C. Bowman, S.R. Doty and S. Martin

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Abstract. We give an algorithm for working out the indecomposable direct summands in a Krull–Schmidt decomposition of a tensor product of two simple modules for $G = \mathrm{SL}_3$ in characteristics 2 and 3. It is shown that there is a finite family of modules such that every such indecomposable summand is expressible as a twisted tensor product of members of that family.

Along the way we obtain the submodule structure of various Weyl and tilting modules. Some of the tilting modules that turn up in characteristic 3 are not rigid; these seem to provide the first example of non-rigid tilting modules for algebraic groups. These non-rigid tilting modules lead to examples of non-rigid projective indecomposable modules for Schur algebras, as shown in the Appendix.

Higher characteristics (for SL_3) will be considered in a later paper.

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1. Introduction

We begin by explaining our motivation, which may be formulated for an arbitrary semisimple algebraic group in positive characteristic.

1.1. Let G be a semisimple, simply connected linear algebraic group over an algebraically closed field K of positive characteristic p. We fix a Borel subgroup B and a maximal torus T with $T \subset B \subset G$ and we let B determine the *negative* roots. We write $X = X(T)$ for the character group of T and let X^+ denote the set of dominant weights. By G-module we always mean a rational G-module, i.e. a $K[G]$-comodule, where $K[G]$ is the coordinate algebra of G. For each $\lambda \in X^+$ we have the following (see [17]) finite dimensional G-modules:

$\mathrm{L}(\lambda)$	simple module of highest weight λ;
$\Delta(\lambda)$	Weyl module of highest weight λ;
$\nabla(\lambda)$	$= \mathrm{ind}_B^G K_\lambda$; dual Weyl module of highest weight λ;
$\mathrm{T}(\lambda)$	indecomposable tilting module of highest weight λ

where K_λ is the 1-dimensional B-module upon which T acts by the character λ with the unipotent radical of B acting trivially. The simple modules $\mathrm{L}(\lambda)$ are contravariantly self-dual. The module $\nabla(\lambda)$ has simple socle isomorphic to $\mathrm{L}(\lambda)$; the module $\Delta(\lambda)$ is isomorphic to ${}^\tau\nabla(\lambda)$, the contravariant dual of $\nabla(\lambda)$, hence has simple head isomorphic to $\mathrm{L}(\lambda)$.

The central problem which interests us is as follows.

Problem 1. Describe the indecomposable direct summands of an arbitrary tensor product of the form $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$, for $\lambda, \mu \in X^+$.

As usual, a superscript $M^{[j]}$ on a G-module M indicates that the structure has been twisted by the jth power of the Frobenius endomorphism on G. By Steinberg's tensor product theorem, there are twisted tensor product factorizations

$$\mathrm{L}(\lambda) \simeq \mathrm{L}(\lambda^0) \otimes \mathrm{L}(\lambda^1)^{[1]} \otimes \mathrm{L}(\lambda^2)^{[2]} \otimes \cdots ;$$
$$\mathrm{L}(\mu) \simeq \mathrm{L}(\mu^0) \otimes \mathrm{L}(\mu^1)^{[1]} \otimes \mathrm{L}(\mu^2)^{[2]} \otimes \cdots$$

where $\lambda = \sum \lambda^j p^j$, $\mu = \sum \mu^j p^j$ are the p-adic expansions (unique) such that each λ^j, μ^j belongs to the restricted region

$$X_1 = \{\nu \in X^+ \mid \langle \alpha^\vee, \nu \rangle \leqslant p-1 \text{ for all simple roots } \alpha\}.$$

Putting these factorizations into the original tensor product we obtain

$$\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu) \simeq \textstyle\bigotimes_{j \geqslant 0} \left(\mathrm{L}(\lambda^1) \otimes \mathrm{L}(\mu^1)\right)^{[j]} \tag{1.1.1}$$

and thus we see that in Problem 1 one should first study the case where both highest weights in question are restricted.

Assume that Problem 1 has been solved for all pairs of restricted weights (note that this is a *finite* problem for any given G). Let $\mathfrak{F} = \mathfrak{F}(G)$ be the set of isomorphism classes of indecomposable direct summands appearing in some $\mathrm{L} \otimes \mathrm{L}'$, for a pair L, L' of restricted simple G-modules. Let $[L \otimes L' : I]$ be the multiplicity of $I \in \mathfrak{F}$ as a direct summand of $L \otimes L'$. Then one can express each tensor product $\mathrm{L}(\lambda^j) \otimes \mathrm{L}(\mu^j)$ as a finite direct sum of indecomposable modules

$$\mathrm{L}(\lambda^j) \otimes \mathrm{L}(\mu^j) \simeq \bigoplus_{I \in \mathfrak{F}} [\mathrm{L}(\lambda^j) \otimes \mathrm{L}(\mu^j) : I]\; I. \tag{1.1.2}$$

Thus, the original tensor product $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ has a decomposition of the form

$$\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu) \simeq \textstyle\bigotimes_{j \geqslant 0} \; \bigoplus_{I \in \mathfrak{F}} [\mathrm{L}(\lambda^j) \otimes \mathrm{L}(\mu^j) : I]\; I^{[j]}$$

and by interchanging the order of the product and sum we obtain the decomposition

$$\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu) \simeq \bigoplus \left(\textstyle\prod_{j \geqslant 0} [\mathrm{L}(\lambda^j) \otimes \mathrm{L}(\mu^j) : I_j]\right) \textstyle\bigotimes_{j \geqslant 0} I_j^{[j]} \tag{1.1.3}$$

where the direct sum is taken over the set of all finite sequences $(I_0, I_1, I_2, \dots)$ of members of $\mathfrak{F}$.

This gives a direct sum decomposition of $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ in terms of twisted tensor products of modules in $\mathfrak{F}$. If all such twisted tensor products are themselves indecomposable as G-modules, then we have in some sense solved Problem 1 for general λ, μ. Even when this isn't true we have still obtained a first approximation towards a solution to Problem 1. This leads us to the following secondary set of problems:

Problem 2. Given G,

(a) classify the members of the family $\mathfrak{F} = \mathfrak{F}(G)$ and compute the multiplicities $[L \otimes L' : I]$ for $I \in \mathfrak{F}$, L, L' restricted;

(b) determine conditions under which a twisted tensor product of members from $\mathfrak{F}$ remains indecomposable;

(c) determine the module structure of the members of $\mathfrak{F}$.

Let G_r denote the kernel of the rth iterate of the Frobenius, let G_rT denote the inverse image of T under the same map, and let $\widehat{\mathrm{Q}}_r(\lambda)$ denote the G_rT-injective hull of $\mathrm{L}(\lambda)$ for any $\lambda \in X_r$, where

$$X_r := \{\nu \in X^+ \mid \langle \alpha^\vee, \nu \rangle \leqslant p^r - 1 \text{ for all simple roots } \alpha\}.$$

Let h denote the Coxeter number of G. If $p \geqslant 2h - 2$ then $\widehat{\mathrm{Q}}_r(\mu)$ has (for any $\mu \in X_r$) a G-module structure; this structure is unique in the sense that any two such G-module structures are equivalent. (These statements are expected to hold for all p.) Concerning Problem 2(b) we observe the following.

Lemma. *Assume that $p \geqslant 2h - 2$ or that if $p < 2h - 2$ then $\widehat{\mathrm{Q}}_1(\mu)$ has a unique G-module structure for all $\mu \in X_1$. If each member of the sequence $(I_j)_{j \geqslant 0}$ $(I_j \in \mathfrak{F})$ has simple G_1T-socle with restricted highest weight then the twisted tensor product $\bigotimes_{j \geqslant 0} I_j^{[j]}$ is indecomposable as a G-module. Hence P is indecomposable.*

Proof. By assumption the socle of I_j is simple, as a G_1T-module, hence has the form $L(\mu(j))$ for some $\mu(j) \in X_1$. Hence the module I_j embeds in the G_1T-injective hull $\widehat{\mathrm{Q}}_1(\mu(j))$ of $\mathrm{L}(\mu(j))$, for each j, so $P := I_0 \otimes I_1^{[1]} \otimes \cdots \otimes I_m^{[m]}$ embeds in $Q := \widehat{\mathrm{Q}}_1(\mu(0)) \otimes \widehat{\mathrm{Q}}_1(\mu(1))^{[1]} \otimes \cdots \otimes \widehat{\mathrm{Q}}_1(\mu(m))^{[m]}$. By [17, II.11.16 Remark 2] the module Q has a G-module structure and is isomorphic to $\widehat{\mathrm{Q}}_r(\mu)$, where $\mu = \sum_j \mu(t_j)p^j$. Since $\widehat{\mathrm{Q}}_r(\mu)$ has simple G_rT-socle $\mathrm{L}(\mu)$ it follows that P also has simple G_rT-socle $\mathrm{L}(\mu)$, and thus has simple G-socle $\mathrm{L}(\mu)$. □

We note that in Types A_1 and A_2 ($G = \mathrm{SL}_2$, SL_3) it is known that $\widehat{\mathrm{Q}}_1(\mu)$ has a unique G-module structure for all $\mu \in X_1$, for any p. In the case $G = \mathrm{SL}_2$ (studied in [11]) it turns out that for any p the members of $\mathfrak{F}$ are always indecomposable tilting modules with simple G_1T-socle of restricted highest weight, so the determination of the family $\mathfrak{F}$ and the multiplicities $[L \otimes L' : I]$ leads in that case to a complete solution of Problem 1 for all pairs of dominant weights. The purpose of this paper is to examine the next most complicated case, namely the case $G = \mathrm{SL}_3$. In that case, we will see that all members of $\mathfrak{F}$ have simple G_1T-socle of restricted highest weight when $p = 2$, and this holds with only two exceptions when $p = 3$, so the decomposition (1.1.3) is decisive in characteristic 2 and provides a great deal of information in characteristic 3.

Furthermore, although in characteristic 3 the summands in (1.1.3) are not always indecomposable, by analyzing the further splittings which arise, we show that there is a finite family $\mathfrak{F}'$, closely related to $\mathfrak{F}$, such that every indecomposable direct summand of $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ is isomorphic to a twisted tensor product of members of $\mathfrak{F}'$. Thus, we obtain a complete solution to Problem 1 in characteristics 2 and 3.

1.2. The paper is organized as follows. In Section 2 we recall known facts that we use. Our main technique is to compute structure of certain Weyl modules (using a computer when necessary) and use that structure to deduce structural information on certain tilting modules. The main results obtained by our computations are given in Sections 3 and 4. To be specific, the structure of the relevant Weyl modules is given in 3.1 and 4.1, while the main results on tensor products — including description of the family $\mathfrak{F}$, multiplicities $[L \otimes L' : I]$ for restricted simples L, L' and $I \in \mathfrak{F}$, and structure of members of $\mathfrak{F}$ (in most cases) — are summarized in 3.2 and 4.2. One will also find worked examples in those sections.

In characteristic 2 all members of $\mathfrak{F}(\mathrm{SL}_3)$ are tilting modules with simple G_1T-socle of restricted highest weight, so the decomposition (1.1.3) gives a complete answer to Problem 1 for all pairs of dominant weights. This is similar to what happens for $G = \mathrm{SL}_2$. Moreover, each member of $\mathfrak{F}(\mathrm{SL}_3)$ in this case is rigid (a module is called *rigid* if its radical and socle filtrations coincide) and can be described by a strong diagram in the sense of [1]. Recall that in [1] a module diagram is a directed graph depicting the radical series of the module, in such a way that vertices correspond to composition factors and edges to non-split extensions, and a strong diagram is one in which the diagram also determines the socle series. (One should consult [1] for precise statements.)

Characteristic 3 is more complicated. (As standard notation, we write (a, b) for a highest weight of the form $a\varpi_1 + b\varpi_2$ where ϖ_1, ϖ_2 are the usual fundamental weights.) First, all but two of the members of $\mathfrak{F}(\mathrm{SL}_3)$ have simple G_1T-socle of restricted highest weight. The two exceptional cases are in fact simple modules of highest weights $(5, 2)$ and $(2, 5)$ that are not restricted, and so one is forced to consider possible further splitting of summands in (1.1.3), in cases where one or both of these modules appears in a twisted tensor product on the right hand side. (This happens only if the tensor square of the Steinberg module occurs in some factor in the right hand side of (1.1.1).) In all cases those further splittings can be worked out; see Proposition 4.3. This leads to the finite family $\mathfrak{F}'$ discussed in the last paragraph of 1.1.

Furthermore, in characteristic 3 it turns out that four members of $\mathfrak{F}(\mathrm{SL}_3)$ — namely the tilting modules $\mathrm{T}(3,3)$, $\mathrm{T}(4,3)$, $\mathrm{T}(3,4)$, and $\mathrm{T}(4,4)$ — are not rigid and do not have strong Alperin diagrams. The structure of one of the simplest of these examples, $\mathrm{T}(4,3)$, is analyzed in detail in the Appendix by C.M. Ringel, using different methods. Although not itself projective, Ringel shows that $\mathrm{T}(4,3)$ is a quotient of the corresponding projective indecomposable for an appropriate Schur algebra, and thus he produces an example of a non-rigid projective indecomposable module for that Schur algebra. (See [13], [20], [3], [4] for background on Schur algebras.) The other non-rigid modules are subject to a similar analysis.

Preliminary calculations indicate that members of $\mathfrak{F}(\mathrm{SL}_3)$ are again rigid in characteristics higher than 3. The observed anomalies in characteristic 3 are associated with the fact that some of the Weyl modules which turn up are too close to the upper wall of the "lowest p^2-alcove" and thus have composition factors with multiplicity greater than 1. (Those multiplicities follow, e.g. from [10], from knowledge of composition factor multiplicities in baby Verma modules, which are well known in this case.) The simple characters for SL_3 have been known for a long time (see e.g., [15], [16]).

Our results overlap somewhat with [18], [19] although our methods are different and we push the calculations further. Larger characteristics, for which some calculations become in a sense independent of p, will be treated in a future paper.

This paper has been circulating for some time in various forms, and since the first version was made available, the preprint [2] has appeared, in which further examples of non-rigid tilting modules for algebraic groups are obtained.

2. Preliminaries

We recall some general facts that will be used in our calculations.

2.1. Let us recall Pillen's Theorem [21, §2, Corollary A] (see also [5, Theorem (2.5)]). Write St_r for the rth Steinberg module $\mathrm{L}((p^r-1)\rho) = \Delta((p^r-1)\rho) = \mathrm{T}((p^r-1)\rho)$. Then for $\lambda \in X_r$ the tilting module $\mathrm{T}(2(p^r-1)\rho + w_0\lambda)$ is isomorphic to the indecomposable G-component of $\mathrm{St}_r \otimes \mathrm{L}((p^r-1)\rho + w_0\lambda)$ containing the weight vectors of highest weight $2(p^r-1)\rho + w_0\lambda$.

2.2. In general the formal character of a tilting module is not known; even for SL_3, as far as we are aware this remains an open problem. The following general result of Donkin (see [7, Proposition 5.5]) computes the formal character of certain tilting modules. Let $\lambda, \mu \in X^+$ and assume that $(\lambda, \alpha_0^\vee) \leqslant p$, where α_0 is the highest short root. Then:

$$\operatorname{ch} \mathrm{T}((p-1)\rho + \lambda) = \operatorname{ch} \mathrm{L}((p-1)\rho) \sum_{\nu \in W\lambda} e(\nu) \tag{2.2.1}$$

and for any $\nu \in X^+$,

$$(\mathrm{T}((p-1)\rho + \lambda + p\mu) : \nabla(\nu)) = \sum_{\xi \in N(\nu)} (\mathrm{T}(\mu) : \nabla(\xi)) \tag{2.2.2}$$

where $N(\nu) = \{\xi \in X^+ : \nu + \rho - p(\xi + \rho) \in W\lambda\}$. Furthermore, in Lemma 5 of Section 2.1 in [8], the characters of the tilting modules which are projective and indecomposable as G_1-modules are computed explicitly, for $G = \mathrm{SL}_3$.

2.3. Another useful general fact (that will be used repeatedly) is the observation that tilting modules are contravariantly self-dual:

$$^{\tau}\mathrm{T}(\lambda) \simeq \mathrm{T}(\lambda) \tag{2.3.1}$$

for all $\lambda \in X^+$. This is because (by [17, II.2.13]) contravariant duality interchanges $\Delta(\mu)$ and $\nabla(\mu)$, so $^{\tau}\mathrm{T}(\lambda)$ is again indecomposable tilting, of the same highest weight.

2.4. Finally, there is a twisted tensor product theorem for tilting modules, assuming that Donkin's conjecture [5, Conjecture (2.2)] is valid or that $p \geqslant 2h - 2$. (It is well known [17, II.11.16, Remark 2] that the conjecture is valid for all p in case $G = \mathrm{SL}_3$.) For our purposes, it is convenient to reformulate the tensor product theorem in the following form. First we observe that, given $\lambda \in X^+$ satisfying the condition

$$\langle \lambda, \alpha^\vee \rangle \geqslant p - 1, \quad \text{for all simple roots } \alpha, \tag{2.4.1}$$

there exist unique weights λ', μ such that

$$\lambda = \lambda' + p\mu, \qquad \lambda' \in (p-1)\rho + X_1,\ \mu \in X^+. \tag{2.4.2}$$

This is easy to see: for each λ_i in $\lambda = \sum \lambda_i \varpi_i$ where the ϖ_i are the fundamental weights, express $\lambda_i - (p-1)$ (uniquely) in the form $\lambda_i - (p-1) = r_i + ps_i$ with $0 \leqslant r_i \leqslant p-1$. Then set $\lambda' = (p-1)\rho + \sum r_i \varpi_i$ and $\mu = \sum s_i \varpi_i$.

Now by induction on m using (2.4.1) and (2.4.2) one shows that every $\lambda \in X^+$ has a unique expression in the form

$$\lambda = \textstyle\sum_{j=0}^{m} a_j(\lambda)\, p^j \tag{2.4.3}$$

with $a_0(\lambda), \dots, a_{m-1}(\lambda) \in (p-1)\rho + X_1$ and $\langle a_m(\lambda), \alpha^\vee \rangle < p - 1$ for at least one simple root α.

Given $\lambda \in X^+$, express λ in the form (2.4.3). Assume Donkin's conjecture holds if $p < 2h - 2$. Then there is an isomorphism of G-modules

$$\mathrm{T}(\lambda) \simeq \bigotimes_{j=0}^{m} \mathrm{T}(a_j(\lambda))^{[j]}. \tag{2.4.4}$$

To prove this one uses induction and [17, Lemma II.E.9] (which is a slight reformulation of [5, Proposition (2.1)]).

3. Results for $p = 2$

For the rest of the paper we take $G = \mathrm{SL}_3$. Conventions: Dominant weights are written as ordered pairs (a, b) of non-negative integers; one should read (a, b) as an abbreviation for $a\varpi_1 + b\varpi_2$ where ϖ_1, ϖ_2 are the fundamental weights, defined by the condition $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$. *When describing module structure, we shall always identify a simple module* $\mathrm{L}(\lambda)$ *with its highest weight* λ*.* Whenever possible we will depict the structure by giving an Alperin diagram (see [1] for definitions) with edges directed downwards, except in the uniserial case, where we will write $M = [L_s, L_{s-1}, \dots, L_1]$ for a module M with unique composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_{s-1} \subset M_s = M$ such that $L_j \simeq M_j/M_{j-1}$ is simple for each j.

3.1. Structure of certain Weyl modules for $p = 2$.

The results given below were computer generated, using GAP [12] code available on the second author's web page. (Some cases are obtainable from [9].)

The restricted region X_1 in this case consists of the weights of the form (a, b) with $0 \leqslant a, b \leqslant 1$, and we have

$$\Delta(0,0) = \mathrm{L}(0,0),\ \Delta(1,0) = \mathrm{L}(1,0),$$
$$\Delta(0,1) = \mathrm{L}(0,1),\ \Delta(1,1) = \mathrm{L}(1,1).$$

These are all tilting modules. Thus it follows immediately that all the members of $\mathfrak{F}$ are tilting.

The structure of the other Weyl modules we need is depicted below. The uniserial modules have structure

$$\Delta(2,0) = [(2,0),(0,1)],\ \Delta(0,2) = [(0,2),(1,0)]$$
$$\Delta(3,0) = [(3,0),(0,0)],\ \Delta(0,3) = [(0,3),(0,0)]$$
$$\Delta(2,1) = [(2,1),(0,2),(1,0)],\ \Delta(1,2) = [(1,2),(2,0),(0,1)].$$

Finally, the structure of $\Delta(2,2)$ is given by the diagram

$$\Delta(2,2) = \boxed{\begin{tikzcd}[column sep=small, row sep=small] & (2,2) \arrow[dl, dash] \arrow[dr, dash] & \\ (0,3) \arrow[dr, dash] & & (3,0) \arrow[dl, dash] \\ & (0,0) & \end{tikzcd}} .$$

We worked these out using explicit calculations in the hyperalgebra, by methods similar to those of [14], [22].

3.2. Restricted tensor product decompositions for $p = 2$. The indecomposable decompositions of restricted tensor products for $p = 2$ is as follows. (We omit any decomposition of the form $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ where one of λ, μ is zero.) There is an involution on G-modules which on weights is the map $\lambda \to -w_0(\lambda)$, where w_0 is the longest element of the Weyl group. (In Type A this comes from a graph automorphism of the Dynkin diagram.) We refer to this involution as *symmetry*, and we will often omit calculations and results that can be obtained by symmetry from a calculation or result already given.

Proposition. *Suppose $p = 2$.*

(a) *The indecomposable direct summands of tensor products of non-trivial restricted simple* SL_3*-modules are as follows:*
  (1) $\mathrm{L}(1,0) \otimes \mathrm{L}(1,0) \simeq \mathrm{T}(2,0)$; $\mathrm{L}(0,1) \otimes \mathrm{L}(0,1) \simeq \mathrm{T}(0,2)$;
  (2) $\mathrm{L}(1,0) \otimes \mathrm{L}(0,1) \simeq \mathrm{T}(1,1) \oplus \mathrm{T}(0,0)$;
  (3) $\mathrm{L}(1,0) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(2,1)$; $\mathrm{L}(0,1) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(1,2)$;
  (4) $\mathrm{L}(1,1) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(2,2) \oplus 2\mathrm{T}(1,1)$.

  Thus the family $\mathfrak{F}(\mathrm{SL}_3)$ is in this case given by $\mathfrak{F} = \{\mathrm{T}(a,b) : 0 \leqslant a, b \leqslant 2\}$.

(b) *The structure of the uniserial members of $\mathfrak{F}$ is given as follows:*
  $\mathrm{T}(0,0) = [(0,0)]$*;* $\mathrm{T}(1,0) = [(1,0)]$*;* $\mathrm{T}(1,1) = [(1,1)]$*;*
  $\mathrm{T}(2,0) = [(0,1), (2,0), (0,1)]$.

  The structure diagrams of $\mathrm{T}(2,1)$*,* $\mathrm{T}(2,2)$ *are displayed below:*

$$\begin{tikzcd}[column sep=small, row sep=small] & (1,0) \arrow[d, dash] & \\ & (0,2) \arrow[dl, dash] \arrow[dr, dash] & \\ (2,1) \arrow[dr, dash] & & (1,0) \arrow[dl, dash] \\ & (0,2) \arrow[d, dash] & \\ & (1,0) & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small, row sep=small] & & (0,0) \arrow[dl, dash] \arrow[dr, dash] & & \\ & (0,3) \arrow[dl, dash] \arrow[dr, dash] & & (3,0) \arrow[dl, dash] \arrow[dr, dash] & \\ (0,0) \arrow[dr, dash] & & (2,2) \arrow[dl, dash] \arrow[dr, dash] & & (0,0) \arrow[dl, dash] \\ & (0,3) \arrow[dr, dash] & & (3,0) \arrow[dl, dash] & \\ & & (0,0) & & \end{tikzcd}$$

  and the structure diagrams of $\mathrm{T}(0,1)$*,* $\mathrm{T}(0,2)$*, and* $\mathrm{T}(1,2)$ *are obtained by symmetry from cases already listed.*

(c) *Each member of $\mathfrak{F}$ has simple G_1T-socle (and head) with highest weight belonging to the restricted region X_1.*

The proof is given in 3.3. First we consider consequences and give some examples. Recall that a dominant weight is called minuscule if the weights of the corresponding Weyl module form a single Weyl group orbit. For $G = \mathrm{SL}_3$ the minuscule weights are $(0,0)$, $(1,0)$, and $(0,1)$.

Corollary. *Let $p = 2$. Given arbitrary dominant weights λ, μ write $\lambda = \sum \lambda^j p^j$, $\mu = \sum \mu^j p^j$ with $\lambda^j, \mu^j \in X_1$ for all $j \geqslant 0$.*

(a) *In the decomposition* (1.1.3)*, each term in the direct sum is indecomposable. Hence the indecomposable direct summands of $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ are expressible as a twisted tensor product of members of $\mathfrak{F}$. Conversely, every twisted tensor product of members of $\mathfrak{F}$ occurs in some $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$.*

(b) *$\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ is indecomposable if and only if for each $j \geqslant 0$ the unordered pair $\{\lambda^j, \mu^j\}$ is one of the cases $\{(1,0),(1,0)\}$, $\{(0,1),(0,1)\}$, $\{(1,0),(1,1)\}$, $\{(0,1),(1,1)\}$ or one of λ^j, μ^j is the trivial weight $(0,0)$.*

(c) *Let m be the maximum j such that at least one of λ^j, μ^j is non-zero. Then $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ is indecomposable tilting, isomorphic to $\mathrm{T}(\lambda + \mu)$, if and only if:* (i) *for each $0 \leqslant j \leqslant m-1$, one of λ^j, μ^j is minuscule and the other is the Steinberg weight* $(1,1)$*, and* (ii) *$\{\lambda^m, \mu^m\}$ is one of the cases listed in part* (b)*.*

Proof. Part (a) follows from (1.1.3) and Lemma 1.1. Part (b) follow from the proposition and the discussion preceding (1.1.3), which shows that each $\mathrm{L}(\lambda^j) \otimes \mathrm{L}(\mu^j)$ must be itself indecomposable in order for $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ to be indecomposable. Then we get part (c) from part (b) by applying Donkin's tensor product theorem (2.4.4). $\square$

Examples. (i) To illustrate the procedure in part (a) of the corollary, we work out a specific example:

$$\begin{aligned}
&\mathrm{L}(7,2) \otimes \mathrm{L}(6,3) \\
&\qquad \simeq \big(\mathrm{L}(1,0) \otimes \mathrm{L}(0,1)\big) \otimes \big(\mathrm{L}(1,1) \otimes \mathrm{L}(1,1)\big)^{[1]} \otimes \big(\mathrm{L}(1,0) \otimes \mathrm{L}(1,0)\big)^{[2]} \\
&\qquad \simeq \big(\mathrm{T}(1,1) \oplus \mathrm{T}(0,0)\big) \otimes \big(\mathrm{T}(2,2) \oplus 2\mathrm{T}(1,1)\big)^{[1]} \otimes \mathrm{T}(2,0)^{[2]} \\
&\qquad \simeq \mathrm{T}(13,5) \oplus 2\mathrm{T}(6,2)^{[1]} \oplus 2\mathrm{T}(11,3) \oplus 2\mathrm{T}(5,1)^{[1]}.
\end{aligned}$$

In the calculation, the first line follows from Steinberg's tensor product theorem, the second is from the proposition, and to get the last line we applied Donkin's tensor product theorem (2.4.4), after interchanging the order of sums and products.

(ii) We have $\mathrm{L}(3,0)\otimes\mathrm{L}(3,2) \simeq \big(\mathrm{L}(1,0)\otimes\mathrm{L}(1,0)\big)\otimes\big(\mathrm{L}(1,0)\otimes\mathrm{L}(1,1)\big)^{[1]} \simeq \mathrm{T}(2,0)\otimes \mathrm{T}(2,1)^{[1]}$, which is indecomposable but not tilting. This illustrates the procedure in part (b) of the corollary.

(iii) We have $\mathrm{L}(3,0)\otimes\mathrm{L}(3,1) \simeq \big(\mathrm{L}(1,0)\otimes\mathrm{L}(1,1)\big)\otimes\big(\mathrm{L}(1,0)\otimes\mathrm{L}(1,0)\big)^{[1]} \simeq \mathrm{T}(2,1)\otimes \mathrm{T}(2,0)^{[1]} \simeq \mathrm{T}(6,1)$, illustrating part (c) of the corollary.

(iv) It is not the case that every indecomposable tilting module occurs as a direct summand of some tensor product of two simple modules. For instance, neither $\mathrm{T}(3,0)$ nor $\mathrm{T}(0,3)$ (both of which are uniserial of length 3) can appear as one of the indecomposable direct summands on the right hand side of (1.1.3). This follows from (2.4.4). More generally, this applies to any non-simple tilting module of the form $\mathrm{T}(a,b)$ with one of a,b equal to zero and the other greater than 2.

3.3. We now consider the proof of Proposition 3.2. First we compute the composition factor multiplicities of the restricted tensor products. Let $\chi_p(\lambda)$ be the formal character of $\mathrm{L}(\lambda)$. Then:

(1) $\chi_p(1,0)\cdot\chi_p(1,0) = \chi_p(2,0) + 2\chi_p(0,1)$;
(2) $\chi_p(1,0)\cdot\chi_p(0,1) = \chi_p(1,1) + \chi_p(0,0)$;
(3) $\chi_p(1,0)\cdot\chi_p(1,1) = \chi_p(2,1) + 2\chi_p(0,2) + 3\chi_p(1,0)$;
(4) $\chi_p(0,1)\cdot\chi_p(0,1) = \chi_p(0,2) + 2\chi_p(1,0)$;
(5) $\chi_p(0,1)\cdot\chi_p(1,1) = \chi_p(1,2) + 2\chi_p(2,0) + 3\chi_p(0,1)$;
(6) $\chi_p(1,1)\cdot\chi_p(1,1) = \chi_p(2,2) + 2\chi_p(0,3) + 2\chi_p(3,0) + 2\chi_p(1,1) + 4\chi_p(0,0)$.

Since $\mathrm{L}(1,0) = \mathrm{T}(1,0)$, it follows that $\mathrm{L}(1,0)\otimes\mathrm{L}(1,0)$ is tilting. It must have $\mathrm{T}(2,0)$ as a direct summand by highest weight considerations. But $\mathrm{T}(2,0)$ is contravariantly self-dual with $\mathrm{L}(0,1)$ in the socle, so it follows that $\mathrm{L}(0,1)$ appears with multiplicity at least 2 as a composition factor of $\mathrm{T}(2,0)$. Now character considerations force the structure to be given by

$$\mathrm{L}(1,0)\otimes\mathrm{L}(1,0) \simeq \mathrm{T}(2,0)$$

where $\mathrm{T}(2,0) = [(0,1),(2,0),(0,1)]$. By symmetry we also have

$$\mathrm{L}(0,1)\otimes\mathrm{L}(0,1) \simeq \mathrm{T}(0,2)$$

where $\mathrm{T}(0,2) = [(1,0),(0,2),(1,0)]$.

$\mathrm{L}(1,0)\otimes\mathrm{L}(0,1)$ is tilting and has a direct summand isomorphic to $\mathrm{T}(1,1) = \mathrm{L}(1,1)$. By character considerations it follows that there is one other indecomposable summand, namely $\mathrm{T}(0,0) = \mathrm{L}(0,0)$. Hence

$$\mathrm{L}(1,0)\otimes\mathrm{L}(0,1) \simeq \mathrm{T}(0,0)\oplus\mathrm{T}(1,1).$$

$\mathrm{L}(1,0)\otimes\mathrm{L}(1,1)$ is tilting and has a direct summand $\mathrm{T}(2,1)$. Self-duality of $\mathrm{T}(2,1)$ forces a copy of $\mathrm{L}(1,0)$ at the top, extending $\mathrm{L}(0,2)$. This, along with the structure of the Weyl modules and known Ext information forces the structure of $\mathrm{T}(2,1)$ to be as given in the statement of Proposition 3.2(b), and also forces

$$\mathrm{L}(1,0) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(2,1).$$

By symmetry we obtain also

$$\mathrm{L}(0,1) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(1,2).$$

Finally, $\mathrm{L}(1,1) \otimes \mathrm{L}(1,1)$ is tilting, with a direct summand isomorphic to $\mathrm{T}(2,2)$. The highest weights of all simple composition factors of the tensor product are in the same linkage class, excepting $(1,1)$, which appears with multiplicity 2. So two copies of $\mathrm{T}(1,1)$ split off. Moreover, $\mathrm{T}(2,2)$ has a submodule isomorphic to $\Delta(2,2)$, thus contains $\mathrm{L}(0,0)$ in the socle. This forces another copy of $\mathrm{L}(0,0)$ at the top of $\mathrm{T}(2,2)$, and this along with known Ext information and the structure of the Weyl modules forces the structure of $\mathrm{T}(2,2)$ to be as given in Proposition 3.2(b), and also forces

$$\mathrm{L}(1,1) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(2,2) \oplus 2\mathrm{T}(1,1).$$

All the claims in Proposition 3.2(a), (b) are now clear. It remains to verify the claim in (c). It is known that Donkin's conjecture holds for SL_3, as discussed at the beginning of 2.4, so $\mathrm{T}((p-1)\rho+\lambda)$ is as a G_1T-module isomorphic to $\widehat{\mathrm{Q}}_1((p-1)\rho+w_0\lambda)$ for any $\lambda \in X_1$. Thus $\mathrm{T}(2,1)$, $\mathrm{T}(1,2)$, and $\mathrm{T}(2,2)$ each has a simple G_1T-socle of restricted highest weight. For $\mathrm{T}(2,0)$ and $\mathrm{T}(0,2)$ one can argue by contradiction, using the fact [5, Proposition (1.5)] that truncation to an appropriate Levi subgroup L maps indecomposable tilting modules for G onto indecomposable tilting modules for L. Thus $\mathrm{T}(2,0)$ and $\mathrm{T}(0,2)$ truncate to $\mathrm{T}(2)$ for $L \simeq \mathrm{SL}_2$, which is known to have simple L_1T-socle and length three. If $\mathrm{T}(2,0)$ or $\mathrm{T}(0,2)$ did not have simple G_1T-socle then the same would be true of the truncation, since no composition factors are killed under truncation. Claim (c) for the remaining cases is trivial.

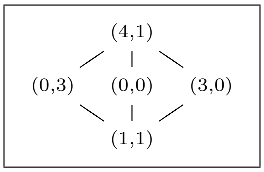
4. Results for $p = 3$

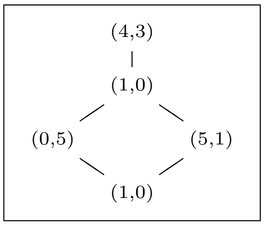
In characteristic 3 several of the Weyl modules one must consider are non-generic due to the proximity of their highest weight to the upper wall of the lowest p^2-alcove. This leads ultimately to examples of non-rigid tilting modules. Another complication is that the G_1T-socles of two direct summands of the tensor square of the Steinberg module fail to be simple.

4.1. Structure of certain Weyl modules for $p = 3$. We record the structure of certain Weyl modules needed later. The uniserial Weyl modules that turn up in our tensor product decompositions have structure given by

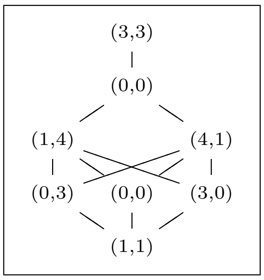
$$\Delta(0,0) = \mathrm{L}(0,0),\ \Delta(1,0) = \mathrm{L}(1,0),\ \Delta(2,0) = \mathrm{L}(2,0),$$
$$\Delta(2,1) = \mathrm{L}(2,1),\ \Delta(2,2) = \mathrm{L}(2,2),\ \Delta(5,2) = \mathrm{L}(5,2),$$
$$\Delta(1,1) = [(1,1),(0,0)],\ \Delta(3,0) = [(3,0),(1,1)],$$
$$\Delta(4,0) = [(4,0),(0,2)],\ \Delta(3,1) = [(3,1),(1,2)],$$
$$\Delta(5,0) = [(5,0),(0,1)],\ \Delta(5,1) = [(5,1),(1,0)],$$
$$\Delta(3,2) = [(3,2),(1,3),(2,1)],\ \Delta(6,0) = [(6,0),(4,1),(0,0)]$$
$$\Delta(4,2) = [(4,2),(0,4),(2,0)],\ \Delta(6,2) = [(6,2),(4,3),(1,0),(5,1)].$$

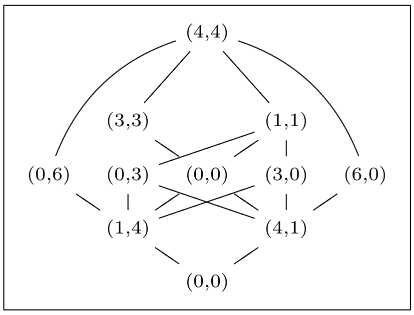
We note that the structure of $\Delta(6,2)$ is needed only in the Appendix. The non-uniserial cases we need have structure

$\Delta(4,1) =$ 

, $\Delta(4,3) =$ 

,

$\Delta(3,3) =$ 

, $\Delta(4,4) =$ 

.

As for the case $p = 2$, these structures were obtained by explicit calculations in the hyperalgebra, using GAP to do the calculations.

4.2. Restricted tensor product decompositions for $p = 3$. The indecomposable decompositions of restricted tensor products for $p = 3$ is given below. We omit any decomposition of the form $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ where one of λ, μ is zero, and we omit all cases that follow by applying symmetry to a case already listed.

Proposition. *Let $p = 3$.*

(a) *The indecomposable direct summands of tensor products of non-trivial restricted simple* SL_3*-modules are as follows:*
   (1) $\mathrm{L}(1,0) \otimes \mathrm{L}(1,0) \simeq \mathrm{T}(2,0) \oplus \mathrm{T}(0,1)$;
   (2) $\mathrm{L}(1,0) \otimes \mathrm{L}(0,1) \simeq \mathrm{T}(1,1)$;
   (3) $\mathrm{L}(1,0) \otimes \mathrm{L}(2,0) \simeq \mathrm{T}(3,0)$;

(4) $\mathrm{L}(1,0) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(2,1) \oplus \mathrm{T}(0,2)$;
(5) $\mathrm{L}(1,0) \otimes \mathrm{L}(0,2) \simeq \mathrm{T}(1,2) \oplus \mathrm{T}(0,1)$;
(6) $\mathrm{L}(1,0) \otimes \mathrm{L}(2,1) \simeq \mathrm{T}(3,1) \oplus \mathrm{T}(2,0)$;
(7) $\mathrm{L}(1,0) \otimes \mathrm{L}(1,2) \simeq \mathrm{T}(2,2) \oplus \mathrm{T}(0,3)$;
(8) $\mathrm{L}(1,0) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(3,2)$;
(9) $\mathrm{L}(2,0) \otimes \mathrm{L}(2,0) \simeq \mathrm{T}(4,0) \oplus \mathrm{T}(2,1)$;
(10) $\mathrm{L}(2,0) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(3,1) \oplus \mathrm{T}(0,1)$;
(11) $\mathrm{L}(2,0) \otimes \mathrm{L}(0,2) \simeq \mathrm{T}(2,2) \oplus \mathrm{T}(1,1)$;
(12) $\mathrm{L}(2,0) \otimes \mathrm{L}(2,1) \simeq \mathrm{T}(4,1) \oplus \mathrm{T}(2,2)$;
(13) $\mathrm{L}(2,0) \otimes \mathrm{L}(1,2) \simeq \mathrm{T}(3,2) \oplus \mathrm{T}(0,2) \oplus \mathrm{T}(1,0)$;
(14) $\mathrm{L}(2,0) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(4,2) \oplus \mathrm{T}(2,3)$;
(15) $\mathrm{L}(1,1) \otimes \mathrm{L}(1,1) \simeq \mathrm{T}(2,2) \oplus \mathrm{T}(0,0) \oplus \mathrm{M}$;
(16) $\mathrm{L}(1,1) \otimes \mathrm{L}(2,1) \simeq \mathrm{T}(3,2) \oplus \mathrm{T}(4,0) \oplus \mathrm{T}(1,0)$;
(17) $\mathrm{L}(1,1) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(3,3) \oplus \mathrm{T}(2,2)$;
(18) $\mathrm{L}(2,1) \otimes \mathrm{L}(2,1) \simeq \mathrm{T}(4,2) \oplus \mathrm{T}(5,0) \oplus \mathrm{T}(2,3) \oplus \mathrm{T}(3,1)$;
(19) $\mathrm{L}(2,1) \otimes \mathrm{L}(1,2) \simeq \mathrm{T}(3,3) \oplus 2\mathrm{T}(2,2) \oplus \mathrm{T}(1,1)$;
(20) $\mathrm{L}(2,1) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(4,3) \oplus 2\mathrm{T}(3,2) \oplus \mathrm{T}(2,4)$;
(21) $\mathrm{L}(2,2) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(4,4) \oplus \mathrm{T}(3,3) \oplus \mathrm{T}(5,2) \oplus \mathrm{T}(2,5) \oplus 3\mathrm{T}(2,2)$.

Thus the family $\mathfrak{F}$ *is in this case given by the twenty-five tilting modules* $\{\mathrm{T}(a,b) : 0 \leqslant a, b \leqslant 4\}$ *along with the six "exceptional" modules*

$$\{\mathrm{T}(5,0), \mathrm{T}(0,5), \mathrm{T}(5,2), \mathrm{T}(2,5), \mathrm{L}(1,1), \mathrm{M}\}.$$

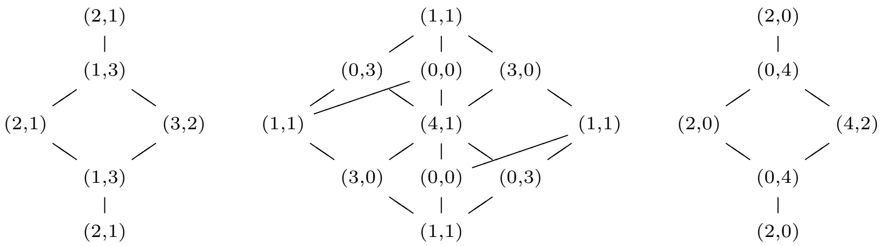
All members of $\mathfrak{F}$ *except* $\mathrm{L}(1,1)$ *and* M *are tilting modules.*

(b) *The uniserial members of* $\mathfrak{F}$ *have the following structure:*

$\mathrm{T}(0,0) = [(0,0)]$; $\mathrm{T}(1,0) = [(1,0)]$; $\mathrm{T}(2,0) = [(2,0)]$;
$\mathrm{T}(1,1) = [(0,0),(1,1),(0,0)]$; $\mathrm{T}(2,1) = [(2,1)]$;
$\mathrm{T}(4,0) = [(0,2),(4,0),(0,2)]$; $\mathrm{T}(3,1) = [(1,2),(3,1),(1,2)]$;
$\mathrm{T}(2,2) = [(2,2)]$; $\mathrm{T}(5,0) = [(0,1),(5,0),(0,1)]$; $\mathrm{T}(5,2) = [(5,2)]$.

The structure of the non-uniserial rigid members of $\mathfrak{F}$ *is given below (symmetric cases omitted):*

$$\begin{tikzcd}[column sep=small] & (1,1) \arrow[dl, dash] \arrow[dr, dash] & \\ (3,0) \arrow[dr, dash] & & (0,0) \arrow[dl, dash] \\ & (1,1) & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (1,1) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ (3,0) \arrow[dr, dash] & (0,0) \arrow[d, dash] & (0,3) \arrow[dl, dash] \\ & (1,1) & \end{tikzcd}$$



all of which are tilting modules excepting the module M *(which does not have a highest weight) pictured at the upper right. Finally, there are four members of* $\mathfrak{F}$*, namely* $\mathrm{T}(3,3)$*,* $\mathrm{T}(4,3)$*,* $\mathrm{T}(3,4)$*, and* $\mathrm{T}(4,4)$*, whose structure is not rigid, which are not pictured. Analysis of their structure requires other methods (see the Appendix).*

(c) *Each member of* $\mathfrak{F}$ *except* $\mathrm{T}(5,2) = \mathrm{L}(5,2)$*,* $\mathrm{T}(2,5) = \mathrm{L}(2,5)$ *has simple* G_1T*-socle (and head) of highest weight belonging to* X_1*.*

Remark. The Alperin diagram for $\mathrm{T}(4,1)$ given above is one of several possibilities. When a module has a direct sum of two or more copies of the same simple on a given socle layer, there may be more than one diagram.

The proof of the proposition will be given in 4.4–4.12. First we consider some consequences and look at a few examples.

Corollary. *Let* $p = 3$*. Given arbitrary dominant weights* λ, μ *write* $\lambda = \sum \lambda^j p^j$*,* $\mu = \sum \mu^j p^j$ *with each* $\lambda^j, \mu^j \in X_1$*.*

(a) *In the decomposition* (1.1.3)*, each term in the direct sum not involving a tensor factor of the form* $\mathrm{T}(5,2)$*,* $\mathrm{T}(2,5)$ *is indecomposable.*

(b) $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ *is indecomposable if and only if for each* $j \geqslant 0$ *the unordered pair* $\{\lambda^j, \mu^j\}$ *is one of the cases* $\{(1,0),(0,1)]$*,* $\{(1,0),(2,0)]$*,* $\{(1,0),(2,2)]$*,* $\{(0,1),(0,2)]$*,* $\{(0,1),(2,2)]$ *or one of* λ^j*,* μ^j *is the zero weight* $(0,0)$*.*

(c) *Let* m *be the maximum* j *such that at least one of* λ^j*,* μ^j *is non-zero. Then* $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ *is indecomposable tilting, isomorphic to* $\mathrm{T}(\lambda + \mu)$*, if and only if:* (i) *for each* $0 \leqslant j \leqslant m-1$*, one of* λ^j, μ^j *is minuscule and the other is the Steinberg weight* $(2,2)$*, and* (ii) $\{\lambda^m, \mu^m\}$ *is one of the cases listed in part* (b)*.*

Proof. The proof is entirely similar to the proof of the corresponding result in the $p = 2$ case. We leave the details to the reader. $\square$

Examples. (i) We work out the indecomposable direct summands of $\mathrm{L}(5,4) \otimes \mathrm{L}(4,5)$, using information from part (a) of the proposition and following the procedure of

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$$
\begin{aligned}
&\mathrm{L}(5,4)\otimes\mathrm{L}(4,5)\\
&\qquad\simeq \big(\mathrm{L}(2,1)\otimes\mathrm{L}(1,1)^{[1]}\big)\otimes\big(\mathrm{L}(1,2)\otimes\mathrm{L}(1,1)^{[1]}\big)\\
&\qquad\simeq \big(\mathrm{L}(2,1)\otimes\mathrm{L}(1,2)\big)\otimes\big(\mathrm{L}(1,1)\otimes\mathrm{L}(1,1)\big)^{[1]}\\
&\qquad\simeq \big(\mathrm{T}(3,3)\oplus 2\mathrm{T}(2,2)\oplus\mathrm{T}(1,1)\big)\otimes\big(\mathrm{T}(2,2)\oplus\mathrm{T}(0,0)\oplus M\big)^{[1]}\\
&\qquad\simeq \big(\mathrm{T}(3,3)\otimes\mathrm{T}(2,2)^{[1]}\big)\oplus\big(\mathrm{T}(3,3)\otimes\mathrm{T}(0,0)^{[1]}\big)\oplus\big(\mathrm{T}(3,3)\otimes\mathrm{M}^{[1]}\big)\\
&\qquad\qquad\oplus 2\big(\mathrm{T}(2,2)\otimes\mathrm{T}(2,2)^{[1]}\big)\oplus 2\big(\mathrm{T}(2,2)\otimes\mathrm{T}(0,0)^{[1]}\big)\oplus 2\big(\mathrm{T}(2,2)\otimes\mathrm{M}^{[1]}\big)\\
&\qquad\qquad\oplus\big(\mathrm{T}(1,1)\otimes\mathrm{T}(2,2)^{[1]}\big)\oplus\big(\mathrm{T}(1,1)\otimes\mathrm{T}(0,0)^{[1]}\big)\oplus\big(\mathrm{T}(1,1)\otimes\mathrm{M}^{[1]}\big)\\
&\qquad\simeq \mathrm{T}(9,9)\oplus\mathrm{T}(3,3)\oplus\big(\mathrm{T}(3,3)\otimes\mathrm{M}^{[1]}\big)\oplus 2\mathrm{T}(8,8)\oplus 2\mathrm{T}(2,2)\\
&\qquad\qquad\oplus 2\big(\mathrm{T}(2,2)\otimes\mathrm{M}^{[1]}\big)\oplus\mathrm{T}(7,7)\oplus\mathrm{T}(1,1)\oplus\big(\mathrm{T}(1,1)\otimes\mathrm{M}^{[1]}\big).
\end{aligned}
$$

We applied (2.4.4) to get the last line of the calculation.

(ii) Illustrating part (b) of the corollary we have $\mathrm{L}(3,1)\otimes\mathrm{L}(1,3)\simeq\mathrm{L}(0,1)\otimes\mathrm{L}(1,0)^{[1]}\otimes\mathrm{L}(1,0)\otimes\mathrm{L}(0,1)^{[1]}\simeq\mathrm{T}(1,1)\otimes\mathrm{T}(1,1)^{[1]}$, which is indecomposable but not tilting.

(iii) To illustrate part (c) of the corollary we have for instance $\mathrm{L}(4,0)\otimes\mathrm{L}(8,8)\simeq\mathrm{T}(12,8)$ or $\mathrm{L}(5,2)\otimes\mathrm{L}(5,4)\simeq\mathrm{T}(10,6)$.

4.3. We now discuss the problem of computing the indecomposable direct summands (and their multiplicities) of $\mathrm{L}(\lambda)\otimes\mathrm{L}(\mu)$ for arbitrary $\lambda,\mu\in X^+$, in the more difficult case where a direct summand on the right hand side of (1.1.3) is not necessarily indecomposable.

It will be convenient to introduce the notation $\mathfrak{F}_0$ for the set $\mathfrak{F}-\{\mathrm{T}(5,2),\mathrm{T}(2,5)\}$. Then Corollary 4.2(a) says that a direct summand $S=\bigotimes_{j\geqslant 0} I_j^{[j]}$ in (1.1.3) is indecomposable whenever all its tensor factors I_j belong to $\mathfrak{F}_0$.

Consider a summand $S=\bigotimes_{j\geqslant 0} I_j^{[j]}$ in (1.1.3) which is possibly not indecomposable. By Corollary 4.2(a), such a summand must have one or more tensor multiplicands of the form $\mathrm{T}(5,2)$ or $\mathrm{T}(2,5)$. Suppose that in the summand in question I_k is $\mathrm{T}(5,2)$ or $\mathrm{T}(2,5)$. We use the fact that $\mathrm{T}(5,2)=\mathrm{L}(5,2)\simeq\mathrm{L}(2,2)\otimes\mathrm{L}(1,0)^{[1]}$, and similarly $\mathrm{T}(2,5)=\mathrm{L}(2,5)\simeq\mathrm{L}(2,2)\otimes\mathrm{L}(0,1)^{[1]}$. Thus we are forced to consider the possible splitting of $\mathrm{L}(1,0)\otimes I_{k+1}$ or $\mathrm{L}(0,1)\otimes I_{k+1}$ in 'degree' $k+1$. (By 'degree' here we just mean the level of j in the twisted tensor product occurring in a direct summand of the right-hand-side of (1.1.3).) There are two cases.

We consider first the case where I_{k+1} is not tilting, i.e., I_{k+1} is either $\mathrm{L}(1,1)$ or M. So we need to split $\mathrm{L}(1,0)\otimes\mathrm{L}(1,1)$, $\mathrm{L}(0,1)\otimes\mathrm{L}(1,1)$, $\mathrm{L}(1,0)\otimes\mathrm{M}$ or $\mathrm{L}(0,1)\otimes\mathrm{M}$. The first two cases are already covered by Corollary 4.2(a), so we just need to consider the last two. But a simple calculation with characters and consideration of linkage classes shows that

$$\begin{aligned} \mathrm{L}(1,0)\otimes \mathrm{M} &\simeq \mathrm{T}(3,1)\oplus \mathrm{T}(1,0)\oplus \mathrm{T}(4,0);\\ \mathrm{L}(0,1)\otimes \mathrm{M} &\simeq \mathrm{T}(1,3)\oplus \mathrm{T}(0,1)\oplus \mathrm{T}(0,4)\end{aligned} \tag{4.3.1}$$

and the summands are once again members of $\mathfrak{F}$ with restricted socles, so these cases present no problem.

We are left with the case where I_{k+1} is tilting. Then this splitting can be computed since $\mathrm{L}(1,0)=\mathrm{T}(1,0)$ and $\mathrm{L}(0,1)=\mathrm{T}(0,1)$ are tilting, so we are just splitting a tensor product of two tilting modules into a direct sum of indecomposable tilting modules, which can always be done. This new decomposition produces only tilting modules in the family $\mathfrak{F}$ except when I_{k+1} is one of the following cases:

$$\mathrm{T}(5,0),\mathrm{T}(4,1),\mathrm{T}(4,2),\mathrm{T}(5,2),\mathrm{T}(4,3),\text{ and }\mathrm{T}(4,4)$$

or one of their symmetric cousins. Up to lower order terms which again belong to $\mathfrak{F}$, these possibilities, when tensored by $\mathrm{L}(1,0)$ or $\mathrm{L}(0,1)$, produce the new tilting modules

$$\mathrm{T}(6,0),\mathrm{T}(5,1),\mathrm{T}(6,2),\mathrm{T}(5,3),\text{ and }\mathrm{T}(5,4) \tag{4.3.2}$$

and of course their symmetric versions. Now by Donkin's tensor product theorem we have a twisted tensor product decomposition for the last three of these, in terms of members of $\mathfrak{F}$:

$$\begin{aligned} \mathrm{T}(6,2)&\simeq \mathrm{T}(3,2)\otimes \mathrm{T}(1,0)^{[1]},\\ \mathrm{T}(5,3)&\simeq \mathrm{T}(2,3)\otimes \mathrm{T}(1,0)^{[1]},\\ \mathrm{T}(5,4)&\simeq \mathrm{T}(2,4)\otimes \mathrm{T}(1,0)^{[1]}.\end{aligned} \tag{4.3.3}$$

Hence, those summands and their symmetric versions present no problem. Finally, if $\mathrm{T}(5,0)$ or $\mathrm{T}(4,1)$ is tensored by $\mathrm{L}(1,0)$ then, modulo lower order terms which belong to $\mathfrak{F}$, we obtain the new summands $\mathrm{T}(6,0)$ and $\mathrm{T}(5,1)$ which are not members of $\mathfrak{F}$ and do not admit a twisted tensor product decomposition. However, these summands must have simple restricted G_1T-socles, since they are embedded in $\mathrm{T}(4,4)$ and $\mathrm{T}(4,3)$, respectively. This is shown by translation arguments, similar to those in 4.12 ahead. Thus we have proved the following result.

Proposition. *Let $p=3$ and $G=\mathrm{SL}_3$.*

(a) *Any tensor product of the form $\mathrm{L}(1,0)\otimes I$ or $\mathrm{L}(0,1)\otimes I$, where I is an indecomposable tilting module in $\mathfrak{F}$, is expressible as a twisted tensor product of modules which are either tilting modules in $\mathfrak{F}_0$ or are one of the "extra" modules $\mathrm{T}(6,0)$, $\mathrm{T}(5,1)$, $\mathrm{T}(0,6)$ or $\mathrm{T}(1,5)$.*

(b) *The extra modules have simple restricted G_1T-socles.*

(c) *For general* $\lambda, \mu \in X^+$*, the indecomposable direct summands of* $\mathrm{L}(\lambda) \otimes \mathrm{L}(\mu)$ *are all expressible as twisted tensor products of modules from the family*

$$\begin{aligned}\mathfrak{F}' &= \mathfrak{F}_0 \cup \{\mathrm{T}(6,0), \mathrm{T}(5,1), \mathrm{T}(0,6), \mathrm{T}(1,5)\} \\ &= \{\mathrm{T}(a,b) \colon 0 \leqslant a, b \leqslant 4\} \cup \\ &\qquad \{\mathrm{T}(6,0), \mathrm{T}(5,1), \mathrm{T}(5,0), \mathrm{T}(0,6), \mathrm{T}(1,5), \mathrm{T}(0,5), \mathrm{L}(1,1), \mathrm{M}\}.\end{aligned}$$

Note that all members of $\mathfrak{F}'$ *have simple* G_1T*-socle of restricted highest weight.*

Example. We consider an example where the direct summands on the right hand side of (1.1.3) are not all indecomposable:

$$\begin{aligned}\mathrm{L}(2,2) \otimes \mathrm{L}(5,2) &\simeq \mathrm{L}(2,2) \otimes \mathrm{L}(2,2) \otimes \mathrm{L}(1,0)^{[1]} \\ &\simeq \big(\mathrm{T}(4,4) \oplus \mathrm{T}(3,3) \oplus \mathrm{T}(5,2) \oplus \mathrm{T}(2,5) \oplus 3\mathrm{T}(2,2)\big) \otimes \mathrm{L}(1,0)^{[1]} \\ &\simeq \mathrm{T}(7,4) \oplus \mathrm{T}(6,3) \oplus \mathrm{T}(8,2) \oplus \mathrm{T}(2,5) \oplus \mathrm{T}(5,5) \oplus 3\mathrm{T}(5,2).\end{aligned}$$

The second line comes from equation (21) in Proposition 4.2(a), and to get the last line one applies (2.4.4) repeatedly, using Proposition 4.2(a) again as needed. For instance, using equation (1) from Proposition 4.2(a) we have

$$\begin{aligned}\mathrm{T}(5,2) \otimes \mathrm{L}(1,0)^{[1]} &\simeq \mathrm{L}(2,2) \otimes \big(\mathrm{L}(1,0) \otimes \mathrm{L}(1,0)\big)^{[1]} \\ &\simeq \mathrm{L}(2,2) \otimes \big(\mathrm{T}(2,0) \oplus \mathrm{T}(0,1)\big)^{[1]} \\ &\simeq \mathrm{T}(8,2) \oplus \mathrm{T}(2,5)\end{aligned}$$

and using equation (2) from Proposition 4.2(a) we have

$$\begin{aligned}\mathrm{T}(2,5) \otimes \mathrm{L}(1,0)^{[1]} &\simeq \mathrm{L}(2,2) \otimes \big(\mathrm{L}(0,1) \otimes \mathrm{L}(1,0)\big)^{[1]} \\ &\simeq \mathrm{L}(2,2) \otimes \mathrm{T}(1,1)^{[1]} \simeq \mathrm{T}(5,5).\end{aligned}$$

4.4. We now embark upon the proof of Proposition 4.2. First we compute the composition factor multiplicities of the restricted tensor products. (Recall that $\chi_p(\lambda) = \mathrm{L}(\lambda)$ is the formal character of $\mathrm{L}(\lambda)$.)

(1) $\chi_p(1,0) \cdot \chi_p(1,0) = \chi_p(2,0) + \chi_p(0,1)$;
(2) $\chi_p(1,0) \cdot \chi_p(0,1) = \chi_p(1,1) + 2\chi_p(0,0)$;
(3) $\chi_p(1,0) \cdot \chi_p(2,0) = \chi_p(3,0) + 2\chi_p(1,1) + \chi_p(0,0)$;
(4) $\chi_p(1,0) \cdot \chi_p(1,1) = \chi_p(2,1) + \chi_p(0,2)$;
(5) $\chi_p(1,0) \cdot \chi_p(0,2) = \chi_p(1,2) + \chi_p(0,1)$;
(6) $\chi_p(1,0) \cdot \chi_p(2,1) = \chi_p(3,1) + 2\chi_p(1,2) + \chi_p(2,0)$;
(7) $\chi_p(1,0) \cdot \chi_p(1,2) = \chi_p(2,2) + \chi_p(0,3) + 2\chi_p(1,1) + \chi_p(0,0)$;
(8) $\chi_p(1,0) \cdot \chi_p(2,2) = \chi_p(3,2) + 2\chi_p(1,3) + 3\chi_p(2,1)$;
(9) $\chi_p(2,0) \cdot \chi_p(2,0) = \chi_p(4,0) + \chi_p(2,1) + 2\chi_p(0,2)$;
(10) $\chi_p(2,0) \cdot \chi_p(1,1) = \chi_p(3,1) + 2\chi_p(1,2) + \chi_p(0,1)$;
(11) $\chi_p(2,0) \cdot \chi_p(0,2) = \chi_p(2,2) + \chi_p(1,1) + 2\chi_p(0,0)$;

(12) $\chi_p(2,0)\cdot\chi_p(2,1) = \chi_p(4,1) + \chi_p(2,2) + 2\chi_p(0,3) + 2\chi_p(3,0) + 4\chi_p(1,1) + 2\chi_p(0,0)$;

(13) $\chi_p(2,0)\cdot\chi_p(1,2) = \chi_p(3,2) + 2\chi_p(1,3) + 3\chi_p(2,1) + \chi_p(0,2) + \chi_p(1,0)$;

(14) $\chi_p(2,0)\cdot\chi_p(2,2) = \chi_p(4,2) + \chi_p(2,3) + 2\chi_p(0,4) + 2\chi_p(3,1) + 3\chi_p(1,2) + 3\chi_p(2,0)$;

(15) $\chi_p(1,1)\cdot\chi_p(1,1) = \chi_p(2,2) + \chi_p(0,3) + \chi_p(3,0) + 2\chi_p(1,1) + 2\chi_p(0,0)$;

(16) $\chi_p(1,1)\cdot\chi_p(2,1) = \chi_p(3,2) + 2\chi_p(1,3) + \chi_p(4,0) + 3\chi_p(2,1) + 2\chi_p(0,2) + \chi_p(1,0)$;

(17) $\chi_p(1,1)\cdot\chi_p(2,2) = \chi_p(3,3) + 2\chi_p(1,4) + 2\chi_p(4,1) + \chi_p(2,2) + 4\chi_p(0,3) + 4\chi_p(3,0) + 6\chi_p(1,1) + 5\chi_p(0,0)$;

(18) $\chi_p(2,1)\cdot\chi_p(2,1) = \chi_p(4,2) + \chi_p(2,3) + 2\chi_p(0,4) + \chi_p(5,0) + 3\chi_p(3,1) + 5\chi_p(1,2) + 3\chi_p(2,0) + 2\chi_p(0,1)$;

(19) $\chi_p(2,1)\cdot\chi_p(1,2) = \chi_p(3,3) + 2\chi_p(1,4) + 2\chi_p(4,1) + 2\chi_p(2,2) + 4\chi_p(0,3) + 4\chi_p(3,0) + 7\chi_p(1,1) + 7\chi_p(0,0)$;

(20) $\chi_p(2,1)\cdot\chi_p(2,2) = \chi_p(4,3) + \chi_p(2,4) + 2\chi_p(0,5) + 2\chi_p(5,1) + 2\chi_p(3,2) + 4\chi_p(1,3) + 2\chi_p(4,0) + 6\chi_p(2,1) + 3\chi_p(0,2) + 5\chi_p(1,0)$;

(21) $\chi_p(2,2)\cdot\chi_p(2,2) = \chi_p(4,4) + \chi_p(2,5) + 2\chi_p(0,6) + \chi_p(5,2) + 3\chi_p(3,3) + 6\chi_p(1,4) + 2\chi_p(6,0) + 6\chi_p(4,1) + 3\chi_p(2,2) + 8\chi_p(0,3) + 8\chi_p(3,0) + 11\chi_p(1,1) + 15\chi_p(0,0)$.

It is important to proceed inductively through the cases, so that the structure of smaller tilting modules is available by the time the argument reaches the higher, more complicated, cases. We order the cases as listed in part (a) of Proposition 4.2. In each case, one starts by partitioning the composition factors into blocks. This amounts to looking at linkage classes of the highest weights of those composition factors.

In cases (1)–(9), (11)–(14) the argument is entirely similar to the arguments already used in characteristic 2, in the proof of Proposition 3.2. In these cases we know that the tensor product in question is tilting, and it turns out that each linkage class determines an indecomposable direct summand. This uses the contravariant self-duality of the tilting modules and the structural information in 4.1 for the Weyl modules, which forces a lower bound on the composition length of the tilting module in question, and it turns out that this lower bound agrees with the upper bound provided by the linkage class.

As an example, let us examine the argument in the case (12), for the tensor product $\mathrm{L}(2,0)\otimes\mathrm{L}(2,1)$. The linkage classes are

$$\{(2,2)\}\cup\{(4,1),(3,0),(0,3),(1,1),(0,0)\}.$$

By highest weight considerations, we must have a single copy of $\mathrm{T}(4,1)$ in $\mathrm{L}(2,0)\otimes\mathrm{L}(2,1)$. Linkage forces a copy of $\mathrm{L}(2,2)=\mathrm{T}(2,2)$ to split off as well. Now $\mathrm{T}(4,1)$

has a submodule isomorphic to $\Delta(4,1)$, so $\mathrm{L}(1,1)$ is contained in its socle. By self-duality of $\mathrm{T}(4,1)$, we must have another copy of $\mathrm{L}(1,1)$ in the top of $\mathrm{T}(4,1)$, so we are forced to put a copy of $\Delta(1,1)$ at the top of $\mathrm{T}(4,1)$. Looking at the structure of $\Delta(4,1)$ in 5.1.6, we see that $\mathrm{T}(4,1)$ must also have at least one copy of $\Delta(3,0)$ and $\Delta(0,3)$ in its Δ-filtration. At this point we are finished, since this accounts for all available composition factors (with their multiplicities) from the linkage class, so we conclude that $\mathrm{L}(2,0)\otimes\mathrm{L}(2,1)\simeq\mathrm{T}(4,1)\oplus\mathrm{T}(2,2)$. The structure of $\mathrm{T}(4,1)$ is nearly forced, because of its self-duality, the fact that all the Ext groups between simple factors is known, and the fact that $\mathrm{T}(4,1)$ must have both Δ and ∇-filtrations. In 4.12 we will see that $\mathrm{T}(0,3)$ is isomorphic to a submodule of $\mathrm{T}(4,1)$, which finishes the determination of the structure of $\mathrm{T}(4,1)$.

4.5. In case (10), one cannot immediately conclude that $\mathrm{L}(2,0)\otimes\mathrm{L}(1,1)$ is tilting since $\mathrm{L}(1,1)$ is not tilting, so we must proceed differently. However, we observe the following, which immediately implies that in fact our tensor product is tilting.

Lemma. *Let V be a simple Weyl module and let $\Delta(\lambda)$ be a Weyl module of highest weight λ. If the composition factors of $V\otimes\operatorname{rad}\Delta(\lambda)$ and $V\otimes L(\lambda)$ lie in disjoint blocks, then $V\otimes L(\lambda)$ is tilting.*

Proof. $V\otimes\Delta(\lambda)$ has a Δ-filtration, by the Wang–Donkin–Mathieu result (see [17, II.4.21]). Now as $V\otimes\operatorname{rad}\Delta(\lambda)$ and $V\otimes L(\lambda)$ have no common linkage classes there can be no non-trivial extensions between these modules, by the linkage principle. Thus $V\otimes\Delta(\lambda)=\big(V\otimes\operatorname{rad}\Delta(\lambda)\big)\oplus\big(V\otimes L(\lambda)\big)$. As $V\otimes\Delta(\lambda)$ has a Δ-filtration this implies $V\otimes L(\lambda)$ does also. As it is the tensor product of two simple (therefore contravariantly self dual) modules it is itself contravariantly self dual and so has a ∇-filtration. Therefore it is tilting. $\square$

Now we may proceed as usual. Looking at the character of $\mathrm{L}(2,0)\otimes\mathrm{L}(1,1)$ we find that there are two linkage classes for the highest weights of the composition factors, namely $\{(0,1)\}$ and $\{(3,1),(1,2)\}$. Since the multiplicity of $\mathrm{L}(0,1)$ is 1, it must give a simple tilting summand $\mathrm{T}(0,1)$. Now $\mathrm{T}(3,1)$ must be a summand by highest weight consideration, and the usual argument forces it to have at least composition length three, which forces equality of the upper and lower bounds, so the structure is $\mathrm{T}(3,1)=[(1,2),(3,1),(1,2)]$ and we have $\mathrm{L}(2,0)\otimes\mathrm{L}(1,1)\simeq\mathrm{T}(3,1)\oplus\mathrm{T}(0,1)$. This takes care of case (10) in our list.

Case (16) follows similarly, making use again of the above lemma to conclude that $\mathrm{L}(1,1)\otimes\mathrm{L}(2,1)$ is tilting. We note that at this stage we may assume that the structure of $\mathrm{T}(3,2)$ and $\mathrm{T}(4,0)$ are already known, since they come up in the earlier cases (8), (9). So one easily concludes from this and the linkage classes that $\mathrm{L}(1,1)\otimes\mathrm{L}(2,1)\simeq\mathrm{T}(3,2)\oplus\mathrm{T}(4,0)\oplus\mathrm{T}(1,0)$.

4.6. We now consider case (15). Since $\mathrm{L}(1,1)$ is not tilting, it is unclear whether or not $\mathrm{L}(1,1) \otimes \mathrm{L}(1,1)$ is tilting. In fact it is not, and analysis of this case is more difficult. First, looking at the character and the linkage classes (there are two) we observe that a copy of the Steinberg module $\mathrm{T}(2,2) = \mathrm{L}(2,2)$ splits off as a direct summand. The remaining composition factors of the tensor product all lie in the same linkage class, but it turns out that a copy of the trivial module splits off, as we show below.

From properties of duals and previous calculations it follows that

$$\begin{aligned} &\dim_K \mathrm{Hom}_G(\mathrm{L}(0,0), \mathrm{L}(1,1) \otimes \mathrm{L}(1,1)) \\ &\qquad = \dim_K \mathrm{Hom}_G(\mathrm{L}(0,0) \otimes \mathrm{L}(1,1), \mathrm{L}(1,1)) \\ &\qquad = \dim_K \mathrm{Hom}_G(\mathrm{L}(1,1), \mathrm{L}(1,1)) = 1; \end{aligned} \tag{1}$$

$$\begin{aligned} &\dim_K \mathrm{Hom}_G(\mathrm{T}(1,1), \mathrm{L}(1,1) \otimes \mathrm{L}(1,1)) \\ &\qquad = \dim_K \mathrm{Hom}_G(\mathrm{L}(1,0) \otimes \mathrm{L}(0,1), \mathrm{L}(1,1) \otimes \mathrm{L}(1,1)) \\ &\qquad = \dim_K \mathrm{Hom}_G(\mathrm{L}(1,1) \otimes \mathrm{L}(0,1), \mathrm{L}(1,1) \otimes \mathrm{L}(0,1)) \\ &\qquad = \dim_K \mathrm{Hom}_G(\mathrm{L}(1,2) \oplus \mathrm{L}(2,0), \mathrm{L}(1,2) \oplus \mathrm{L}(2,0)) = 2; \end{aligned} \tag{2}$$

$$\begin{aligned} &\dim_K \mathrm{Hom}_G(\mathrm{L}(3,0), \mathrm{L}(1,1) \otimes \mathrm{L}(1,1)) \\ &\qquad = \dim_K \mathrm{Hom}_G(\mathrm{L}(1,1) \otimes \mathrm{L}(3,0), \mathrm{L}(1,1)) \\ &\qquad = \dim_K \mathrm{Hom}_G(\mathrm{L}(4,1), \mathrm{L}(1,1)) = 0 \end{aligned} \tag{3}$$

and, by symmetry, an equality similar to (3) holds, in which $(3,0)$ is replaced by $(0,3)$. We also observe that

$$\mathrm{Hom}_G(\mathrm{L}(1,1), \mathrm{L}(1,1) \otimes \mathrm{L}(1,1)) \simeq \mathrm{Hom}_G(\mathrm{L}(1,1) \otimes \mathrm{L}(1,1), \mathrm{L}(1,1)) \tag{4}$$

By (1), (3), and (4) we see that the socle of $\mathrm{L}(1,1) \otimes \mathrm{L}(1,1)$ is either: (a) $\mathrm{L}(2,2) \oplus \mathrm{L}(0,0)$, or (b) $\mathrm{L}(2,2) \oplus \mathrm{L}(0,0) \oplus \mathrm{L}(1,1)$.

From the structure of the Weyl modules in question we know (see e.g. [17, II.4.14]) all the Ext^1 groups between the simple modules of interest here. Combining this with self-duality would force the structure of the non-simple direct summand of $\mathrm{L}(1,1) \otimes \mathrm{L}(1,1)$ to be given by one of the following diagrams:

$$\begin{tikzcd}[column sep=small, row sep=small]
& (0,0) \arrow[d, dash] & \\
& (1,1) \arrow[dl, dash] \arrow[dr, dash] & \\
(3,0) \arrow[dr, dash] & & (0,3) \arrow[dl, dash] \\
& (1,1) \arrow[d, dash] & \\
& (0,0) &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small, row sep=small]
& (1,1) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
(3,0) \arrow[dr, dash] & (0,0) \arrow[d, dash] & (0,3) \arrow[dl, dash] \\
& (1,1) &
\end{tikzcd}$$

where the left diagram corresponds with possibility (a) and the right with possibility (b). However, the left diagram would contradict (2). Hence, possibility (a) is in fact ruled out, and we are left with possibility (b). It follows that $\mathrm{L}(1,1)\otimes\mathrm{L}(1,1) \simeq \mathrm{L}(2,2) \oplus \mathrm{L}(0,0) \oplus \mathrm{M}$, as claimed.

4.7. There are just five cases remaining in the proof of Proposition 4.2, namely cases (17)–(21). We now consider case (17). The module $\mathrm{L}(1,1) \otimes \mathrm{L}(2,2)$ is tilting by Lemma 4.5, so by highest weight considerations $\mathrm{T}(3,3)$ is a direct summand. This is also justified by Pillen's Theorem (see 2.1). The character of $\mathrm{T}(3,3)$ may be computed by (2.2.2), which shows that it has a Δ-filtration with Δ-factors isomorphic to

$$\Delta(3,3),\ \Delta(4,1),\ \Delta(1,4),\ \Delta(3,0),\ \Delta(0,3),\ \Delta(1,1)$$

each occurring with multiplicity one. This accounts for all the composition factors appearing in the character of $\mathrm{L}(1,1) \otimes \mathrm{L}(2,2)$, except for one copy of the Steinberg module $\mathrm{T}(2,2) = \mathrm{L}(2,2)$. Hence we conclude that

$$\mathrm{L}(1,1) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(3,3) \oplus \mathrm{T}(2,2).$$

4.8. $\mathrm{L}(2,1) \otimes \mathrm{L}(2,1)$ is tilting since $\mathrm{L}(2,1)$ is, so by highest weight considerations a copy of $\mathrm{T}(4,2)$ splits off as a direct summand. The structure of $\mathrm{T}(4,2)$ was determined in a previous case of the proof. Subtracting its character from the character of $\mathrm{L}(2,1)\otimes\mathrm{L}(2,1)$, we see that the highest weight of what remains is $(5,0)$, so a copy of $\mathrm{T}(5,0)$ must split off as well. The linkage class of $(5,0)$ contains only two weights $\{(5,0),(0,1)\}$ and from this and the known structure of the Weyl modules it follows easily that $\mathrm{T}(5,0)$ is uniserial with structure $\mathrm{T}(5,0) = [(0,1),(5,0),(0,1)]$. Now highest weight and character considerations force the remaining summands to be one copy of $\mathrm{T}(2,3)$ and one copy of $\mathrm{T}(3,1)$. Hence

$$\mathrm{L}(2,1) \otimes \mathrm{L}(2,1) \simeq \mathrm{T}(4,2) \oplus \mathrm{T}(5,0) \oplus \mathrm{T}(2,3) \oplus \mathrm{T}(3,1).$$

We note we can assume that $\mathrm{T}(3,1)$ and $\mathrm{T}(2,3)$ are known at this point, since they arise in earlier cases of the proof. (Actually, to be precise $\mathrm{T}(2,3)$ doesn't arise in any earlier case, but its symmetric cousin $\mathrm{T}(3,2)$ does.)

4.9. $\mathrm{L}(2,1) \otimes \mathrm{L}(1,2)$ is tilting since both $\mathrm{L}(2,1)$ and $\mathrm{L}(1,2)$ are, so by highest weight considerations a copy of $\mathrm{T}(3,3)$ splits off as a direct summand. The character of $\mathrm{T}(3,3)$ was computed already in 4.7, so by character considerations one easily deduces that

$$\mathrm{L}(2,1) \otimes \mathrm{L}(1,2) \simeq \mathrm{T}(3,3) \oplus 2\mathrm{T}(2,2) \oplus \mathrm{T}(1,1).$$

Of course, the character of $\mathrm{T}(1,1)$ is already known by an earlier case of the proof.

4.10. $\mathrm{L}(2,1) \otimes \mathrm{L}(2,2)$ is tilting since both $\mathrm{L}(2,1)$ and $\mathrm{L}(2,2)$ are, so by highest weight considerations a copy of $\mathrm{T}(4,3)$ splits off as a direct summand. From [8, §2.1, Lemma 5] we compute its Δ-factors to be

$$\Delta(4,3),\ \Delta(5,1),\ \Delta(0,5),\ \Delta(1,0).$$

One sees also that $\mathrm{T}(4,3)$ has simple socle of highest weight $(1,0)$ by arguments similar to those in 4.7. From character computations one now shows that

$$\mathrm{L}(2,1) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(4,3) \oplus 2\mathrm{T}(3,2) \oplus \mathrm{T}(2,4).$$

The structure of $\mathrm{T}(3,2)$ is available by a previous case of the proof, and the structure of $\mathrm{T}(2,4)$ follows by symmetry from that of $\mathrm{T}(4,2)$, again a previous case.

4.11. $\mathrm{L}(2,2)\otimes\mathrm{L}(2,2)$ is tilting since $\mathrm{L}(2,2)$ is, so by highest weight considerations a copy of $\mathrm{T}(4,4)$ must split off as a direct summand. The Δ-factor multiplicities of $\mathrm{T}(4,4)$ are computed by [8, §2.1, Lemma 5] to be

$$\Delta(4,4),\ \Delta(6,0),\ \Delta(0,6),\ \Delta(3,3),\ \Delta(4,1),\ \Delta(1,4),\ \Delta(1,1),\ \Delta(0,0)$$

each of multiplicity one. From this, using the character of $\mathrm{L}(2,2)\otimes\mathrm{L}(2,2)$ it follows by highest weight considerations, after subtracting the character of $\mathrm{T}(4,4)$, that a copy of $\mathrm{T}(3,3)$ must also split off as a direct summand. Then it easily follows that

$$\mathrm{L}(2,2) \otimes \mathrm{L}(2,2) \simeq \mathrm{T}(4,4) \oplus \mathrm{T}(3,3) \oplus \mathrm{T}(5,2) \oplus \mathrm{T}(2,5) \oplus 3\mathrm{T}(2,2)$$

where $\mathrm{T}(5,2) = \mathrm{L}(5,2)$, $\mathrm{T}(2,5) = \mathrm{L}(2,5)$, and $\mathrm{T}(2,2) = \mathrm{L}(2,2)$.

At this point the proof of Proposition 4.2(a), (b) is complete.

4.12. It remains to prove the claim in part (c) of Proposition 4.2. It is known that Donkin's conjecture holds for SL_3, as discussed at the beginning of 2.4, so $\mathrm{T}((p-1)\rho+\lambda)$ is as a G_1T-module isomorphic to $\widehat{\mathrm{Q}}_1((p-1)\rho + w_0\lambda)$ for any $\lambda \in X_1$. Thus $\mathrm{T}(a,b)$ has simple G_1T-socle of restricted highest weight, for any $2 \leqslant a,b \leqslant 4$. Moreover, the claim is true of $\mathrm{T}(0,0)$, $\mathrm{T}(1,0)$, $\mathrm{T}(2,0)$, $\mathrm{T}(2,1)$, $\mathrm{L}(1,1)$ and their symmetric counterparts, since these are all simple G-modules of restricted highest weight.

For $\lambda = (1,1)$ and $(5,0)$ one easily checks by direct computation that $\Delta(\lambda)$, which is a non-split extension between two simple G-modules, remains non-split upon restriction to G_1T. It then follows that $\mathrm{T}(\lambda)$ has simple G_1T-socle of restricted highest weight in each case.

For $\lambda = (4,0)$ and $(3,1)$ one could argue as in the preceding paragraph, or restrict to an appropriate Levi subgroup, as in the last paragraph of 3.3.

The remaining cases, up to symmetry, are $\mathrm{T}(3,0)$, $\mathrm{T}(4,1)$, and M. We apply the translation principle [17, II.E.11]. Observe (from their structure) that $\mathrm{T}(0,2)$ embeds in $\mathrm{T}(4,0)$, which in turn embeds in $\mathrm{T}(2,4)$. Picking $\lambda = (0,0)$ and

$\mu = (-1, 1)$ in the closure of the bottom alcove, observe that applying the (exact) functor T_μ^λ to these embeddings, we obtain embeddings of $\mathrm{T}(0,3)$ in $\mathrm{T}(4,1)$, and $\mathrm{T}(4,1)$ in $\mathrm{T}(3,3)$. Since $\mathrm{T}(3,3)$ has simple G_1T-socle of restricted highest weight, it follows that the same holds for $\mathrm{T}(0,3)$ and $\mathrm{T}(4,1)$. The cases $\mathrm{T}(3,0)$ and $\mathrm{T}(1,4)$ are treated by the symmetric argument. Finally, we observe that $\dim_K \mathrm{Hom}_{G_1T}(\mathrm{L}(0,0), \mathrm{L}(1,1) \otimes \mathrm{L}(1,1)) = 1$, by a calculation similar to 4.6(1). This, along with 4.6, shows that M remains indecomposable on restriction to G_1T, with socle and head isomorphic to $\mathrm{L}(1,1)$, and with 7 copies of $\mathrm{L}(0,0)$ in the middle Loewy layer. The proof of Proposition 4.2 is complete.

4.13. Discussion. We now discuss the remaining issue in characteristic 3: the structure of the tilting modules $\mathrm{T}(\lambda)$ for $\lambda = (3,3)$, $(4,3)$, $(3,4)$, and $(4,4)$. These tilting modules are in fact S-modules for the Schur algebra $S = S_K(3,r)$ in degree $r = 9, 10, 11, 12$, respectively. (See [13], [20] for background on Schur algebras.)

Thus, in order to study the structure of $\mathrm{T}(\lambda)$ one may employ techniques from the theory of finite dimensional quasi-hereditary algebras. Now the simplest cases (in terms of number of composition factors) are $\mathrm{T}(4,3)$ for $S(3,10)$ and $\mathrm{T}(3,4)$ for $S(3,11)$. As these modules are symmetric, it makes sense to focus on the smaller Schur algebra $S(3,10)$ and thus $\mathrm{T}(4,3)$. In fact, it is enough to understand the block A of $S(3,10)$ consisting of the six weights $(10,0)$, $(6,2)$, $(4,3)$, $(5,1)$, $(0,5)$, and $(1,0)$. (It is easily seen that this is a complete linkage class of dominant weights in $S(3,10)$, for instance by drawing the alcove diagrams.) To construct $\mathrm{T}(4,3)$ we must “glue” together the Δ-factors in a way that results in a contravariantly self-dual module. Looking at the diagrams in Figure 1 below picturing the various Weyl

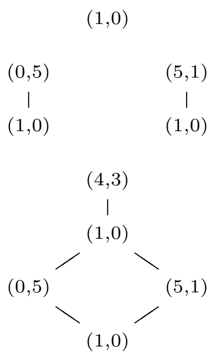

FIGURE 1. Weyl filtration factors of $\mathrm{T}(4,3)$

modules in the filtration, we see that it is impossible to do this in a rigid way. There are three copies of $\mathrm{L}(1,0)$ above the middle factor $\mathrm{L}(4,3)$ and only two below. Thus, there must be two copies of $\mathrm{L}(1,0)$ lying immediately above $\mathrm{L}(4,3)$ when viewing the radical series, and two copies lying immediately below $\mathrm{L}(4,3)$ when viewing the socle series. This implies that $\mathrm{T}(4,3)$ is not rigid. To understand the structure of $\mathrm{T}(4,3)$ one may apply Gabriel's theorem to find a quiver and relations presentation for the basic algebra of the block A, or an appropriate quasi-hereditary quotient thereof. This is carried out in the Appendix. The other cases could be treated similarly.

Note that none of $\mathrm{T}(4,3)$, $\mathrm{T}(3,4)$, $\mathrm{T}(3,3)$, or $\mathrm{T}(4,4)$ is projective as an S-module, because if so, the reciprocity law $(P(\lambda)\colon \Delta(\mu)) = [\nabla(\mu)\colon L(\lambda)]$ (see e.g. [6, Proposition A2.2]) would be violated.

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APPENDIX: THE SL_3-MODULE $T(4,3)$ FOR $p=3$

C. M. Ringel

Let k be an algebraically closed field of characteristic $p = 3$. Following Bowman, Doty and Martin, we consider rational SL_3-modules with composition factors $L(\lambda)$, where λ is one of the weights $(1,0)$, $(0,5)$, $(5,1)$, $(4,3)$, $(6,2)$. Dealing with a dominant weight (a,b), or the simple module $L(a,b)$, we usually will write just ab. The corresponding Weyl module, dual Weyl module, or tilting module, will be denoted by $\Delta(ab)$, $\nabla(ab)$ and $T(ab)$, respectively.

The paper [BDM] by Bowman, Doty and Martin describes in detail the structure of the modules $\Delta(\lambda), \nabla(\lambda)$ for $\lambda = 10,\ 05,\ 51,\ 43,\ 62$ and also $T(10),\ T(05),\ T(51)$ and it provides the factors of a Δ-filtration for $T(43)$. This module $T(43)$ is still quite small (it has length 10), but its structure is not completely obvious at first sight. The main aim of this appendix is to explain the shape of this module.

Let us call a finite set I of dominant weights (or of simple modules) an *ideal* provided for any $\lambda \in I$ all composition factors of $T(\lambda)$ belong to I. The category of modules with all composition factors in an ideal I is a highest weight category with weight set I, thus can be identified with the module category of a basic quasi-hereditary algebra which we denote by $A(I)$. In order to analyse the module $T(43)$, we need to look at the ideal $I = \{10, 05, 51, 43\}$, thus at the algebra $A(10, 05, 51, 43)$.

In order to determine the precise relations for $A(10, 05, 51, 43)$, we will have to look also at the module $T(62)$, see section 4. Note that $\{10,\ 05,\ 51,\ 43,\ 62\}$ is again an ideal, thus we deal with the algebra $A(10, 05, 51, 43, 62)$.

The use of quivers and relations for presenting a basic finite dimensional algebras was initiated by Gabriel around 1970, the text books [ARS] and [ASS] can be used as a reference. The class of quasi-hereditary algebras was introduced by Scott and Cline-Parshall-Scott; for basic properties one may refer to [DR] and [R2]. The author is grateful to S. Doty and R. Farnsteiner for fruitful discussions and helpful suggestions concerning the material presented in the appendix.

1. The main result

Deviating from [BDM], we will consider **right** modules. Thus, given a finite-dimensional algebra A, an indecomposable projective A-module is of the form eA with e a primitive idempotent. The algebras to be considered will be factor algebras of path algebras of quivers and the advantage of looking at right modules will be that in this way we can write the paths in the quiver as going from left to right.

Proposition. *The algebra* $A(10,05,51,43)$ *is isomorphic to the path algebra of the quiver*

$$Q = Q(10,05,51,43) \qquad \begin{tikzcd} & 05 \arrow[dl, shift left, "\alpha'"] & \\ 10 \arrow[ur, shift left, "\alpha"] \arrow[rr, shift left, "\gamma"] \arrow[dr, shift left, "\beta"] & & 43 \arrow[ll, shift left, "\gamma'"] \\ & 51 \arrow[ul, shift left, "\beta'"] & \end{tikzcd}$$

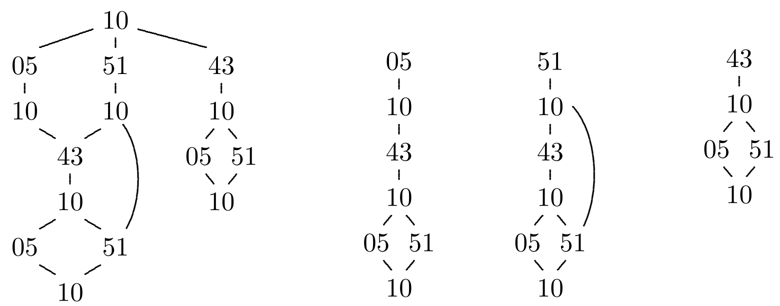
modulo the ideal generated by the following relations

$$\alpha'\alpha = 0, \quad \alpha'\beta = 0, \quad \beta'\alpha = 0, \quad \beta'(1-\gamma\gamma')\beta = 0,$$
$$\gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - \beta\beta') = 0, \quad (\alpha\alpha' - \beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0.$$

We are going to give some comments before embarking on the proof.

(1) Since the quiver $Q(10,05,51,43)$ is bipartite, say with a $(+)$-vertex 10 and three $(-)$-vertices $05,\ 51,\ 43$, possible relations between vertices of the same parity involve paths of even lengths, those between vertices with different parity involve paths of odd lengths. Our convention for labelling arrows between a $(+)$-vertex a and a $(-)$-vertex b is the following: we use a greek letter for the arrow $a \to b$ and add a dash for the arrow $b \to a$.

(2) The assertion of the proposition can be visualized by drawing the shape of the indecomposable projective A-modules. The indecomposable projective A-module with top λ will be denoted by $P(\lambda) = e_\lambda A$, where e_λ is the primitive idempotent corresponding to λ, and we will denote the radical of A by J.



These are the coefficient quivers of the indecomposable projective A-modules with respect to suitable bases. In addition, the proposition asserts that all the non-zero coefficents can be chosen to be equal to 1. Note that this means that A has a basis $\mathcal{B}$ which consists of a complete set of primitive and orthogonal idempotents as well as of elements from the radical J, and such that $\mathcal{B}$ is multiplicative (this means: if $u, v \in \mathcal{B}$, then either $uv = 0$ or else $uv \in \mathcal{B}$).

For the convenience of the reader, let us recall the notion of a coefficient quiver (see for example [R3]): By definition, a representation M of a quiver Q over a field k is of the form $M = (M_x; M_\alpha)_{x,\alpha}$; here, for every vertex x of Q, there is given a finite-dimensional k-space M_x, say of dimension d_x, and for every arrow $\alpha : x \to y$, there is given a linear transformation $M_\alpha : M_x \to M_y$. A *basis* $\mathcal{B}$ of M is by definition a subset of the disjoint union of the various k-spaces M_x such that for any vertex x the set $\mathcal{B}_x = \mathcal{B} \cap M_x$ is a basis of M_x. Now assume that there is given a basis $\mathcal{B}$ of M. For any arrow $\alpha : x \to y$, write M_α as a $(d_x \times d_y)$-matrix $M_{\alpha,\mathcal{B}}$ whose rows are indexed by $\mathcal{B}_x$ and whose columns are indexed by $\mathcal{B}_y$. We denote by $M_{\alpha,\mathcal{B}}(b, b')$ the corresponding matrix coefficients, where $b \in \mathcal{B}_x$, $b' \in \mathcal{B}_y$, these matrix coefficients $M_{\alpha,\mathcal{B}}(b, b')$ are defined by $M_\alpha(b) = \sum_{b' \in \mathcal{B}} b' \, M_{\alpha,\mathcal{B}}(b, b')$. By definition, the *coefficient quiver* $\Gamma(M, \mathcal{B})$ of M with respect to $\mathcal{B}$ has the set $\mathcal{B}$ as set of vertices, and there is an arrow (α, b, b') provided $M_{\alpha,\mathcal{B}}(b, b') \neq 0$ (and we call $M_{\alpha,\mathcal{B}}(b, b')$ the corresponding coefficient). If b belongs to $\mathcal{B}_x$, we usually label the vertex b_x by x. If necessary, we label the arrow (α, b, b') by α; but since we only deal with quivers without multiple arrows, the labelling of arrows could be omitted. In all cases considered in the appendix, we can arrange the vertices in such a way that all the arrows point downwards, and then replace arrows by edges. This convention will be used throughout.

Note that there is a long-standing tradition in matrix theory to focus attention to such coefficient quivers (see e.g. [BR]), whereas the representation theory of groups and algebras is quite reluctant to use them.

Looking at the pictures one should be aware that the four upper base elements form a complete set of primitive and orthogonal idempotents, thus these are the generators of the indecomposable projective A-modules. Those directly below generate the radical of A, and they are just the arrows of the quiver (or better: the residue classes of the arrows in the factor algebra of the path algebra modulo the relations). Of course, on the left we see $P(10)$, then $P(05)$ and $P(51)$, and finally, on the right, $P(43)$.

(3) The strange relation $\beta'(1 - \gamma\gamma')\beta = 0$ leads to the curved edge in $P(51)$ as well as in $P(10)$. Note that the submodule lattice of $P(51)$ would not at all be changed when deleting this extra line — but its effect would be seen in $P(10)$. Namely, without this extra line, the socle of $P(10)$ would be of length 3 (namely, top $\mathrm{rad}^2 P(10)$ is the direct sum of three copies of 10, and the two copies displayed in the left part are both mapped under γ to 43, thus there is a diagonal which is mapped under γ to zero; without the curved line, this diagonal would belong to the socle), whereas the socle of $P(10)$ is of length 2.

(4) Looking at the first four relations presented above, one could have the feeling of a certain asymmetry concerning the role of $P(05)$ and $P(51)$, or also of the role of 05 and 51 as composition factors of the radical of $P(51)$. But such a feeling is

misleading as will be seen in the proof. The pretended lack of symmetry concerns also our display of $T(43)$. Sections 7 and 8 will be devoted to a detailed analysis of the module $T(43)$ in order to focus the attention to its hidden symmetries.

(5) Note that all the tilting A-modules are local (and also colocal):

$$\begin{array}{rcl} T(10) & = & P(10)/(\alpha A+\beta A+\gamma A) \\ T(05) & = & P(10)/(\beta A+\gamma A), \\ T(51) & = & P(10)/(\alpha A+\gamma A), \\ T(43) & = & P(10)/\gamma A. \end{array}$$

As we have mentioned, sections 7 and 8 will discus in more detail the module $T(43)$.

(6) A further comment: One may be surprised to see that one can find relations which are not complicated at all: many are monomials, the remaining ones are differences of monomials, always using paths of length at most 4.

2. Preliminaries on algebras and the presentation of algebras using quivers and relations

Let t be a natural number. Recall that the zero module has Loewy length 0 and that a module M is said to have *Loewy length at most* t with $t \geqslant 1$, provided it has a submodule M' of Loewy length at most $t-1$ such that M/M' is semisimple. Given a module M, we denote by $\operatorname{soc}_t M$ the maximal submodule of Loewy length at most t, and by $\operatorname{top}^t M$ the maximal factor module of Loewy length t. Of course, we write $\operatorname{soc} = \operatorname{soc}_1$ and $\operatorname{top} = \operatorname{top}^1$, but also $\operatorname{top}^t M = M/\operatorname{rad}^t M$.

Let A be a finite-dimensional basic algebra with radical J and quiver Q. Let us assume that Q has no multiple arrows (which is the case for all the quivers considered here). For any arrow $\zeta : i \to j$ in Q, we choose an element $\eta(\zeta) \in e_iJe_j \setminus e_iJ^2e_j$; the set of elements $\eta(\zeta)$ will be called a *generator choice* for A. In this way, we obtain a surjective algebra homomorphisms

$$\eta : kQ \to A$$

If ρ is the kernel of η, then $\rho = \bigoplus_{ij} e_i\rho e_j$, and we call a generating set for ρ consisting of elements in $\bigcup_{ij} e_i\rho e_j$ a *set of relations* for A. We are looking for a generator choice for the algebra $A(10, 05, 51, 43)$ which allows to see clearly the structure of $T(43)$. Usually, we will write ζ instead of $\eta(\zeta)$ and hope this will not produce confusion. If $\zeta \in e_iJe_j \setminus e_iJ^2e_j$ belongs to a generator choice, we obviously may replace it by any element of the form $c\zeta + d$ with $0 \neq c \in k$ and $d \in e_iJ^2e_j$ and obtain a new generator choice.

3. The algebra $B = A(10, 05, 51)$

Consider a quasi-hereditary algebra B with quiver being the full subquiver of $Q(10, 05, 51, 43)$ with vertices 10, 05, 51 and with ordering $10 < 05$, $10 < 51$. It is well-known (and easy to see) that B is uniquely determined by these data. The indecomposable projectives have the following shape

$$\begin{array}{c}05\\ \mid\\ 10\end{array} \qquad \begin{array}{c}51\\ \mid\\ 10\end{array} \qquad \begin{array}{ccc} & 10 & \\ \diagup & & \diagdown \\ 05 & & 51 \\ \mid & & \mid \\ 10 & & 10\end{array}$$

What we display are the again coefficient quivers of the indecomposable projective B-modules considered as representations of kQ with respect to a suitable basis.

We see that the algebra B is of Loewy length 3 and that it can be described by the relations:

$$\alpha'\alpha = \alpha'\beta = \beta'\alpha = \beta'\beta = 0.$$

Of course, $\Delta(10) = \nabla(10) = 10$; and the modules $\Delta(05)$, $\Delta(51)$, $\nabla(05)$ and $\nabla(51)$ are serial of length 2, always with 10 as one of the composition factors. This means that the structure of the modules $\Delta(\lambda)$, $\nabla(\lambda)$, for $\lambda = 10,\ 05\ 51$ can be read off from the quiver (but, of course, conversely, the quiver was obtained from the knowledge of the corresponding Δ- and ∇-modules).

Note that $T(05)$ is the only indecomposable module with a Δ-filtration with factors $\Delta(10)$ and $\Delta(05)$, since $\mathrm{Ext}^1(\Delta(10), \Delta(05)) = k$. Similarly, $T(51)$ is the only indecomposable module with a Δ-filtration with factors $\Delta(10)$ and $\Delta(51)$.

Let us remark that the structure of the module category mod B is well-known: using covering theory, one observes that mod B is obtained from the category of representations of the affine quiver of type $\widetilde{A}_{22}$ with a unique sink and a unique source by identifying the simple projective module with the simple injective module. In mod B, there is a family of homogeneous tubes indexed by $k \setminus \{0\}$, the modules on the boundary are of length 4 with top and socle equal to 10 and with $\mathrm{rad}/\mathrm{soc} = 05 \oplus 51$. We will call these modules the *homogeneous B-modules of length* 4. (The representation theory of affine quivers can be found for example in [R1] and [SS]; from covering theory, we need only the process of removing a node, see [M].)

4. The modules $\mathrm{rad}\,\Delta(43)$ and $\nabla(43)/\mathrm{soc}$ are isomorphic

We will use the following information concerning the modules $\Delta(43)$ and $\nabla(43)$, see [BDM]. Both $\mathrm{rad}\,\Delta(43)$ and $\nabla(43)/\mathrm{soc}$ are homogeneous B-modules of length

4, thus the modules $\Delta(43)$ and $\nabla(43)$ have the following shape

$$\Delta(43)\quad \begin{tikzcd} & 43 \arrow[d, dash] & \\ & 10 \arrow[dl, dash] \arrow[dr, dash] & \\ 05 \arrow[dr, dash] & & 51 \arrow[dl, dash] \\ & 10 & \end{tikzcd} \qquad\qquad \nabla(43)\quad \begin{tikzcd} & 10 \arrow[dl, dash] \arrow[dr, dash] & \\ 05 \arrow[dr, dash] & & 51 \arrow[dl, dash] \\ & 10 \arrow[d, dash] & \\ & 43 & \end{tikzcd}$$

Here, we have drawn again coefficient quivers with respect to suitable bases. But note that we do not (yet) claim that all the non-zero coefficients can be chosen to be equal to 1.

In order to show the assertion in the title, we have to expand our considerations taking into account also the weight 62. The existence of an isomorphism in question will be obtained by looking at the tilting module $T(62)$.

In dealing with a tilting module $T(\mu)$, there is a unique submodule isomorphic to $\Delta(\mu)$, and a unique factor module isomorphic to $\nabla(\mu)$. Let $R(\mu) = \operatorname{rad}\,\Delta(\mu)$ and let $Q(\mu)$ be the kernel of the canonical map $\pi : T(\mu) \to \nabla(\mu)/\operatorname{soc}$. Note that $\Delta(\mu) \subseteq Q(\mu)$ (namely, if $\pi(\Delta(\mu))$ would not be zero, then it would be a submodule of $\nabla(\mu)/\operatorname{soc}$ with top equal to μ; however $\nabla(\mu)/\operatorname{soc}$ has no composition factor of the form μ). It follows that $R(\mu) \subset Q(\mu)$ and we call $C(\mu) = Q(\mu)/R(\mu)$ the *core* of the tilting module $T(\mu)$. Also, we see that $\mu = \Delta(\mu)/R(\mu)$ is a simple submodule of $C(\mu)$. In fact, *μ is a direct summand of $C(\mu)$.* Namely, there is $U \subset T(\mu)$ with $T(\mu)/U = \nabla(\mu)$. Then $U \subset Q(\mu)$ and $Q(\mu)/U = \mu$. Since $R(\mu) \subset Q(\mu)$ and $R(\mu)$ has no composition factor of the form μ, it follows that $R(\mu) \subseteq U$. Altogether, we see that $U + \Delta(\mu) = Q(\mu)$ and $U \cap \Delta(\mu) = R(\mu)$. Thus $Q(\mu)/R(\mu) = U/R(\mu) \oplus \Delta(\mu)/R(\mu) = U/R(\mu) \oplus \mu$.

The module $\Delta(62)$ is serial with going down factors 62, 43, 10, 51, and the module $\nabla(62)$ is serial with going down factors 51, 10, 43, 62, see [BDM], 4.1. Also we will use that $T(62)$ has Δ-factors $\Delta(51)$, $\Delta(43)$, $\Delta(62)$, each with multiplicity one (and thus ∇-factors $\nabla(62)$, $\nabla(43)$, $\nabla(51)$). To get the Δ-factors of $T(62)$, one has to use [BDM], (2.2.2) along with the known structure of the Deltas (this requires a small calculation, which is left to the reader.)

The quiver $Q(10,05,51,43,62)$ of $A(10,05,51,43,62)$ is

$$Q(10,05,51,43,62)\qquad \begin{tikzcd} & 05 \arrow[dl, shift left, "\alpha'"] & & \\ 10 \arrow[ur, shift left, "\alpha"] \arrow[rr, shift left, "\gamma"] \arrow[dr, shift left, "\beta"] & & 43 \arrow[ll, shift left, "\gamma'"] \arrow[dr, shift left, "\delta'"] & \\ & 51 \arrow[ul, shift left, "\beta'"] & & 62 \arrow[ul, shift left, "\delta"] \end{tikzcd}$$

with ordering $10 < 05 < 43 < 62$, and $10 < 51 < 43$.

Lemma 1. *The core of* $T(62)$ *is of the form* $(\mathrm{rad}\ \Delta(43)) \oplus 62$ *as well as of the form* $(\nabla(43)/\operatorname{soc}) \oplus 62$.

Corollary. *The modules* $\mathrm{rad}\ \Delta(43)$ *and* $\nabla(43)/\operatorname{soc}$ *are isomorphic.*

Note that it is quite unusual that the modules $\mathrm{rad}\ \Delta(\lambda)$ and $\nabla(\lambda)/\operatorname{soc}$ are isomorphic, for a weight λ.

Proof of Lemma 1. Let $T_1 \subset T_2 \subset T(62)$ be a filtration with factors

$$T_1 = \Delta(62), \quad T_2/T_1 = \Delta(43), \quad T(62)/T_2 = \Delta(51).$$

Now $R(62) = \mathrm{rad}\ \Delta(62) \subset T_1 \subset T_2$, thus we may look at the factor module $T_2/R(62)$ and the exact sequence

$$0 \to 62 \to T_2/R(62) \to \Delta(43) \to 0$$

(with $62 = T_1/R(62)$). We consider the submodule $N = \mathrm{rad}\ \Delta(43)$ of $\Delta(43)$, with factor module $\Delta(43)/N = 43$. We have $\mathrm{Ext}^1(N, 62) = 0$, since $\mathrm{Ext}^1(S, 62) = 0$ for all the composition factors S of N. This implies that there is an exact sequence

$$0 \to N \oplus 62 \to T_2/R(62) \to 43 \to 0.$$

Thus, there is a submodule $U \subset T_2$ with $R(62) \subset U$ such that $U/R(62)$ is isomorphic to $N \oplus 62$ and T_2/U is isomorphic to 43. Since $T(62)/T_2 = \Delta(51)$ is of length 2, we see that $T(62)/U$ is of length 3.

Now consider the canonical map $\pi : T(62) \to \nabla(62)/\operatorname{soc}$. This map vanishes on $R(62)$, thus induces a map $\pi' : T(62)/R(62) \to \nabla(62)/\operatorname{soc}$. Let us look at the submodule $U/R(62)$ of $T(62)/R(62)$. Since the socle of $\nabla(62)/\operatorname{soc}$ is equal to 43, and $U/R(62) = N \oplus 62$ has no composition factor of the form 43, we see that $U/R(62)$ is contained in the kernel of π', and therefore U is contained in the kernel of π.

By definition, the kernel of the canonical map $\pi : T(62) \to \nabla(62)/\operatorname{soc}$ is $Q(62)$, thus we have shown that $U \subseteq Q(62)$. But $T(62)/U$ is of length 3 as is $T(62)/Q(62)$, thus $U = Q(62)$. But this means that $Q(62)/R(62) = U/R(62) = N \oplus 62 = (\mathrm{rad}\ \Delta(43)) \oplus 62$.

The dual arguments show that $Q(62)/R(62) = (\nabla(43)/\operatorname{soc}) \oplus 62$. $\square$

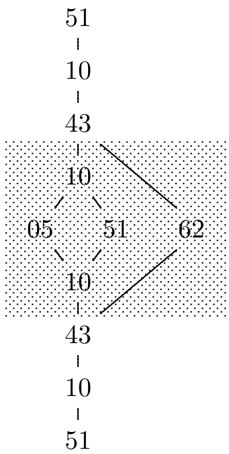
As we have mentioned, the module $N = \mathrm{rad}\ \Delta(43)$ is a B-module, where $B = A(10, 05, 51)$. This algebra B has been discussed in section 3. The coefficient quiver of N is

$$\begin{tikzcd}[arrows=dash] & 10 \arrow[dl] \arrow[dr] & \\ 05 \arrow[dr] & & 51 \arrow[dl] \\ & 10 & \end{tikzcd}$$

Now, choosing a suitable basis of N, we can assume that at least 3 of the non-zero coefficients are equal to 1 and we look at the remaining coefficient, say that for the arrow α. It will be a non-zero scalar c in k. Recall that we have started with a particular generator choice for the algebra B which we can change. If we replace the element $\alpha \in J$ by $\frac{1}{c}\alpha$, then the coefficients needed for N will all be equal to 1.

Remark. Extending the analysis of the Δ- and the ∇-filtrations of $T(43)$, one can show that $T(62)$ is the indecomposable projective $A(10, 05, 51, 43, 62)$-module with top 51 (as well as the indecomposable injective $A(10, 05, 51, 43, 62)$-module with socle 51). As Doty has pointed out, the last assertion follows also from Theorem 5.1 of the DeVisscher-Donkin paper [DD] (that result is based on their Conjecture 5.2 holding, but it is proved in Section 7 of the same paper that the conjecture holds for GL(3); hence it holds also for SL(3)).

Let us add without proof that in this way one may show that the module $T(62)$ has a coefficient quiver of the form



the shaded part being the core of $T(62)$.

5. The module $T(43)$

Lemma 2. *We have* top $T(43) = 10 =$ soc $T(43)$.

Proof. We use that $T(43)$ has Δ-factors $\Delta(10)$, $\Delta(05)$, $\Delta(51)$, $\Delta(43)$ in order to show that top $T(43) = 10$. Since top $T(43)$ is isomorphic to a submodule of the direct sum of the tops of the Δ-factors, it follows that top $T(43)$ is multiplicity free. Since $T(43)$ maps onto $\nabla(43)$, the only composition factor 43 cannot belong to the top.

Actually, it is $N = T(43)/\text{rad}\,\Delta(43)$ which maps onto $\nabla(43)$, and $\nabla(43)$ maps onto $\nabla(05)$ which is serial with top 10 and socle 05; this shows that the only composition factor of the form 05 of N does not belong to top N. Now 05 is not in top N and not in top rad $\Delta(43)$, thus not in top $T(43)$. Similarly, 51 is not in top $T(43)$. It follows that top $T(43) = 10$.

Note that the ∇-factors of $T(43)$ are $\nabla(10)$, $\nabla(05)$, $\nabla(51)$, $\nabla(43)$. Namely, $T(43)$ maps onto $\nabla(43)$, say with kernel N'. The number of composition factors of N' of the form $05, 51, 10$ is $1, 1, 3$, respectively. Since N' has a ∇-filtration, its ∇-factors have to be $\nabla(05), \nabla(51)$ and $\nabla(10)$, each with multiplicity one. In the same way, as we have seen that $T(43)$ has simple top 10, we now see that it also has simple socle 10. $\square$

Let us add also the following remark:

Remark. *The module* $T(43)$ *is a faithful* A*-module.*

Proof. First of all, we show that the modules $T(05)$ and $T(51)$ are both isomorphic to factor modules (and to submodules) of $T(43)$. The Δ-filtration of $T(43)$ shows that $T(43)$ has a factor module with factors $\Delta(10)$ and $\Delta(05)$. Since this factor module is indecomposable, it follows that it is $T(05)$. Similarly, $T(51)$ is a factor module of $T(43)$. (And dually, $T(05)$ and $T(51)$ are also submodules of $T(43)$). Of course, also $T(10)$ is a factor module and a submodule of $T(43)$. It follows that $T(43)$ is faithful, since the direct sum of all tilting modules is always a faithful module (it is a "tilting" module in the sense used in [R2]). $\square$

6. Algebras with quiver $Q(10, 05, 51, 43)$

Assume that we deal with a quasi-hereditary algebra A with quiver $Q(10, 05, 51, 43)$, with ordering $10 < 05 < 43$ and $10 < 51 < 43$ and such that rad $\Delta(43)$ and $\nabla(43)/\operatorname{soc}$ both are homogeneous B-modules of length 4.

Since we know the composition factors of all the A-modules $\nabla(\lambda)$, we can use the reciprocity law in order to see that the indecomposable projective modules have the following Δ-factors (going downwards)

$$\begin{array}{lc} P(43) & \Delta(43) \\ P(05) & \Delta(05) \mid \Delta(43) \\ P(51) & \Delta(51) \mid \Delta(43) \\ P(10) & \Delta(10) \mid \Delta(05) \oplus \Delta(51) \mid \Delta(43) \oplus \Delta(43). \end{array}$$

We see: Since the Loewy length of these factors of $P(10)$ are $1, 2, 4$, the Loewy length of $P(10)$ can be at most 7. Of course, the Loewy length of $P(43) = \Delta(43)$ is 4 and that of $P(05)$ and $P(51)$ is at most 6. It follows that $J^7 = 0$.

Our aim is to contruct a presentation of A by the quiver Q and suitable relations. As we have mentioned, for any arrow $\alpha : i \to j$ in Q we choose an element in

$e_iJe_j \setminus e_iJ^2e_j$ which we denote again by α, in order to obtain a surjective algebra homomorphisms

$$\eta : kQ \to A.$$

Since $J^7 = 0$, we see that *all paths of length 7 in the quiver are zero when considered as elements of A.*

Lemma 3. *Any generator choice for A satisfies the conditions*

$$\alpha'\alpha, \alpha'\beta, \beta'\alpha, \beta'\beta \ \in J^4,$$

$$\gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - c_0\beta\beta') = 0, \quad (\alpha\alpha' - c_1\beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0.$$

for some non-zero scalars $c_0, c_1 \in k$.

Proof. The algebra B considered in section 3 is the factor algebra of A modulo the ideal generated by e_{43}. Since we know that the paths $\alpha'\alpha,\ \alpha'\beta,\ \beta'\alpha,\ \beta'\beta$ are zero in B, they belong to J^4 (any path between vertices of the form 05 and 51 which goes through 43 has length at least 4):

$$\alpha'\alpha, \alpha'\beta, \beta'\alpha, \beta'\beta \ \in J^4.$$

Since $e_{43}Je_{43} = 0$, we have

$$\gamma'\gamma = 0.$$

Also, the shape of $P(43)$ shows that $e_{43}J^3e_{10}$ is one-dimensional, and that the paths $\gamma'\alpha\alpha'$ and $\gamma'\beta\beta'$ both are non-zero, thus they are scalar multiples of each other. Thus, we can assume that

$$\gamma'(\alpha\alpha' - c_0\beta\beta') = 0,$$

with some non-zero scalar c_0. Dually, we have

$$(\alpha\alpha' - c_1\beta\beta')\gamma = 0$$

with some non-zero scalar c_1. (Later, we will use the fact that the modules $\operatorname{rad}\Delta(43)$ and $\nabla(43)/\operatorname{soc}$ are isomorphic, then we can assume that $c_0 = c_1$; also, we will replace one of the arrows $\alpha, \alpha', \beta, \beta'$ by a non-zero scalar multiples, in order to change the coefficient c_0 to 1).

Since $P(43) = \Delta(43)$ is of Loewy length 4, we see that $\gamma'J^3 = 0$, in particular we have

$$\gamma'\alpha\alpha'\gamma = 0$$

(and also that $\gamma'\alpha\alpha'\alpha$ and $\gamma'\alpha\alpha'\beta$ are zero.) $\square$

We have seen in the proof that $\gamma' J^3 = 0$, since $\Delta(43)$ is of Loewy length 4. Dually, since $\nabla(43)$ is of Loewy length 4, we have $J^3\gamma = 0$.

Lemma 4. *A factor algebra of the path algebra of the quiver $Q(10, 05, 51, 43)$ satisfying the relations exhibited in Lemma 3 is generated as a k-space by the elements*

$$\begin{array}{ll}
Q_0 & 10,\ 05,\ 51,\ 43, \\
Q_1 & \alpha,\ \beta,\ \gamma,\ \alpha',\ \beta',\ \gamma', \\
Q_2 & \alpha\alpha',\ \beta\beta',\ \gamma\gamma',\ \alpha'\gamma,\ \beta'\gamma,\ \gamma'\alpha,\ \gamma'\beta, \\
Q_3 & \alpha\alpha'\gamma,\ \gamma\gamma'\alpha,\ \gamma\gamma'\beta,\ \alpha'\gamma\gamma',\ \beta'\gamma\gamma',\ \gamma'\alpha\alpha', \\
Q_4 & \alpha\alpha'\gamma\gamma',\ \gamma\gamma'\alpha\alpha',\ \alpha'\gamma\gamma'\alpha,\ \alpha'\gamma\gamma'\beta,\ \beta'\gamma\gamma'\alpha,\ \beta'\gamma\gamma'\beta, \\
Q_5 & \alpha\alpha'\gamma\gamma'\alpha,\ \alpha\alpha'\gamma\gamma'\beta,\ \alpha'\gamma\gamma'\alpha\alpha',\ \beta'\gamma\gamma'\beta\beta', \\
Q_6 & \alpha\alpha'\gamma\gamma'\alpha\alpha',
\end{array}$$

thus is of dimension at most 34.

Proof. One shows inductively that the elements listed as Q_i generate the factor space J^i/J^{i+1}. This is obvious for $i = 0, 1, 2$, since here we have listed all the paths of length i. For $i = 3$, the missing paths of length 3 are

$$\alpha\alpha'\alpha,\ \alpha\alpha'\beta,\ \beta\beta'\alpha,\ \beta\beta'\beta,\ \gamma\gamma'\gamma,$$

as well as

$$\beta\beta'\gamma,\ \gamma'\beta\beta'.$$

By assumption, the first five belong to J^4, whereas the last two are equal to a non-zero multiple of $\alpha\alpha'\gamma$ and $\gamma'\alpha\alpha'$, respectively.

Next, consider $i \geqslant 4$. We have to take the paths in Q_{i-1} and multiply them from the right by the arrows and see what happens. For $i = 4$, the missing paths are $\gamma\gamma'\beta\beta'$ (it is a multiple of $\gamma\gamma'\alpha\alpha'$), the paths $\alpha'\gamma\gamma'\gamma$ and $\beta'\gamma\gamma'\gamma$ (both involve $\gamma'\gamma$) as well as the right multiples of $\gamma'\alpha\alpha'$ (all belong to J^5).

In the same way, we deal with the cases $i = 5, 6, 7$. In particular, for $i = 7$, we see that $J^7 = J^8$, and therefore $J^7 = 0$. This shows that we have obtained a generating set of the algebra as a k-space. □

7. The algebra $A = A(10, 05, 51, 43)$

Now, let $A = A(10, 05, 51, 43)$.

Lemma 5. *For any generator choice of elements of A, the paths listed in Lemma 4 form a basis of A.*

Proof. Lemma 3 asserts that we can apply Lemma 4. On the other hand, we know that $\dim A = 34$, since we know the dimension of the indecomposable projective A-modules. □

Lemma 6. *The socle of $P(10)$ has length* 2.

Proof. Since $\Delta(43)\oplus\Delta(43)$ is a submodule of $P(10)$, the length of the socle of $P(10)$ is at least 2.

According to Lemma 2, the top of $T(43)$ is equal to 10, thus we see that $T(43)$ is a factor module of $P(10)$, say $T(43)=P(10)/W$ for some submodule W of $P(10)$. The subcategory of modules with a Δ-filtration is closed under kernels of surjective maps [R2], thus W has a Δ-filtration. But W has a composition factor of the form 43, and is of length 5, thus W is isomorphic to $\Delta(43)$ and therefore has simple socle. Quoting again Lemma 2, we know that also $T(43)$ has simple socle, thus the length of the socle of $P(10)$ is at most 2. □

Proof of the proposition. Assume that there is given a generator choice for A. Then $\alpha'\alpha$ belongs to J^4, thus to $e_{05}J^4e_{05}$. The basis of A exhibited in Lemma 4 shows that $e_{05}J^4e_{05}$ is generated by $\alpha'\gamma\gamma'\alpha$, thus we see that $\alpha'\alpha$ has to be a multiple of $\alpha'\gamma\gamma'\alpha$. In the same way, we consider also the elements $\alpha'\beta,\ \beta'\alpha,\ \beta'\beta$ and obtain scalars $c_{aa},c_{ab},c_{ba},c_{bb}$ (some could be zero) such that

$$\begin{aligned}\alpha'\alpha &= c_{aa}\,\alpha'\gamma\gamma'\alpha,\\ \alpha'\beta &= c_{ab}\,\alpha'\gamma\gamma'\beta,\\ \beta'\alpha &= c_{ba}\,\beta'\gamma\gamma'\alpha,\\ \beta'\beta &= c_{bb}\,\beta'\gamma\gamma'\beta.\end{aligned}$$

We show that we can achieve that three of these coefficients are zero: Let

$$\begin{aligned}\alpha'_0 &= \alpha'(1-c_{aa}\gamma\gamma'),\\ \beta'_0 &= \beta'(1-c_{ba}\gamma\gamma'),\\ \beta_0 &= (1-(c_{ab}-c_{aa})\gamma\gamma')\beta,\end{aligned}$$

Then

$$\begin{aligned}\alpha'_0\alpha &= \alpha'(1-c_{aa}\gamma\gamma')\alpha=0,\\ \beta'_0\alpha &= \beta'(1-c_{ba}\gamma\gamma')\alpha=0,\end{aligned}$$

and

$$\begin{aligned}\alpha'_0\beta_0 &= \alpha'(1-c_{aa}\gamma\gamma')(1-(c_{ab}-c_{aa})\gamma\gamma')\beta\\ &= \alpha'(1-c_{aa}\gamma\gamma'-(c_{ab}-c_{aa})\gamma\gamma')\beta\\ &= \alpha'(1-c_{ab}\gamma\gamma')\beta \quad = \quad 0.\end{aligned}$$

In the last calculation, we have deleted the summand in rad^6, since actually $\gamma'\gamma=0$.

This shows that replacing $\alpha',\ \beta,\ \beta'$ by $\alpha'_0,\ \beta_0,\ \beta'_0$, respectively, we can assume that all the parameters c_{aa},c_{ab},c_{ba} are equal to zero.

Thus, we can assume that we deal with the relations:

$$\begin{gathered}\alpha'\alpha=0,\quad \alpha'\beta=0,\quad \beta'\alpha=0,\quad \beta'(1-c_{bb}\gamma\gamma')\beta=0,\\ \gamma'\gamma=0,\quad \gamma'(\alpha\alpha'-c_0\beta\beta')=0,\quad (\alpha\alpha'-c_1\beta\beta')\gamma=0,\quad \gamma'\alpha\alpha'\gamma=0.\end{gathered}$$

Let us show that $c_{bb} \neq 0$. Assume, for the contrary that $c_{bb} = 0$. Then the element $\alpha\alpha' - \beta\beta'$ belongs to the socle of $P(10)$. But of course, also the elements $\alpha\alpha'\gamma\gamma'\alpha\alpha'$ and $\gamma\gamma'\alpha\alpha'$ belong to the socle of $P(10)$, thus the socle of $P(10)$ is of length at least 3. But this contradicts Lemma 6.

We have mentioned already, that the isomorphy of $\operatorname{rad}\Delta(43)$ and $\nabla(43)/\operatorname{soc}$ implies that $c_0 = c_1$. Thus we deal with a set of relations

$$\begin{gathered}\alpha'\alpha = 0, \quad \alpha'\beta = 0, \quad \beta'\alpha = 0, \quad \beta'(1 - c'\gamma\gamma')\beta = 0,\\ \gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - c\beta\beta') = 0, \quad (\alpha\alpha' - c\beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0.\end{gathered}$$

with two non-zero scalars c, c'. It remains a last change of the generator choice: Replace say γ by $\frac{1}{c'}\gamma$ and α by $\frac{1}{c}\alpha$. Then we obtain the wanted presentation. This completes the proof of the Proposition. □

8. The module $T(43)$

As we have mentioned, $T(43)$ is a factor module of $P(10)$, namely $T(43) = P(10)/\gamma A$, thus it has the following coefficient quiver:

$$\begin{tikzcd}
 & 10 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
05 \arrow[d, "\alpha'"'] & & 51 \arrow[d, "\beta'"] \\
10_1 \arrow[dr, "\gamma"] & & 10_2 \arrow[dl, "\gamma"'] \arrow[ddd, bend left, "\beta"] \\
 & 43 \arrow[d, "\gamma'"] & \\
 & 10_3 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
05 \arrow[dr, "\alpha'"'] & & 51 \arrow[dl, "\beta'"] \\
 & 10 &
\end{tikzcd}$$

with all non-zero coefficients being equal to 1.

The picture shows nicely the Δ-filtration of $T(43)$, but, of course, one also wants to see a ∇-filtration. This is the reason why we have labelled the three copies of 10 in the middle (since we exhibit a coefficient quiver, these elements $10_1,\ 10_2,\ 10_3$ are elements of a basis). Consider the subspace

$$V = \langle 10_1, 10_2, 10_3 \rangle$$

of $T(43)$ and the elements $x = 10_1 + 10_3 - 10_2$ and $y = 10_1 - 10_2$ of V. One easily sees the following:

The element x lies in the kernel both of β and γ, and it is mapped under α to the composition factor 05 lying in $\mathrm{soc}_2\, T(43)$. Thus, it provides an embedding of $\nabla(05)$ into $T(43)/\mathrm{soc}$.

The element y lies in the kernel both of α and γ, and it is mapped under β to the composition factor 51 lying in $\mathrm{soc}_2\, T(43)$. Thus, it provides an embedding of $\nabla(51)$ into $T(43)/\mathrm{soc}$.

The sum of the submodules xA and yA is a submodule of $T(43)$ of length 5 with a ∇-filtration with factors going down:

$$\nabla(05) \oplus \nabla(51) \mid \nabla(10).$$

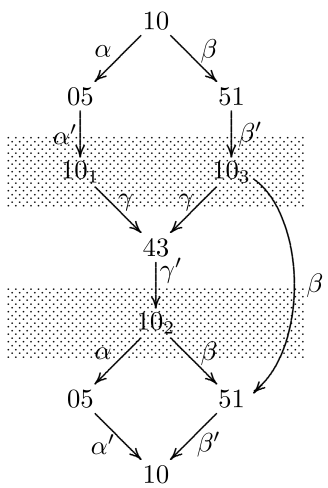
Finally, the factor module $T(43)/(xA + yA)$ is obviously of the form $\nabla(43)$, since its socle is 43 and its length is 5.

Remark. In terms of the basis of A presented above, we also can write:

$$x = \alpha\alpha' + \alpha\alpha'\gamma\gamma' - \beta\beta'$$
$$y = \alpha\alpha' - \beta\beta'.$$

9. A further look at the module $T(43)$

In order to understand the module $T(43)$ better, let us concentrate on the essential part which looks quite strange, namely the three subfactors 10_1, 10_2, 10_3 shaded below:



The three elements 10_1, 10_2 10_3 are displayed in two layers, namely in the radical layers they belong to. If we consider the position of composition factors of the form 10 in the socle layers, we get a dual configuration, since the subspace inside V generated by the difference $10_1 - 10_3$ lies in the kernel of γ and therefore belongs to $\mathrm{soc}_3\, T(43)$.

Let us look at the the space

$$V = 10_1 \oplus 10_2 \oplus 10_3,$$

in more detail, taking into account all the information stored there, namely the endomorphism $\overline{\gamma} = \gamma\gamma'$ as well as the images of the maps to V and the kernels of the maps starting at V. One may be tempted to look at the subspaces

$$\mathrm{Im}(\alpha'),\ \mathrm{Im}(\beta'),\ \mathrm{Ker}(\alpha),\ \mathrm{Ker}(\beta),$$

however, one has to observe that the maps mentioned here are not intrinsically given, but can be replaced by suitable others (as we have done when we were reducing the number of parameters). For example, instead of looking at α', we have to take into account the whole family of maps $\alpha' + c\alpha'\overline{\gamma}$ with $c \in k$. Thus, the intrinsic subspaces to be considered are

$$\begin{aligned}
U_1 &= \mathrm{Im}(\alpha') + \mathrm{Im}(\alpha'\overline{\gamma}) = \mathrm{Im}(\alpha') + \mathrm{Im}(\overline{\gamma}),\\
U_2 &= \mathrm{Im}(\beta') + \mathrm{Im}(\overline{\gamma}),\\
U_3 &= \mathrm{Ker}(\alpha) \cap \mathrm{Ker}\overline{\gamma},\\
U_4 &= \mathrm{Ker}(\beta) \cap \mathrm{Ker}\overline{\gamma},
\end{aligned}$$

as well as $\mathrm{Ker}(\overline{\gamma})$ and $\mathrm{Im}(\overline{\gamma})$. However, since we see that

$$\begin{aligned}
\mathrm{Ker}(\overline{\gamma}) &= U_3 + U_4,\\
\mathrm{Im}(\overline{\gamma}) &= U_1 \cap U_2,
\end{aligned}$$

it is sufficient to consider V with its subspaces $U_1, \dots, U_4$.

This means that we deal with a vector space with four subspaces, thus with a representation of

the 4-subspace quiver $\qquad$ with dimension vector

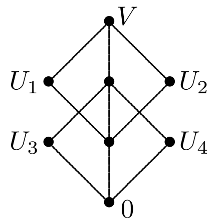
$$\begin{tikzcd}
\circ \arrow[drr] & \circ \arrow[dr] & & \circ \arrow[dl] & \circ \arrow[dll] \\
& & \circ & &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
2 \arrow[drr] & 2 \arrow[dr] & & 1 \arrow[dl] & 1 \arrow[dll] \\
& & 3 & &
\end{tikzcd}$$

A direct calculation shows that we get the following representation:

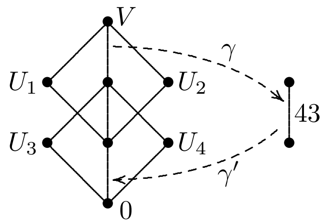
$$\begin{tikzcd}
kk0 \arrow[drr] & 0kk \arrow[dr] & & U_3 \arrow[dl] & U_4 \arrow[dll] \\
& & kkk & &
\end{tikzcd}
\qquad \text{with} \qquad
\begin{aligned}
U_3 &= \langle(1,0,-1)\rangle\\
U_4 &= \langle(1,1,-1)\rangle
\end{aligned}$$

This is an indecomposable representation of the 4-subspace quiver, it belongs to a tube of rank 2 (and is uniquely determined by its dimension vector). Note that its endomorphism ring is a local ring of dimension 2, with radical being the maps $V/(U_3 + U_4) \to U_1 \cap U_2$; and $\overline{\gamma}$ is just such a map. The lattice of subspaces of V

generated by the subspaces U_1, U_2, U_3, U_4 looks as follows:



Let us repeat that $\overline{\gamma} = \gamma\gamma'$ maps $V/(U_3+U_4)$ onto $U_1 \cap U_2$, thus we may indicate the operation of γ and γ' as follows:

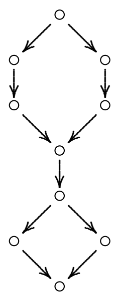


We should stress that the last two pictures show subspace lattices (thus composition factors are drawn as intervals between two bullets), in contrast to the pictures of coefficient quivers, where the composition factors are depicted by their labels (such as 10, 05, 51, ...) and the lines indicate extensions of simple modules.

Note that the core of $T(43)$ is semisimple, namely of the form $10 \oplus 43$, here 10 is just the subfactor $(U_3+U_4)/(U_1 \cap U_2)$.

We hope that the considerations above show well the hidden symmetries of $T(43)$.

Finally, let us remark that the module $T(43)$ has a diagram D in the sense of Alperin (but no strong diagram), namely the following:



(obtained from the coefficient quiver by deleting the by-path β).

10. A related algebra

We have used the Δ-filtration of $T(43)$ in order to show that $T(43)$ has simple socle, and this implied that the coefficient c_{bb} had to be non-zero. In this way, we have obtained the somewhat strange relations presented in the Proposition. Let us now consider the same quiver $Q(10, 05, 51, 43)$, but with the relations

$$\alpha'\alpha = 0, \quad \alpha'\beta = 0, \quad \beta'\alpha = 0, \quad \beta'\beta = 0,$$
$$\gamma'\gamma = 0, \quad \gamma'(\alpha\alpha' - \beta\beta') = 0, \quad (\alpha\alpha' - \beta\beta')\gamma = 0, \quad \gamma'\alpha\alpha'\gamma = 0.$$

The corresponding algebra A' still is quasi-hereditary, and the Δ-modules and the ∇-modules have the same shape as those for the algebra $A = A(10, 05, 51, 43)$. However, now it turns out that the tilting module for 43 is of length 11, with a Δ-filtration of the form

$$\Delta(10) \oplus \Delta(10) \mid \Delta(05) \oplus \Delta(51) \mid \Delta(43)$$

and a similar ∇-filtration.

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C. Bowman
Corpus Christi College
Cambridge, CB2 1RH, England, UK
C.Bowman@dpmms.cam.ac.uk

S.R. Doty
Mathematics and Statistics, Loyola University Chicago,
Chicago, IL 60626, USA
doty@math.luc.edu

S. Martin
Magdalene College,
Cambridge, CB3 0AG, England, UK
S.Martin@dpmms.cam.ac.uk

C.M. Ringel
Fakultät für Mathematik, Universität Bielefeld,
PO Box 100 131, D-33 501 Bielefeld, Germany
ringel@math.uni-bielefeld.de