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LENGTHS OF PARABOLIC SUBGROUPS IN FINITE COXETER GROUPS

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Abstract. The definition of length for an element in a Coxeter group was generalized by Perkins and Rowley [7] to assign a length to subsets of a Coxeter group. Here the lengths of irreducible parabolic subgroups of finite Coxeter groups are determined.

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1. Introduction

Suppose W is a Coxeter group, and let R be a set of fundamental reflections for W (see, for example, [6] for a detailed introduction to finite Coxeter groups and their root systems). Then $W = \langle R \ : \ (rs)^{m_{rs}} = 1, r, s \in R\rangle$ where $m_{rs} \in \mathbb{Z}^+ \cup \{\infty\}$, $m_{rr} = 1$ and $m_{rs} = m_{sr} \geq 2$ for $r, s \in R$ with $r \neq s$. For J a subset of R, $W_J = \langle r \ : \ r \in J\rangle$ is called a standard parabolic subgroup of W. A subgroup of W which is conjugate to a standard parabolic subgroup is referred to as a parabolic subgroup of W. Parabolic subgroups frequently make important guest appearances in other areas of mathematics – see for example [1], [3] and [4].

For $w \in W$ we define $l(w)$, the length of w, to be

$$l(w) = \min\{l \ : \ w = r_1 \cdots r_l \text{ for some } r_i \in R\}$$

if $w \neq 1$, setting $l(1) = 0$. The length function has long been of fundamental importance in the study of Coxeter groups. In [7] the notion of length was extended to assign lengths to subsets of Coxeter groups. The aim of this paper is to determine the possible lengths of parabolic subgroups in finite Coxeter groups. In [5] various results were given about lengths of particular subsets, such as conjugacy classes, cosets and subgroups. In the discussion of subgroups in [5], some results of the current paper were referred to but not proved or even explicitly stated and one,

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Theorem 1.2, was stated (but not proved) as Theorem 3.3 of [5].

In order to describe the way in which the length function for elements of a Coxeter group W is extended to a length function for subsets of W, we need to work with the root system Φ of W. To define Φ we start with a real vector space V which has basis $\Pi = \{\alpha_r \ : \ r \in R\}$ (Π is the set of fundamental roots of W). Then we define a symmetric bilinear form $\langle \ , \ \rangle$ on V by setting, for $r, s \in R$,

$$\langle \alpha_r, \alpha_s \rangle = \begin{cases} -\cos(\pi/m_{rs}) & \text{if } m_{rs} \neq \infty; \\ -1 & \text{otherwise.} \end{cases}$$

Now, for $r, s \in R$ the action of r on α_s, written $r \cdot \alpha_s$, is given by

$$r \cdot \alpha_s = \alpha_s - 2\langle \alpha_r, \alpha_s \rangle \alpha_r.$$

This extends in a natural way to an action of W on V which is both faithful and respects the bilinear form $\langle \cdot , \cdot \rangle$. The root system Φ of W is the following subset of V:

$$\Phi := \{w \cdot \alpha_r : w \in W, r \in R\}.$$

Additionally we have

$$\Phi^+ := \{\sum_{r \in R} \lambda_r \alpha_r \in \Phi : \lambda_r \geq 0 \text{ for each } r \in R\},$$

the set of positive roots, and $\Phi^- := -\Phi^+$, the set of negative roots. It turns out (Section 5.4 of [6]) that Φ is the disjoint union of Φ^+ and Φ^-. For $w \in W$, let $N(w) = \{\alpha \in \Phi^+ : w \cdot \alpha \in \Phi^-\}$. The set $N(w)$ is often called the inversion set of w. A well known, and frequently used, fact is that $l(w) = |N(w)|$ (Proposition 5.6(b) of [6]).

Following [7], for $X \subseteq W$ set

$$N(X) = \{\alpha \in \Phi^+ : x \cdot \alpha \in \Phi^- \text{ for some } x \in X\}.$$

Observe that $N(X) = \cup_{x \in X} N(x)$. We define the length of X to be

$$l(X) = |N(X)|.$$

If X consists of the single element w, then the length of X is just $l(w)$. It was shown in Proposition 1.1 of [7] that for a finite standard parabolic subgroup X of W $l(X) \leq l(X^w)$ for each $w \in W$, with equality if and only if X^w is also a standard parabolic subgroup of W. Here we delve more deeply into the question of possible lengths of conjugates of standard parabolic subgroups of W. From now on W is assumed to be a finite Coxeter group and X a standard parabolic subgroup of W. We define the 'length polynomial' (of the conjugacy class of X), $\Lambda_{X,W}(t)$, to be

$$\Lambda_{X,W}(t) = \sum \lambda_l t^l,$$

where λ_l is the number of conjugates of X having length l. In some of our formulae for the length polynomial we shall encounter $[q]$ where $q \in \mathbb{Q}$, by which we mean the integer part of q. The following lemma shows that in order to determine $\Lambda_{X,W}(t)$ we need only consider the case of irreducible Coxeter groups:

Lemma 1.1. *Let $W_1, \dots, W_m$ be the irreducible components of W and, for $i = 1, \dots, m$, set $X_i = X \cap W_i$. (So $W = W_1 \times \cdots \times W_m$ and $X = X_1 \times \cdots \times X_m$.) Then*

$$\Lambda_{X,W}(t) = \prod_{i=1}^{m} \Lambda_{X_i,W_i}(t).$$

Proof. Write $\Lambda_{X_i,W_i} = \sum \lambda_{i,l} t^l$ for each i. Any conjugate Y of X is of the form $Y = Y_1 \times \cdots \times Y_m$, where each Y_i is conjugate in W_i to X_i. Now $l(Y) = \sum_{i=1}^m l(Y_i)$. Set $l_i = l(Y_i)$ for each i. Then $l(Y) = l$ if and only if $\sum_{i=1}^m l_i = l$. The number of conjugates of X_i having length l_i is λ_{i,l_i}. Hence the number of conjugates Y of X with $l(Y) = l$ (that is, the coefficient of t^l in $\Lambda_{X,W}$) is

$$\sum_{\substack{l_1, \dots, l_m : \\ l_1 + \cdots + l_m = l}} \prod_{i=1}^{m} \lambda_{i,l_i}.$$

This is equal to the coefficient of t^l in $\prod_{i=1}^m \Lambda_{X_i,W_i}(t)$. Therefore $\Lambda_{X,W}(t) = \prod_{i=1}^m \Lambda_{X_i,W_i}(t)$. □

Our first main result concerns the case when W is a classical Weyl group with X and W of the same type. We will review the notation for the groups A_n, B_n and D_n in Sections 3 – 5.

Theorem 1.2.(*i*) *If $W \cong A_n$ and $X \cong A_i$, then*

$$\Lambda_{X,W}(t) = t^{i(i+1)/2} \sum_{\rho=0}^{n-i} (n + 1 - i - \rho) \begin{pmatrix} \rho + i - 1 \\ i - 1 \end{pmatrix} t^{(i+1)\rho}.$$

(*ii*) *If $W \cong B_n$ and $X \cong B_i$, then*

$$\Lambda_{X,W}(t) = t^{i^2} \sum_{\rho=0}^{n-i} \begin{pmatrix} \rho + i - 1 \\ i - 1 \end{pmatrix} t^{2i\rho}.$$

(iii) If $W \cong D_n$ and $X \cong D_i$, then

$$\Lambda_{X,W}(t) = t^{i(i-1)} \sum_{\rho=0}^{n-i} \begin{pmatrix} \rho + i - 1 \\ i - 1 \end{pmatrix} t^{2i\rho}.$$

Theorem 1.2 is an immediate consequence of Propositions 3.2, 4.1 and 5.1.

As an example, letting $W \cong A_3$, we have

$$\Lambda_{A_1,A_3} = 3t + 2t^3 + t^5; \;\; \Lambda_{A_2,A_3} = 2t^3 + 2t^6.$$

When X is of a different type from W we have Theorem 1.3 below. The signed cycle notation appearing in the theorem will be explained in Section 4.

Theorem 1.3.*(i) If $W \cong B_n$, and $X \cong A_1$, then*

$$\Lambda_{X,W} = \sum_{k=1}^{n} t^{2k-1} \quad \text{if } X = \langle (\bar{1}) \rangle;$$

and

$$\Lambda_{X,W} = \sum_{k=1}^{n-1} (n-k) t^{2k-1} + \sum_{k=1}^{2n-3} t^{2k+1} \left(\left[\tfrac{k+1}{2} \right] - \max\{k+1-n, 0\} \right) \text{ if } X = \langle (12) \rangle.$$

(ii) If $W \cong B_n$, and $X \cong A_i$, $i > 1$, then

$$\begin{aligned} \Lambda_{X,W}(t) \;\; &= \;\; t^{i(i+1)/2} \sum_{\rho=0}^{n-i-1} t^{(i+1)\rho} (n-i-\rho) \tbinom{i+\rho-1}{\rho} \\ &+ t^{(i+1)(i+2)/2} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{y=\max\{\rho-n+i+2,1\}}^{[\rho/2]+1} \tbinom{\rho+i+1-2y}{i-1} \\ &+ t^{(i+1)^2} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{r=1}^{i-1} 2^r \sum_{s=\max\{\rho-n+i+1,0\}}^{[\rho/2]} \tbinom{s+r}{r} \tbinom{\rho-2s+i-r-1}{i-r-1}. \end{aligned}$$

(iii) If $W \cong D_n$ and $X \cong A_i$, then

$$\Lambda_{X,W} = \begin{cases} \displaystyle\sum_{\rho=0}^{n-3} \tbinom{\rho+2}{2} t^{6(1+\rho)} & \text{if } i = 3 \text{ and } X = \langle (12), (23), (\bar{1}\bar{2}) \rangle; \\ t^{n(n-1)/2} + (2^{n-2} - 1) t^{n(n-1)} & \text{if } n \text{ is even and } i = n-1. \end{cases}$$

In all other cases the length polynomial for $W \cong D_n$ and $X \cong A_i$ is given by

$$\begin{aligned}\Lambda_{X,D_n}(t) &= t^{i(i+1)/2} \sum_{\rho=0}^{n-i-1} t^{(i+1)\rho}(n-i-\rho)\binom{i+\rho-1}{\rho} \\ &+ t^{i(i+1)/2} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{y=\max\{\rho-n+i+2,1\}}^{[\rho/2]+1} \binom{\rho+i+1-2y}{i-1} \\ &+ t^{i(i+1)} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{r=1}^{i-1} 2^r \sum_{s=\max\{\rho-n+i+1,0\}}^{[\rho/2]} \binom{s+r}{r}\binom{\rho-2s+i-r-1}{i-r-1}.\end{aligned}$$

Theorem 1.3 will follow from Lemmas 4.2, 5.2 and 5.3 and Theorems 4.11 and 5.7. If X is not an irreducible parabolic subgroup, the calculations appear to become much harder. As an indication of this in Section 6 we look at the simplest reducible parabolic subgroups of $W \cong A_n$, namely those of the type $A_i \times A_j$, and obtain, in Propositions 6.2 and 6.3, formulae for the number of minimal and maximal length conjugates of X. Finally, in Section 7 we give tables showing the distribution of lengths of irreducible parabolic subgroups in the exceptional groups.

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2. Preliminaries

In this section we establish various basic results which will be useful. We recall that a left coset of X in W has a (unique) minimal length element, normally referred to as the minimal left coset representative of that coset. For $\alpha \in \Phi$, $X \cdot \alpha = \{x \cdot \alpha \ : \ x \in X\}$ denotes the X-orbit of α and we use Φ_X, Φ_X^+ to denote, respectively, the subroot system of X and $\Phi_X \cap \Phi^+$. The next proposition brings together some useful facts about the length function and the interaction between $N(w)$ and $N(v)$ for elements w, v of W.

Proposition 2.1.(i) *Let $g \in W$. Then g is a minimal left coset representative of X in W precisely when $l(gx) = l(g) + l(x)$ for each $x \in X$.*
(ii) *Let $w, v \in W$. Then*

$$N(wv) = N(v) \backslash -v^{-1} \cdot N(w) \dot{\cup} v^{-1} \cdot \left(N(w) \backslash N(v^{-1})\right).$$

Moreover, $l(wv) = l(w) + l(v)$ if and only if $N(wv) = N(v) \dot{\cup} v^{-1} \cdot N(w)$.
(iii) *For all $w \in W$, $N(w^{-1}) = -w \cdot N(w)$.*
(iv) *Let $w, v \in W$. Then $l(wv) = l(w) + l(v) - 2|N(w) \cap N(v^{-1})|$.*

Proof.(i) See Section 5.12 of [6].

(*ii*) Let $w, v \in W$ and $\alpha \in \Phi^+$. Then $\alpha \in N(wv)$ if and only if either $v \cdot \alpha \in \Phi^-$ and $w \cdot (v \cdot \alpha) \notin \Phi^+$, or $v \cdot \alpha \in \Phi^+$ and $w \cdot (v \cdot \alpha) \in \Phi^-$. In the first case, this means $\alpha \in N(v)$ and $v \cdot \alpha \notin -N(w)$, so $\alpha \in N(v) \backslash - v^{-1} \cdot N(w)$. In the second case we have $v \cdot \alpha \in N(w)$ and $v^{-1} \cdot (v \cdot \alpha) = \alpha \in \Phi^+$. Hence $v \cdot \alpha \notin N(v^{-1})$. Thus $\alpha \in v^{-1} \cdot (N(w) \backslash N(v^{-1}))$. Therefore

$$N(wv) = N(v) \backslash - v^{-1} \cdot N(w) \dot{\cup} v^{-1} \cdot \left(N(w) \backslash N(v^{-1})\right).$$

Now $N(wv) \subseteq N(v) \dot{\cup} v^{-1} N(w)$, so $|N(wv)| \leq |N(v)| + |N(w)|$ with equality if and only if $N(wv) = N(v) \dot{\cup}\, v^{-1} \cdot N(w)$. Therefore $l(wv) = l(w) + l(v)$ if and only if

$$N(wv) = N(v) \,\dot{\cup}\, v^{-1} \cdot N(w).$$

(*iii*) Let $w \in W$ and $\alpha \in \Phi^+$. Then $\alpha \in N(w^{-1})$ if and only if $w^{-1} \cdot \alpha \in \Phi^-$, which is if and only if $\alpha \in w \cdot \Phi^-$, that is, $\alpha \in w \cdot (-N(w))$. Hence $N(w^{-1}) = -w \cdot N(w)$.

(*iv*) Let $w, v \in W$. Then by part (*ii*)

$$\begin{aligned} l(wv) = |N(wv)| &= |N(v) \backslash - v^{-1} \cdot N(w)| + |v^{-1} \cdot \left(N(w) \backslash N(v^{-1})\right)| \\ &= |N(v)| - |N(v) \cap -v^{-1} \cdot N(w)| + |N(w) \backslash N(v^{-1})| \\ &= |N(v)| - |-v \cdot N(v) \cap N(w)| + |N(w)| - |N(w) \cap N(v^{-1})| \\ &= |N(v)| + N(w)| - 2|N(w) \cap N(v^{-1})| \end{aligned}$$

since by part (*iii*), $-v \cdot N(v) = N(v^{-1})$. Thus $l(wv) = l(w) + l(v) - 2|N(w) \cap N(v^{-1})|$. $\square$

Lemma 2.2. *Let $g \in W$. Then g is a minimal left coset representative of X in W if and only if $N(g) \cap N(X) = \emptyset$.*

Proof. By Proposition 2.1 (*i*), g is a minimal left coset representative of X in W if and only if $l(gx) = l(g) + l(x)$ for each $x \in X$, and by (*iv*) this occurs if and only if $N(g) \cap N(x^{-1}) = \emptyset$ for all $x \in X$. This is if and only if $N(g) \cap N(x) = \emptyset$ for all $x \in X$, which is if and only if $N(g) \cap N(X) = \emptyset$. $\square$

Before we state Proposition 2.4, which underpins many of our later results, we make the following definition. Let g be a minimal left coset representative of X in W. Define $\mathcal{O}(g)$ to be the set of X-orbits Ω of Φ for which $\emptyset \neq \Omega \cap N(g) \neq \Omega$.

Lemma 2.3. *Let g be a minimal left coset representative of X in W. If $\Omega \in \mathcal{O}(g)$, then $\Omega \subseteq \Phi^+$.*

Proof. Let $\Omega \in \mathcal{O}(g)$ and suppose that there exists $\beta \in \Phi^+$ with $-\beta \in \Omega$. By definition of $\mathcal{O}(g)$, there exist $\alpha \in N(g) \cap \Omega$ and $x \in X$ for which $x \cdot \alpha = -\beta$.

That is, $\alpha \in N(g) \cap N(x)$, contradicting the fact that g is a minimal left coset representative of X. Hence $\Omega \subseteq \Phi^+$. $\square$

Proposition 2.4. *Let g be a minimal left coset representative of X in W. Then*

$$N(gXg^{-1}) = g \cdot N(X) \cup \left(\Phi^+ \cap g \cdot \bigcup_{\Omega \in \mathcal{O}(g)} \pm\Omega \right)$$

and

$$l(gXg^{-1}) = l(X) + \sum_{\Omega \in \mathcal{O}(g)} |\Omega|.$$

Proof. For all $x \in X$, Proposition 2.1 (i) and (ii) imply that $N(gx) = N(x)\dot{\cup}x^{-1} \cdot N(g)$ and (since $x^{-1} \in X$) $N(xg^{-1}) = N(g^{-1})\dot{\cup}g \cdot N(x)$. Another application of Proposition 2.1(ii) gives

$$\begin{aligned} N(gxg^{-1}) &= N(g(xg^{-1})) \\ &= \left[N(xg^{-1})\backslash(-(xg^{-1})^{-1} \cdot N(g))\right] \dot{\cup} (xg^{-1})^{-1} \cdot \left[N(g)\backslash N((xg^{-1})^{-1})\right] \\ &= \left[\left(N(g^{-1})\dot{\cup} g \cdot N(x)\right) \backslash(-gx^{-1} \cdot N(g))\right] \\ &\quad \dot{\cup}\, gx^{-1} \cdot \left[N(g)\backslash \left(N(x^{-1})\dot{\cup}\, x \cdot N(g)\right)\right] \\ &= g \cdot \left[(-N(g)\dot{\cup}N(x)) \backslash(-x^{-1} \cdot N(g))\right] \\ &\quad \dot{\cup}\, g \cdot \left\{\left(x^{-1} \cdot N(g)\right) \backslash \left[-N(x)\dot{\cup}N(g)\right]\right\}. \end{aligned}$$

Now

$$N(x) \cap (-x^{-1} \cdot N(g)) = -x^{-1} \cdot \left(N(x^{-1}) \cap N(g)\right) = \emptyset$$

and $x^{-1} \cdot N(g)\backslash - N(x) = x^{-1} \cdot (N(g)\backslash N(x^{-1})) = x^{-1} \cdot N(g)$. So we end up with

$$N(gxg^{-1}) = g \cdot \left[N(x)\dot{\cup} - \left(N(g)\backslash x^{-1} \cdot N(g)\right) \dot{\cup}\, (x^{-1} \cdot N(g))\backslash N(g)\right].$$

Therefore

$$N(gXg^{-1}) = g \cdot \left[N(X) \cup \left(\bigcup_{x \in X} - [N(g)\backslash x \cdot N(g)]\right) \cup [X \cdot N(g)] \backslash N(g)\right].$$

Now $\alpha \in \bigcup_{x \in X} [N(g)\backslash x \cdot N(g)]$ precisely when $\alpha \in N(g)$ and there exist $x \in X$, $\beta \in \Phi \setminus N(g)$ for which $\alpha = x \cdot \beta$. That is, α is in an X-orbit Ω with $\emptyset \neq N(g) \cap \Omega \neq \Omega$. Recalling the definition of $\mathcal{O}(g)$, it now follows that $\bigcup_{x \in X} - [N(g)\backslash x \cdot N(g)] =$

$-N(g) \cap \bigcup_{\Omega \in \mathcal{O}(g)} -\Omega$. We also have $[X \cdot N(g)] \setminus N(g) = \left[\bigcup_{\Omega \in \mathcal{O}(g)} \Omega\right] \setminus N(g)$. Therefore

$$N(gXg^{-1}) = g \cdot N(X) \cup \left(\Phi^+ \cap g \cdot \bigcup_{\Omega \in \mathcal{O}(g)} \pm\Omega\right).$$

Suppose $\Omega \in \mathcal{O}(g)$. If $g \cdot N(X) \cap g \cdot \Omega \neq \emptyset$, then $N(X) \cap \Omega \neq \emptyset$, which implies $\Omega \not\subseteq \Phi^+$, contradicting Lemma 2.3. If $g \cdot N(X) \cap g \cdot (-\Omega) \neq \emptyset$, then $N(X) \not\subseteq \Phi^+$, which is impossible. Therefore $N(gXg^{-1})$ is the disjoint union of $g \cdot N(X)$ and $\left(\Phi^+ \cap g \cdot \bigcup_{\Omega \in \mathcal{O}(g)} \pm\Omega\right)$. Now if $\alpha \in \Omega$ for some $\Omega \in \mathcal{O}(g)$, then $g \cdot \alpha \in \Phi^-$ if and only if $g \cdot (-\alpha) \in \Phi^+$. Therefore

$$\left|\Phi^+ \cap g \cdot \sum_{\Omega \in \mathcal{O}(g)} \pm\Omega\right| = \sum_{\Omega \in \mathcal{O}(g)} |\Omega|.$$

Hence

$$l(gXg^{-1}) = l(X) + \sum_{\Omega \in \mathcal{O}(g)} |\Omega|,$$

so completing the proof. □

We make use of Proposition 2.4 repeatedly in the next three sections. The general strategy for a given conjugacy class of parabolic subgroups of W, where W is A_n, B_n or D_n, will be first to choose a suitable standard parabolic subgroup X in the conjugacy class and to describe in detail the X-orbits of the root system Φ. Then the minimal left coset representatives g of X in W will be examined, and Proposition 2.4 will be used to determine the number of coset representatives g for which gXg^{-1} has a given length. From this information the length polynomial can then be calculated. We remark that Proposition 2.4 could also be used to calculate the length of specific conjugates Y of a given standard parabolic subgroup X, as long as the minimal left coset representative g for which $Y = gXg^{-1}$ is known. (The length of a standard parabolic subgroup X is simply the number of positive roots in the root system of the Coxeter group to which it is isomorphic.)

3. $W \cong A_n$ and $X \cong A_i$

In this section we assume $W \cong A_n \cong \mathrm{Sym}(n+1) = \langle (12), \ldots, (n\ n+1) \rangle$. We will be working in some detail with the root system of W. Let V be an $(n+1)$-dimensional real vector space with orthonormal basis $\varepsilon_1, \ldots, \varepsilon_{n+1}$. Then W acts on V by permuting the indices of the basis vectors. We may choose $(12), \ldots, (n\ n+1)$ as the fundamental reflections of W. Now set the fundamental root corresponding to the fundamental reflection $(i\ i+1)$ to be $\varepsilon_i - \varepsilon_{i+1}$. The set of positive roots is then

$$\Phi_W^+ = \{\varepsilon_x - \varepsilon_y : 1 \leq x < y \leq n+1\}.$$

The only irreducible parabolic subgroups of W are of type A_i. So we assume $X \cong A_i$ for some $i \leq n$. Without loss of generality we may suppose $X = \langle (12), \ldots, (i\ i+1)\rangle$. Then we have

$$\Phi_X^+ = \{\varepsilon_x - \varepsilon_y : 1 \leq x \leq y \leq i+1\}.$$

Define, for $i+1 \leq j \leq n$, $\Omega_{i,j} := \{\varepsilon_1 - \varepsilon_{j+1}, \ldots, \varepsilon_{i+1} - \varepsilon_{j+1}\}$. It is straightforward to check that the X-orbits of Φ are Φ_X, $\pm\Omega_{i,j}$ $(i+1 \leq j \leq n)$ and the fixed points $\pm\{\varepsilon_j - \varepsilon_k\}$ $(i+1 < j < k \leq n+1)$. Let g be a minimal left coset representative of X in W. It is clear that $\Phi_X \notin \mathcal{O}(g)$ and $\{\varepsilon_j - \varepsilon_k\} \notin \mathcal{O}(g)$ for each $i+1 < j < k \leq n+1$. We may modify Proposition 2.4 thus:

$$l(gXg^{-1}) = l(X) + \sum_{\Omega_{i,j} \in \mathcal{O}(g)} (i+1). \tag{1}$$

Now we examine the action of g on $\{1, \ldots, n+1\}$. By Lemma 2.2, g must satisfy $N(g) \cap N(X) = \emptyset$. Now $N(X) = \Phi_X^+$ and clearly $N(g) \cap \Phi_X^+$ is empty exactly when $g(x) < g(y)$ for all $1 \leq x < y \leq i+1$. That is, we must have $g(1) < g(2) < \cdots < g(i+1)$.

Let $j \in \{i+2, \ldots, n+1\}$. Suppose $g(j) < g(1)$. Then for each $k \leq i+1$, $g \cdot (\varepsilon_k - \varepsilon_j) = \varepsilon_{g(k)} - \varepsilon_{g(j)} \in \Phi^-$. Thus $\Omega_{i,j} \subseteq N(g)$. If $g(j) > g(i+1)$, then for each $k \leq i+1$, $g(\varepsilon_k - \varepsilon_j) = \varepsilon_{g(k)} - \varepsilon_{g(j)} \in \Phi^+$ and $\Omega_{i,j} \cap N(g) = \emptyset$. In both cases $\Omega_{i,j} \notin \mathcal{O}(g)$. However if $g(1) < g(j) < g(i+1)$, then $\varepsilon_{i+1} - \varepsilon_j \in N(g) \cap \Omega_{i,j}$ but $\varepsilon_1 - \varepsilon_j \in \Omega_{i,j} \backslash N(g)$, and hence $\Omega_{i,j} \in \mathcal{O}(g)$.

Lemma 3.1. *Let g be a minimal left coset representative of X in W. Then the number $\rho = \rho(g)$ of $j \in \{i+2, \ldots, n+1\}$ with $\Omega_{i,j} \in \mathcal{O}(g)$ is $g(i+1) - g(1) - i$.*

Proof. By the above, $\Omega_{i,j} \in \mathcal{O}(g)$ if and only if $g(1) < g(j) < g(i+1)$. So ρ is given by the number of $j > i+1$ for which $g(j)$ lies between $g(1)$ and $g(i+1)$. Since $g(k)$ must lie between $g(1)$ and $g(i+1)$ for each $2 \leq k \leq i$, we see that $\rho = g(i+1) - g(1) - i$. □

Proposition 3.2. *Let $A_i \cong X \leq W \cong A_n$. Then each conjugate of X has length $l(X) + \rho(i+1)$ for some ρ with $0 \leq \rho \leq n-i$. Define $F(A_i, A_n, \rho)$ to be the number* *of conjugates of X having length $l(X) + \rho(i+1)$, for $0 \le \rho \le n-i$. Then*

$$F(A_i, A_n, \rho) = (n+1-i-\rho)\begin{pmatrix} i+\rho-1 \\ i-1 \end{pmatrix}.$$

Proof. By (1) above, the length of any conjugate of X may differ from the length of X only by a multiple of $(i+1)$, because $|\Omega_{i,j}| = i+1$ for each j and these are the only orbits that can contribute to $\mathcal{O}(g)$. So any conjugate has length $l(X) + \rho(i+1)$ for some ρ between 0 and $n-i$. By Lemma 3.1, the number of conjugates gXg^{-1} having length $l(X)+\rho(i+1)$ is simply the number of g such that $\rho(g) = g(i+1) - g(1) - i = \rho$. Now $g(1) < g(2) < \ldots < g(i+1)$, so once we have chosen $g(1)$, we have $g(i+1) = g(1) + i + \rho$ and then $g(2), \ldots, g(i)$ are uniquely determined by choosing $i-1$ of the $i-1+\rho$ numbers between $g(1)$ and $g(i+1)$. Once $g(1), \ldots, g(i+1)$ have been chosen, there are $((n+1)-(i+1))! = (n-i)!$ ways to assign the other $g(j)$. Thus for each possible $g(1)$ there are $(n-i)!\begin{pmatrix} i+\rho-1 \\ i-1 \end{pmatrix}$ possible g. Now $g(1) \ge 1$ and $g(1) + \rho + i = g(i+1) \le n+1$. That is $1 \le g(1) \le n+1-i-\rho$. So the number of coset representatives g giving conjugates of length $l(X)+\rho(i+1)$ is $(n-i)!\,(n+1-i-\rho)\begin{pmatrix} i+\rho-1 \\ i-1 \end{pmatrix}$. The normalizer $N_W(X)$ of X in W has order $(n-i)!(i+1)!$. Hence $|N_W(X)|/|X| = (n-i)!$, so each conjugate arises from $(n-i)!$ cosets. We must therefore divide by $(n-i)!$ to arrive at the formula given. $\square$

4. $W \cong B_n$

In this section we assume $W \cong B_n$ with X an irreducible standard parabolic subgroup of W. We may think of B_n as the group of signed permutations of n objects (see, for example, [6]). Define the i^{th} 'sign change' to be the element sending i to $-i$ and fixing all other j. The set of such elements generates a group of order 2^n, isomorphic to $(\mathbb{Z}_2)^n$, and B_n is just the semidirect product of this group with $\mathrm{Sym}(n)$. For ease of notation, we will use the following concise way of expressing elements of B_n. We write a permutation in $\mathrm{Sym}(n)$ (including 1-cycles), and add a minus sign above the number i if the i^{th} sign change is to be applied. We adopt the convention of reading the sign first; that is, if $w = (1\ \bar{2}\ 3)(4) \in B_4$ then $w(1) = 2$, $w(2) = -3$, $w(3) = 1$ and $w(4) = 4$.

Let V be an n-dimensional real vector space with orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. Then W acts on V as follows: for $w \in W$, $w \cdot \varepsilon_i = \varepsilon_{w(i)}$ if $w(i) > 0$ and $-\varepsilon_{|w(i)|}$ otherwise. It can be shown that the reflections $(\bar{1}), (12), \ldots, (n-1\ n)$ generate W

and satisfy the appropriate Coxeter relations, and we may take the vector ε_1 along with $\varepsilon_2-\varepsilon_1,\ldots,\varepsilon_n-\varepsilon_{n-1}$ as the set of fundamental roots. The set of positive roots is then

$$\Phi^+ = \{\varepsilon_y \pm \varepsilon_x : 1 \leq x < y \leq n\} \cup \{\varepsilon_x : 1 \leq x \leq n\}.$$

Since X is an irreducible parabolic subgroup of W, either $X \cong A_i$ or $X \cong B_i$, some i. We deal with the latter case first, so $X = \langle(\overline{1}), (12), \ldots, (i-1\ i)\rangle$. The set of positive roots of X is

$$\Phi_X^+ = \{\varepsilon_y \pm \varepsilon_x : 1 \leq x < y \leq i\} \cup \{\varepsilon_x : 1 \leq x \leq i\}.$$

with X having two orbits on Φ_X. The remaining X-orbits of Φ are $\pm\Omega_{i,j}$ where $\Omega_{i,j} := \{\varepsilon_j \pm \varepsilon_1, \ldots, \varepsilon_j \pm \varepsilon_i\}$ for $i < j \leq n$, and the fixed points $\pm\{\varepsilon_j\}$ for $i < j \leq n$ and, for $i < j < k \leq n$, $\pm\{\varepsilon_k - \varepsilon_j\}$. Let g be a minimal left coset representative of X in W. Then, using Proposition 2.4, we have

$$l(gXg^{-1}) = l(X) + \sum_{\Omega_{i,j} \in \mathcal{O}(g)} 2i. \tag{2}$$

Since g is a minimal left coset representative, by Lemma 2.2 $g(x) > 0$ for each $1 \leq x \leq i$ (since $\varepsilon_x \in \Phi_X^+ = N(X)$) and also that $g(1) < g(2) < \ldots < g(i)$. Now suppose $g(j) > 0$, for some $j > i$. Then $g\cdot(\varepsilon_j + \varepsilon_1) \in \Phi^+$, so $\Omega_{i,j}$ is not contained in $N(g)$. But $\Omega_{i,j} \cap N(g) = \emptyset$ when $g(j) > g(i)$. Conversely when $g(j) < 0$, $\Omega_{i,j} \cap N(g)$ is never empty, but $\Omega_{i,j} \subseteq N(g)$ when $-g(j) > g(i)$. In other words, $\Omega_{i,j} \in \mathcal{O}(g)$ if and only if $|g(j)| < g(i)$. We may now prove

Proposition 4.1. *Let $B_i \cong X \leq W \cong B_n$. Then each conjugate of X has length $l(X) + 2i\rho$ for some $0 \leq \rho \leq n-i$. Define $F(B_i, B_n, \rho)$ to be the number of conjugates of X having length $l(X) + 2i\rho$, for $0 \leq \rho \leq n-i$. Then*

$$F(B_i, B_n, \rho) = \binom{\rho + i - 1}{i-1}.$$

Proof. By (2) the length of any conjugate of X differs from $l(X)$ by a multiple of $2i$, because $|\Omega_{i,j}| = 2i$ for all j. Since there are $n-i$ subsets $\Omega_{i,j}$, we get $l(gXg^{-1}) = l(X) + 2i\rho$ for some ρ, $0 \leq \rho \leq n-i$ (g a minimal left coset representative for X in W). By the previous paragraph, $\Omega_{i,j} \in \mathcal{O}(g)$ if and only if $|g(j)| < g(i)$. Thus the number $\rho(g)$ of orbits $\Omega_{i,j}$ with $\Omega_{i,j} \in \mathcal{O}(g)$ is $\rho(g) = g(i) - i$. For each g with $\rho(g) = \rho$ then, we have $g(i) = \rho + i$. Now, since $g(1) < \ldots < g(i)$, there are $\binom{g(i)-1}{i-1} = \binom{\rho+i-1}{i-1}$ choices for $g(1), \ldots, g(i-1)$. Once $g(1), \ldots, g(i)$

are chosen, there are $2^{(n-i)}(n-i)!$ choices for the other $g(j)$ (each can have a '+' or a '−' above it).

Therefore, for each ρ, the number of coset representatives leading to conjugates of length $l(X)+2i\rho$ is $2^{(n-i)}(n-i)!\begin{pmatrix}\rho+i-1\\ i-1\end{pmatrix}$. However $|N_W(X)|/|X| = 2^{(n-i)}(n-i)!$ (where $N_W(X)$ is the normalizer of X in W) so we obtain $F(i,n,\rho) = \begin{pmatrix}\rho+i-1\\ i-1\end{pmatrix}$. □

We now turn our attention to the case when $X \cong A_i$, for some i. When $i=1$ there are two (non-conjugate) possibilities for X, namely $\langle(\bar{1})\rangle$ and $\langle(12)\rangle$. Clearly for any involution w, $l(\langle w\rangle) = l(w)$, so we are in effect looking at the lengths of reflections in a conjugacy class. We have

Lemma 4.2.(i) *If* $X = \langle(\bar{1})\rangle$, *then*

$$\Lambda_{X,W} = \sum_{k=1}^{n} t^{2k-1}.$$

(ii) *If* $X = \langle(12)\rangle$, *then*

$$\Lambda_{X,W} = \sum_{k=1}^{n-1}(n-k)t^{2k-1} + \sum_{k=1}^{2n-3} t^{2k+1}\left(\left[\tfrac{k+1}{2}\right] - \max\{k+1-n, 0\}\right).$$

Proof.(i) For $1 \le k \le n$, $N((\bar{k})) = \{e_k \pm e_x : 1 \le x \le k-1\} \cup \{e_k\}$. Hence $l((\bar{k})) = 2k-1$, and part (i) follows immediately.

(ii) Let $X = \langle(12)\rangle$. Then the conjugacy class of X is $\{\langle(ij)\rangle, \langle(\bar{i}\,\bar{j})\rangle : 1 \le i < j \le n\}$. For $1 \le i < j \le n$, $N((ij)) = \{e_y - e_i : i < y < j\} \cup \{e_j - e_x : i < x < j\} \cup \{e_j - e_i\}$. Thus $l((ij)) = 2(j-i)-1$. For each k between 1 and $n-1$ there are $n-k$ transpositions (ij) with $(j-i)=k$. Hence the contribution to $\Lambda_{X,W}$ from subgroups $\langle(ij)\rangle$ is $\sum_{k=1}^{n-1}(n-k)t^{2k-1}$. We also note that

$$\begin{aligned} N((\bar{i}\,\bar{j})) &= \{e_i, e_j\} \cup \{e_y + e_i : 1 \le y \le j, y \ne i\} \cup \{e_i - e_x : 1 \le x < i\} \\ &\quad \cup \{e_j + e_x : 1 \le x < i\} \cup \{e_j - e_x : 1 \le x < j, x \ne i\}, \end{aligned}$$

so $l((\bar{i}\,\bar{j})) = 2(i+j)-3 = 2(i+j-2)+1$. For $1 \le k \le 2n-3$, the number of pairs i,j with $1 \le i < j \le n$ for which $k = i+j-2$ is the number of pairs $i, k+2-i$ with $1 \le i < k+2-i \le n$. This is just the number of positive integers i with $k+2-n \le i \le \frac{k+1}{2}$, which is given by $\lfloor \frac{k+1}{2} \rfloor - \max\{k+1-n, 0\}$. Hence the contribution to $\Lambda_{X,W}$ from subgroups $\langle(\overline{i}\,\overline{j})\rangle$ is $\sum_{k=1}^{2n-3} t^{2k+1}\left(\left[\frac{k+1}{2}\right] - \max\{k+1-n, 0\}\right)$. This completes the proof of part (ii).

□

For the rest of this section we assume $X \cong A_i$ with $i > 1$. So we may suppose $X = \langle(12), \ldots, (i\, i+1)\rangle$. The X-orbits of Φ are Φ_X, $\pm\Delta_1$, $\pm\Delta_2, \pm\Omega_{i,j}(i+1 < j \leq n), \pm\Theta_{i,j}(i+1 < j \leq n), \pm\{\varepsilon_j\}(i+1 < j \leq n)$ and $\pm\{\varepsilon_j \pm \varepsilon_k\}$ $(i+1 < k < j \leq n)$. Here

$\Phi_X^+ = \{\varepsilon_y - \varepsilon_x : 1 \leq x < y \leq i+1\}$;
$\Delta_1 = \{\varepsilon_y + \varepsilon_x : 1 \leq x < y \leq i+1\}$;
$\Delta_2 = \{\varepsilon_x : 1 \leq x \leq i+1\}$;
$\Omega_{i,j} = \{\varepsilon_j - \varepsilon_1, \ldots, \varepsilon_j - \varepsilon_{i+1}\}$ (where $i+1 < j \leq n$); and
$\Theta_{i,j} = \{\varepsilon_j + \varepsilon_1, \ldots, \varepsilon_j + \varepsilon_{i+1}\}$ (where $i+1 < j \leq n$).

Let $Y = \langle(\overline{1}), (12), \ldots, (i\, i+1)\rangle$. Then every minimal left coset representative g of X in W may be written $g = g_Y g_X$ where g_X is a minimal left coset representative of X in Y and g_Y is a minimal left coset representative of Y in W. Furthermore, by Proposition 2.1, $N(g) = g_X^{-1} \cdot N(g_Y) \dot{\cup} N(g_X)$.

Now $g_X \cdot \alpha \in \Phi^+$ for each $\alpha = \varepsilon_y - \varepsilon_x \in \Phi_X^+$. It follows that if $g_X(y) < 0$, then $g_X(x) < 0$ and $|g_X(y)| < |g_X(x)|$, whereas if $g_X(x) > 0$ then $g_X(y) > 0$ and $|g_X(x)| < |g_X(y)|$. We may now state the following lemma:

Lemma 4.3. *The set of minimal left coset representatives of X in Y consists of those g_X for which there exists a $J \in \{0, 1, \ldots, i+1\}$ such that*

(i) *for $1 \leq x < y \leq J$, $g_X(x) < 0$, $g_X(y) < 0$ and $|g_X(y)| < |g_X(x)|$; and*
(ii) *for $J < x < y \leq i+1$, $g_X(x) > 0$, $g_X(y) > 0$ and $|g_X(x)| < |g_X(y)|$.*

The set of minimal left coset representatives of Y in W consists of all those g_Y for which both $g_Y(k) > 0$ for each $1 \leq k \leq i+1$ and $g_Y(1) < \ldots < g_Y(i+1)$.

Definition 4.4. Let $g = g_Y g_X$ be a minimal left coset representative of X in W. We define $u(g) = \min\{|g_X(1)|, |g_X(i+1)|\}$.

Note that for any minimal left coset representative g of X in W we have

$$\max\{|g_X(1)|, |g_X(i+1)|\} = i+1.$$

Lemma 4.5. *For g a minimal left coset representative of X in W, $\Delta_1 \notin \mathcal{O}(g)$ if and only if $u(g) = 1$, and $\Delta_2 \notin \mathcal{O}(g)$ if and only if $J = 0$ or $J = i+1$.*

Proof. Recall that $\Delta_1 = \{\varepsilon_y + \varepsilon_x : 1 \leq x < y \leq i+1\}$. Now $\Delta_1 \notin \mathcal{O}(g)$ when either $\Delta_1 \subseteq N(g)$ or $\Delta_1 \cap N(g) = \emptyset$. Suppose $\Delta_1 \subseteq N(g)$. Then at least one of $g(x) < 0$, $g(y) < 0$ holds for each pair $1 \leq x < y \leq i+1$. In addition, if $g(y) > 0$, then $|g(x)| > |g(y)|$. By Lemma 4.3, $g_Y(k) > 0$ for all $1 \leq k \leq i+1$, and $g_Y(1) < \cdots < g_Y(i+1)$. Therefore at least one of $g_X(x) < 0$, $g_X(y) < 0$ holds, and if $g_X(y) > 0$, then $|g_X(x)| > |g_X(y)|$. Now parts (i) and (ii) of Lemma 4.3 imply that either $J = i+1$ (and then $|g_X(i+1)| = 1$ is automatic), or that both $J = i$ and $|g_X(i+1)| = 1$. Likewise $\Delta_1 \cap N(g) = \emptyset$ if and only if either $J = 0$ or $J = 1$ and $|g_X(1)| = 1$. Similar arguments apply to Δ_2. $\square$

Proposition 4.6. *Let $F_1(A_i, B_n, \rho)$, for $0 \leq \rho \leq n-i-1$, be the number of conjugates of X arising from minimal left coset representatives g with $J \in \{0, i+1\}$ and length $l(X) + (i+1)\rho$. Then*

$$F_1(A_i, B_n, \rho) = (n-i-\rho)\begin{pmatrix} i+\rho-1 \\ \rho \end{pmatrix}.$$

Proof. If g is a minimal left coset representative with $J \in \{0, i+1\}$ then $g_X X g_X^{-1} = X$. Without loss of generality then, assume $g_X = 1$. Then $g = g_Y$, so $g(x) > 0$ and $g(x) < g(y)$ for each $x < y$ in $\{1, \ldots, i+1\}$. Suppose $g(j) < 0$ for some $j > i+1$. Then $\Omega_{i,j} \subseteq N(g)$, but $\Theta_{i,j} \subseteq N(g)$ if and only if $|g(j)| > g(i+1)$, and $\Theta_{i,j} \cap N(g) = \emptyset$ if and only if $|g(j)| < g(1)$. If, on the other hand, $g(j) > 0$ then $\Theta_{i,j} \cap N(g) = \emptyset$, and $\Omega_{i,j} \notin \mathcal{O}(g)$ if and only if either $g(j) < g(1)$ or $g(i+1) < g(j)$. Thus for each j with $g(1) < g(j) < g(i+1)$ we add exactly one orbit of size $i+1$ to $\mathcal{O}(g)$. Hence $F_1(A_i, B_n, \rho)$ is simply the number of g such that $\rho(g) = \rho = g(i+1) - g(1) - i$. Using an argument analogous to that in the proof of Proposition 3.2, bearing in mind that $|N_W(Y)|/|Y| = 2^{n-i-1}(n-i-1)!$, we obtain the formula $F_1(A_i, B_n, \rho) = (n-i-\rho)\begin{pmatrix} i+\rho-1 \\ \rho \end{pmatrix}$, as required. $\square$

We assume for the moment that $g = g_X g_Y$ is a minimal left coset representative of X in W for which $1 \leq J \leq i$. Now define

$$\Omega'_{i,j} := g_X(\Omega_{i,j}) = \{\varepsilon_j + \varepsilon_{|g_X(1)|}, \ldots, \varepsilon_j + \varepsilon_{|g_X(J)|}, \varepsilon_j - \varepsilon_{|g_X(J+1)|}, \ldots, \varepsilon_j - \varepsilon_{|g_X(i+1)|}\}$$

and $\Theta'_{i,j} := g_X(\Theta_{i,j})$; that is,

$$\Theta'_{i,j} = \{\varepsilon_j - \varepsilon_{|g_X(1)|}, \ldots, \varepsilon_j - \varepsilon_{|g_X(J)|}, \varepsilon_j + \varepsilon_{|g_X(J+1)|}, \ldots, g(\varepsilon_j) + \varepsilon_{|g_X(i+1)|}\}.$$

Observe that $N(g) \cap \Omega_{i,j} = \left(g_X^{-1} N(g_Y) \dot{\cup} N(g_X)\right) \cap \Omega_{i,j}$. But $N(g_X) \subseteq \Phi_Y^+$ and $\Omega_{i,j} \subseteq \Phi^+ \backslash \Phi_Y^+$. Thus $N(g) \cap \Omega_{i,j} = g_X^{-1}\left(N(g_Y) \cap \Omega'_{i,j}\right)$. Similarly $N(g) \cap \Theta_{i,j} = g_X^{-1}\left(N(g_Y) \cap \Theta'_{i,j}\right)$. Hence $\Omega_{i,j} \in \mathcal{O}(g)$ unless $N(g_Y) \cap \Omega'_{i,j} = \emptyset$ or $\Omega'_{i,j} \subseteq N(g_Y)$, and similarly for $\Theta_{i,j}$.

By Lemma 4.3, $g_Y(k) > 0$ for each $k \in \{1, 2, \ldots, i+1\}$ and g_Y preserves the order of these numbers. So if $g_Y(j) > 0$ then $\Omega_{i,j} \in \mathcal{O}(g)$ unless $\Omega'_{i,j} \cap N(g_Y) = \emptyset$, which happens if and only if $g_Y(j) > g_Y(|g_X(i+1)|)$. If $g_Y(j) < 0$ then $\Omega_{i,j} \in \mathcal{O}(g)$ unless $\Omega'_{i,j} \subseteq N(g_Y)$, which occurs if and only if $g_Y(j) > g_Y(|g_X(1)|)$. Similarly, if $g_Y(j) > 0$ then $\Theta_{i,j} \in \mathcal{O}(g)$ unless $\Theta_{i,j} \cap N(g) = \emptyset$, if and only if $g_Y(j) > g_Y(|g_X(1)|)$. If $g_Y(j) < 0$ then $\Theta_{i,j} \in \mathcal{O}(g)$ unless $\Theta'_{i,j} \subseteq N(g_Y)$, which is if and only if $g_Y(j) > g_Y(|g_X(i+1)|)$.

Recalling definition 4.4, it is now easily seen that for each j we add two orbits (of size $i+1$) to $\mathcal{O}(g)$ if $g_Y(j) < g_Y(u(g))$, one orbit if $g_Y(u(g)) < g_Y(j) < g_Y(i+1)$ and none if $g_Y(i+1) < g_Y(j)$. The following lemma therefore is immediate.

Lemma 4.7. *Let g be a minimal left coset representative of X in W, for which $1 \leq J \leq i$. Let $\rho(g)$ be the total number of $\Omega_{i,j}$ and $\Theta_{i,j}$ in $\mathcal{O}(g)$. Then $\rho(g) = g_Y(u(g)) - u(g) + g_Y(i+1) - (i+1)$.*

Proposition 4.8. *Let $0 \leq \rho \leq 2(n-i)$. The number of minimal left coset representatives g with $1 \leq J \leq i$, $\rho(g) = \rho$ and $u(g) = u$ is*

$$2^{u-1} \sum_{y=\max\{\rho-n+i+1+u,u\}}^{u+[\rho/2]} \begin{pmatrix} y-1 \\ u-1 \end{pmatrix} \begin{pmatrix} \rho+u+i-2y \\ i-u \end{pmatrix}.$$

Proof. We begin by counting all g_Y with $\rho(g) = \rho$ and $u(g) = \min\{|g_X(1)|, |g_X(i+1)|\} = u$ for fixed u and ρ. By Lemma 4.7 $g_Y(u) + g_Y(i+1) = \rho + u + i + 1$. Now $u \leq g_Y(u)$ and $i+1 \leq g_Y(i+1) \leq n$, so $g_Y(u) + n \geq \rho + u + i + 1$ and therefore $g_Y(u) \geq \max\{\rho - n + i + 1 + u, u\}$. Also we have

$$\begin{aligned} \rho + u + i + 1 &= g_Y(u) + g_Y(i+1) \\ &= 2g_Y(u) + (g_Y(i+1) - g_Y(u)) \\ &\geq 2g_Y(u) + (i+1-u). \end{aligned}$$

Therefore $\rho \geq 2g_Y(u)$. In summary, $g_Y(u)$ may take all integer values between $\max\{\rho - n + i + 1 + u, u\}$ and $u + [\rho/2]$. For $g_Y(u) = y$, there are $\begin{pmatrix} y-1 \\ u-1 \end{pmatrix}$ ways to assign the set $g_Y(|g_X(k)|)$ for $|g_X(k)| < u$, $\begin{pmatrix} \rho+u+i-2y \\ i-u \end{pmatrix}$ ways to assign the set $g_Y(|g_X(k)|)$ for $u < |g_X(k)| < i+1$ and finally $2^{n-i-1}(n-i-1)!$ ways to assign $g_Y(j)$ for all other j. Thus, for each u there are

$$2^{n-i-1}(n-i-1)! \sum_{y=\max\{\rho-n+i+1+u,u\}}^{u+[\rho/2]} \binom{y-1}{u-1} \binom{\rho+u+i-2y}{i-u}$$

choices for g_Y. For a fixed u and ρ we must now calculate the number of possible g_X. Suppose $u = |g_X(1)|$. Then by Lemma 4.3, once we have allocated the numbers $|g_X(2)|, \ldots, |g_X(J)|$, g_X will have been uniquely determined. The number of g_X for which $|g_X(1)| = u$ is therefore

$$\sum_{J=1}^{u} \binom{u-1}{J-1} = \sum_{l=0}^{u-1} \binom{u-1}{l} = 2^{u-1}.$$

A symmetrical calculation shows that the number of g_X for which $|g_X(i+1)| = u$ is also 2^{u-1}; the total number of appropriate g_X is then 2^u. Combining the calculations for g_X and g_Y and noting that $|N_W(X)|/|X| = 2^{n-i}(n-i-1)!$ gives the formula in the statement of Proposition 4.8. $\square$

At this point we must use Lemma 4.5 to split the coset representatives g up according to whether or not Δ_1, $\Delta_2 \in \mathcal{O}(g)$. Suppose $u(g) = 1$ (we are still assuming that $1 \leq J \leq i$). Then $\Delta_1 \notin \mathcal{O}(g)$ but $\Delta_2 \in \mathcal{O}(g)$.

Proposition 4.9. *Let $F_2(A_i, B_n, \rho)$ be the number of conjugates of X arising from minimal left coset representatives g with $1 \leq J \leq i$, $u(g) = 1$ and $\rho(g) = \rho$. Then*

$$F_2(A_i, B_n, \rho) = \sum_{y=\max\{\rho-n+i+2,1\}}^{[\rho/2]+1} \binom{\rho+i+1-2y}{i-1}.$$

Moreover if $u(g) = 1$ and $\rho(g) = \rho$, then $l(gXg^{-1}) = l(X) + (\rho+1)(i+1)$.

Proof. The result is immediate upon substitution of $u(g) = 1$ into the formula derived in Proposition 4.8. $\square$

By Lemma 4.5, if $u(g) > 1$ then both Δ_1 and Δ_2 are in $\mathcal{O}(g)$.

Proposition 4.10. *Let $F_3(A_i, B_n, \rho)$ be the number of conjugates of X arising from minimal left coset representatives g with $1 \leq J \leq i$, $u(g) > 1$ and $\rho(g) = \rho$. Then*

$$F_3(A_i, B_n, \rho) = \sum_{r=1}^{i-1} 2^r \sum_{s=\max\{\rho-n+i+1,0\}}^{[\rho/2]} \binom{s+r}{r} \binom{\rho-2s+i-r-1}{i-r-1}.$$

and if $\rho(g) = \rho$ then $l(gXg^{-1}) = 2l(X) + (\rho+1)(i+1)$.

Proof. By Proposition 4.8, we have

$$\begin{aligned} F_3(A_i,B_n,\rho) &= \sum_{u=2}^{i} 2^{u-1} \sum_{y=\max\{\rho-n+i+1+u,u\}}^{[\rho/2]+u} \binom{y-1}{u-1}\binom{\rho+u+i-2y}{i-u} \\ &= \sum_{r=1}^{i-1} 2^{r} \sum_{y=\max\{\rho-n+i+2+r,r+1\}}^{[\rho/2]+r+1} \binom{y-1}{r}\binom{\rho+r+1+i-2y}{i-r-1} \\ &= \sum_{r=1}^{i-1} 2^{r} \sum_{s=\max\{\rho-n+i+1,0\}}^{[\rho/2]} \binom{s+r}{r}\binom{\rho+i-r-1-2s}{i-r-1} \end{aligned}$$

□

Combining Propositions 4.6, 4.9 and 4.10 we obtain

Theorem 4.11. *Suppose $A_i \cong X \le W \cong B_n$ with $i > 1$. Then the length polynomial is*

$$\begin{aligned} \Lambda_{A_i,B_n}(t) &= t^{i(i+1)/2} \sum_{\rho=0}^{n-i-1} t^{(i+1)\rho}(n-i-\rho)\binom{i+\rho-1}{\rho} \\ &+ t^{(i+1)(i+2)/2} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{y=\max\{\rho-n+i+2,1\}}^{[\rho/2]+1} \binom{\rho+i+1-2y}{i-1} \\ &+ t^{(i+1)^2} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{r=1}^{i-1} 2^{r} \sum_{s=\max\{\rho-n+i+1,0\}}^{[\rho/2]} \binom{s+r}{r}\binom{\rho-2s+i-r-1}{i-r-1}. \end{aligned}$$

5. $W \cong D_n$

Now we assume $W \cong D_n$, $n \ge 4$, and X is a standard irreducible parabolic subgroup of W. Then X is of type A_i for some $1 \le i < n$ or type D_i for some $4 \le i \le n$. It is useful to think of D_n as the subgroup (of index 2) of B_n generated by $\mathrm{Sym}(n)$ and the elements of $(\mathbb{Z}_2)^n$ involving an even number of sign changes. We use the same notation for elements of D_n as for elements of B_n. For example $(\bar{1}\ \bar{2})(3\ 4)$ is an element of D_4 because it has two (an even number) of sign changes, indicated by minus signs above the numbers 1 and 2. However $(\bar{1}\ \bar{2})(\bar{3}\ 4)$ has three sign changes so is not an element of D_4.

Let V be an n-dimensional real vector space with orthonormal basis $\varepsilon_1,\ldots,\varepsilon_n$. Then W acts on V in the same way as B_n does, as follows: for $w \in W$, $w\cdot\varepsilon_i = \varepsilon_{w(i)}$ if $w(i) > 0$ and $-\varepsilon_{|w(i)|}$ otherwise. The elements $(12),\ldots,(n-1\ n)$ and $(\bar{1}\bar{2})$ can be shown to generate W and obey all the relations in the Coxeter graph for W. The fundamental root corresponding to $(i\ i+1)$ (for $1 \le i < n$) is $\varepsilon_{i+1} - \varepsilon_i$ and the fundamental root corresponding to $(\bar{1}\bar{2})$ is $\varepsilon_2 + \varepsilon_1$. The set of positive roots for W is

$$\Phi^+ = \{\varepsilon_y \pm \varepsilon_x : 1 \leq x < y \leq n\}.$$

We begin our analysis with the case where $X \cong D_i$, some i. Hence

$$X = \langle (\bar{1}\bar{2}), (12), \ldots, (i-1\ i) \rangle$$

with

$$\Phi^+_X = \{\varepsilon_y \pm \varepsilon_x : 1 \leq x < y \leq i\}.$$

For $i < j \leq n$ define $\Omega_{i,j} := \{\varepsilon_j \pm \varepsilon_1, \ldots, \varepsilon_j \pm \varepsilon_i\}$. Since the X-orbits of Φ are Φ_X, $\pm\Omega_{i,j}$ (for $i < j \leq n$) and the fixed points $\pm\{\varepsilon_k - \varepsilon_j\}$ (for $i < j < k \leq n$), we have, analogously to (2),

$$l(gXg^{-1}) = l(X) + \sum_{\Omega_{i,j} \in \mathcal{O}(g)} 2i. \tag{3}$$

Let g be a minimal left coset representative of X in W. By considering $|g(j)|$ and using the same reasoning as in the previous section we deduce that $\Omega_{i,j} \in \mathcal{O}(g)$ if and only if $|g(j)| < g(i)$. We write $\rho(g)$ for the number of $\Omega_{i,j}$ with $\Omega_{i,j} \in \mathcal{O}(g)$.

Proposition 5.1. *Let $D_i \cong X \leq W \cong D_n$. Each conjugate of X has length $l(X) + 2i\rho$ for some $0 \leq \rho \leq n-i$. Define $F(D_i, D_n, \rho)$ to be the number of conjugates of X having length $l(X) + 2i\rho$ (for $0 \leq \rho \leq n-i$). Then*

$$F(D_i, D_n, \rho) = \begin{pmatrix} \rho + i - 1 \\ i - 1 \end{pmatrix}.$$

Proof. For each appropriate ρ we first calculate the number of minimal left coset representatives g of X in W for which $\rho(g) = \rho$. If $\rho(g) = \rho$ then we show, by identical reasoning to that employed in Proposition 4.1, that there are $\begin{pmatrix} \rho + i - 1 \\ i - 1 \end{pmatrix}$ choices for $g(1), \ldots g(i-1)$. Once $g(1), \ldots, g(i)$ are chosen, there are $2^{(n-i-1)}(n-i)!$ choices for the other $g(j)$ (the sign above each $g(j)$ in the signed permutation expression for g can be '+' or '−', but there must be an even number of '−' signs, so we have a factor of $2^{(n-i-1)}$ rather than $2^{(n-i)}$). However each conjugate arises from $2^{(n-i-1)}(n-i)!$ coset representatives. Thus $F(D_i, D_n, \rho) = \begin{pmatrix} \rho + i - 1 \\ i - 1 \end{pmatrix}$, so completing the proof. □

It remains to consider the situation where X is a standard parabolic subgroup of W with $X \cong A_i$, for some i. There are three possibilities here: either X is conjugate to $\langle(12),\ldots,(i\,i+1)\rangle$ or X is conjugate to $\langle(\bar{1}\bar{2}),(23),\ldots,(i\,i+1)\rangle$, or $i=3$ and $X=\langle(12),(23),(\bar{1}\bar{2})\rangle$. However $\langle(\bar{1}\bar{2}),(23),\ldots,(i\,i+1)\rangle$ can be mapped to $\langle(12),\ldots,(i\,i+1)\rangle$ via a length-preserving automorphism (induced from a non-trivial automorphism of the Coxeter graph). Therefore, unless $i=3$ and X is conjugate to $\langle(12),(23),(\bar{1}\bar{2})\rangle$ (which we deal with in Lemma 5.2) we may assume without loss of generality that $X=\langle(12),\ldots,(i\,i+1)\rangle$.

Lemma 5.2. *Suppose $A_3\cong X<W\cong D_n$, and that $X=\langle(12),(23),(\bar{1}\bar{2})\rangle$. Then the length polynomial is*

$$\Lambda_{X,D_n}(t)=\sum_{\rho=0}^{n-3}\binom{\rho+2}{2}t^{6(1+\rho)}$$

Proof. The normalizer of X has order $2{\cdot}4!{\cdot}2^{n-4}(n-3)!$. Thus X has $\binom{n}{3}$ conjugates. For $4\le j\le n$ define $\Psi_j=\{e_j\pm e_1,e_j\pm e_2,e_j\pm e_3\}$. The orbits of X are $\Phi_X=\{e_2\pm e_1,e_3\pm e_2,e_3\pm e_1\}$, $\pm\Psi_j$ for $4\le j\le n$ and the fixed points $\pm\{e_j\pm e_k\}$ for $4\le k<j\le n$. Let g be a minimal left coset representative of X in W. The only orbits that can contribute to $\mathcal{O}(g)$ are the Ψ_j. So we may modify Proposition 2.4 thus:

$$l(gXg^{-1})=6+6|\{j\in\{4,\ldots,n\}:\Psi_j\in\mathcal{O}(g)\}|. \tag{4}$$

Now $N(g)\cap\Phi_X=\emptyset$. If $g(x)<0$ and $g(y)<0$ for $1\le x<y\le 3$, then $g(e_x+e_y)\in\Phi^-$, which is impossible. So if $g(2)<0$ or $g(3)<0$, then $g(1)>0$, which implies either e_2-e_1 or e_3-e_1 is in $N(g)$. Hence $g(2)>0$ and $g(3)>0$. Similar observations show that the condition $0<|g(1)|<g(2)<g(3)$ is necessary and sufficient for g to be a minimal left coset representative. We next consider which Ψ_j lie in $\mathcal{O}(g)$. If $|g(j)|>g(3)$, then either $\Psi_j\subseteq N(g)$ or $\Psi_j\cap N(g)=\emptyset$. However if $|g(j)|<g(3)$, then $\Psi_j\in\mathcal{O}(g)$. Therefore the number $\rho(g)$ of Ψ_j with $\Psi_j\in\mathcal{O}(g)$ is $g(3)-3$. We now fix ρ and determine the number of possible g with $\rho(g)=\rho$. Now $g(3)=3+\rho$, so there are now $\binom{2+\rho}{2}$ choices for $|g(1)|$ and $g(2)$, and there are then $(n-3)!$ ways to assign the other $|g(k)|$, meaning that (since we know $g(2)>0$ and an even number of k have $g(k)<0$), there are $2^{n-3}\binom{2+\rho}{2}(n-3)!$ possible coset representatives g giving conjugates of X with length $6+6\rho$ (by Equation 4). However each conjugate arises from $2^{n-3}(n-3)!$ cosets. We must therefore divide by $2^{n-3}(n-3)!$ to get that there are $\binom{2+\rho}{2}$ conjugates of X with length $6+6\rho=6(1+\rho)$. Therefore the length polynomial is $\Lambda_{X,D_n}(t)=\sum_{\rho=0}^{n-3}\binom{\rho+2}{2}t^{6(1+\rho)}$. $\square$

Note that the length polynomial for A_3 in D_4 is given by either of Lemma 5.2 and Lemma 5.3 below. We also remark that the situation of Lemma 5.2 does not arise in B_n because $\langle(12),(23),(\bar{1}\bar{2})\rangle$ is not a parabolic subgroup of B_n.

For the rest of this section, we assume that X is not conjugate to $\langle(12),(23),(\bar{1}\bar{2})\rangle$. Then without loss of generality we may set $X = \langle(12),\ldots,(i\ i+1)\rangle$. The coset representatives are the same as in the case $X \cong A_i$ and $W \cong B_n$, with the proviso that the each representative must of course be an element of D_n. That is, it must have an even number of '-' signs in its expression as a signed permutation. So there are half as many coset representatives; but in nearly all cases there are the same number of conjugates. To see this, note that if $i < n-1$ and g is an element of B_n conjugating X to X_1, then the product $g(\bar{n})$ also conjugates X to X_1 and exactly one of g and $g(\bar{n})$ is an element of D_n. If $i = n-1$ and n is odd, then for any $g \in B_n$, g and $g(\bar{1})\cdots(\bar{n})$ conjugate X to the same subgroup and again, exactly one of these elements lies in D_n. Therefore if n is odd or $i < n-1$, then X has the same number of conjugates in D_n as it does in B_n. The case where n is even and $i = n-1$ is dealt with in the next lemma.

Lemma 5.3. *Suppose n is even and $A_{n-1} \cong X < W \cong D_n$. Then the length polynomial is*

$$\Lambda_{A_{n-1},D_n}(t) = t^{n(n-1)/2} + (2^{n-2}-1)t^{n(n-1)}.$$

Proof. The normalizer of $X = \langle(12),\ldots,(n{-}1\ n)\rangle$ is generated by X and $(\bar{1})\cdots(\bar{n})$ and has order $2n!$. Hence X has $\frac{|D_n|}{2n!} = 2^{n-2}$ conjugates. The X-orbits of Φ are Φ_X and $\pm\Delta$ where $\Delta = \{\varepsilon_x + \varepsilon_y : 1 \leq x < y \leq n\}$. Clearly $\Delta \in \mathcal{O}(g)$ for any non-trivial minimal coset representative g. Thus, by Proposition 2.4, $l(gXg^{-1}) = |\Phi^+| = n(n{-}1)$. Hence $\Lambda_{A_{n-1},D_n}(t) = t^{n(n-1)/2}+(2^{n-2}-1)t^{n(n-1)}$. □

For the rest of this section we will assume that either $i < n-1$ or n is odd (or both), and that X is not conjugate to $\langle(12),(23),(\bar{1}\bar{2})\rangle$. So we can assume that $X = \langle(12),(23),\ldots,(i\ i+1)\rangle$. Put $\Delta_1 = \{\varepsilon_x + \varepsilon_y : 1 \leq x < y \leq i+1\}$ and, for $i+1 < j \leq n$, $\Omega_{i,j} = \{\varepsilon_j - \varepsilon_1, \ldots, \varepsilon_j - \varepsilon_{i+1}\}$ and $\Theta_{i,j} = \{\varepsilon_j + \varepsilon_1, \ldots, \varepsilon_j + \varepsilon_{i+1}\}$. Then the X-orbits of Φ are Φ_X, $\pm\Delta_1$, $\pm\Omega_{i,j}$ $(i+1 < j \leq n)$, $\pm\Theta_{i,j}$ and the fixed points $\pm\{\varepsilon_j \pm \varepsilon_k\}$ $(i+1 < k < j \leq n)$. Since X has the same number of conjugates in D_n as it does in B_n, the formulae of Propositions 4.6, 4.9 and 4.10 will therefore still apply. However the lengths of the conjugates will be different, because of the absence of Δ_2. Using the notation of Section 4, for a coset representative g, if $J = 0$ or $J = i+1$ then $\Delta_1 \notin \mathcal{O}(g)$. Therefore the length $l(gXg^{-1})$ differs from $l(X)$ by $(i+1)\rho(g)$ where $\rho(g)$ is defined to be the total number of $\Omega_{i,j}$ and $\Theta_{i,j}$ belonging to $\mathcal{O}(g)$. As before at most one of $\Theta_{i,j} \in \mathcal{O}(g)$, $\Omega_{i,j} \in \mathcal{O}(g)$ holds. So we have

Proposition 5.4. *For $0 \leq \rho \leq n-i-1$, let $F_1(A_i, D_n, \rho)$ be the number of conjugates of X arising from minimal left coset representatives g with $J \in \{0, i+1\}$ and length $l(X) + (i+1)\rho$. Then*

$$F_1(A_i, D_n, \rho) = (n-i-\rho) \left(\begin{array}{c} i+\rho-1 \\ \rho \end{array} \right).$$

Now if $u(g) = 1$, where $u(g)$ is defined as in Definition 4.4, we again have $\Delta_1 \notin \mathcal{O}(g)$, so if $\rho(g) = \rho$, $l(gXg^{-1}) = l(X) + (i+1)\rho$. Thus

Proposition 5.5. *Let $F_2(A_i, D_n, \rho)$ be the number of conjugates of X arising from minimal left coset representatives g with $u(g) = 1$ and $\rho(g) = \rho$. Then*

$$F_2(A_i, D_n, \rho) = \sum_{y=\max\{\rho-n+i+2,1\}}^{[\rho/2]+1} \left(\begin{array}{c} \rho+i+1-2y \\ i-1 \end{array} \right)$$

and $l(gXg^{-1}) = l(X) + (i+1)\rho$.

Finally if $u(g) > 1$ then $\Delta_1 \in \mathcal{O}(g)$, so we get

Proposition 5.6. *Let $F_3(A_i, D_n, \rho)$ be the number of conjugates of X arising from minimal left coset representatives g with $u(g) > 1$ and $\rho(g) = \rho$. Then*

$$F_3(A_i, D_n, \rho) = \sum_{r=1}^{i-1} 2^r \sum_{s=\max\{\rho-n+i+1,0\}}^{[\rho/2]} \left(\begin{array}{c} s+r \\ r \end{array} \right) \left(\begin{array}{c} \rho-2s+i-r-1 \\ i-r-1 \end{array} \right).$$

and if $\rho(g) = \rho$ then $l(gXg^{-1}) = 2l(X) + (i+1)\rho$.

Combining these three results we get:

Theorem 5.7. *Let $A_i \cong X < W \cong D_n$, where either n is odd or $i < n-1$. If X is not conjugate to $\langle (12), (23), (\bar{1}\bar{2}) \rangle$, then the length polynomial is*

$$\begin{aligned} \Lambda_{A_i,D_n}(t) &= t^{i(i+1)/2} \sum_{\rho=0}^{n-i-1} t^{(i+1)\rho} (n-i-\rho) \tbinom{i+\rho-1}{\rho} \\ &+ t^{i(i+1)/2} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{y=\max\{\rho-n+i+2,1\}}^{[\rho/2]+1} \tbinom{\rho+i+1-2y}{i-1} \\ &+ t^{i(i+1)} \sum_{\rho=0}^{2(n-i-1)} t^{(i+1)\rho} \sum_{r=1}^{i-1} 2^r \sum_{s=\max\{\rho-n+i+1,0\}}^{[\rho/2]} \tbinom{s+r}{r} \tbinom{\rho-2s+i-r-1}{i-r-1}. \end{aligned}$$

6. $W \cong A_n$ **and** $X \cong A_i \times A_j$

Conjugates of parabolic subgroups which are not irreducible have a much more complicated distribution of lengths. In this section we obtain formulas for the number of conjugates with minimal and maximal lengths in the case $A_i \times A_j \cong X < W \cong A_n$, to give a flavour of the calculations required.

Let $X_1 = \langle (12), \ldots, (i\; i+1)\rangle \cong A_i$, $X_2 = \langle (i+2\; i+3), \ldots, (l\; l+1)\rangle \cong A_j$ (so $l = i+j+1$) and $X = X_1 \times X_2 < W = \langle (12) \ldots, (n\; n+1)\rangle \cong A_n$.
Define $\Delta = \{\varepsilon_x - \varepsilon_y : 1 \leq x < i+2 \leq y \leq l+1\}$, and, for $l+2 \leq k \leq n+1$, $\Omega_{i,k} = \{\varepsilon_x - \varepsilon_k : 1 \leq x \leq i+1\}$ and $\Theta_{i,k} = \{\varepsilon_x - \varepsilon_k : i+2 \leq x \leq l+1\}$. Then the X-orbits on Φ are Φ_{X_1}, Φ_{X_2}, $\pm\Delta$, $\pm\Omega_{i,k}$, $\pm\Theta_{i,k}$ $(l+2 \leq k \leq n+1)$ and the fixed points $\pm\{\varepsilon_x - \varepsilon_y\}$ $(l+2 \leq x < y \leq n+1)$.

By Lemma 2.2, we quickly see that g is a minimal left coset representative of X in W if and only if $g(x) < g(y)$ for all x, y with $1 \leq x < y \leq i+1$ or $i+2 \leq x < y \leq l+1$.

Now $l(X) = \frac{1}{2}\left(i(i+1) + j(j+1)\right)$. The maximal length of any conjugate would occur when as many orbits as possible are in $\mathcal{O}(g)$, that is all except fixed points and Φ_X. Thus the maximum length, by Proposition 2.4, is

$$|\Phi^+| - \begin{pmatrix} n-i-j-1 \\ 2 \end{pmatrix} = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} - \begin{pmatrix} n-i-j-1 \\ 2 \end{pmatrix}.$$

The next lemma is a consequence of the definitions of $\Delta, \Omega_{i,k}$ and $\Theta_{i,k}$, and the requirements on minimal left coset representatives stated above.

Lemma 6.1. *Let g be a minimal left coset representative of X in W, and $l+2 \leq k \leq n+1$.*

(i) $\Delta \cap N(g) = \emptyset$ *if and only if* $g(i+1) < g(i+2)$.

(ii) $\Delta \subseteq N(g)$ *if and only if* $g(l+1) < g(1)$.

(iii) $\Omega_{i,k} \in \mathcal{O}(g)$ *if and only if* $g(1) < g(k) < g(i+1)$.

(iv) $\Theta_{i,k} \in \mathcal{O}(g)$ *if and only if* $g(i+2) < g(k) < g(l+1)$.

Proposition 6.2. *The number of conjugates of X with length $l(X)$ is*

$$\begin{cases} (n+1-i-j)(n-i-j) & \text{if } i \neq j; \\ \frac{1}{2}(n+1-i-j)(n-i-j) & \text{if } i = j. \end{cases}$$

Proof. Suppose $l(gXg^{-1}) = l(X)$. Then by Lemma 6.1 (i) and (ii), either $g(1) < \cdots < g(i{+}1) < g(i{+}2) < \cdots g(l{+}1)$ or $g(i{+}2) < \cdots < g(l{+}1) < g(1) < \cdots g(i{+}1)$.

Now (iii) and (iv) imply that $g(x+1) = g(1)+x$ for $1 \leq x \leq i$, and $g(x+i+2) = g(i+2)+x$ for $1 \leq x \leq j$ (recall that $l = i+j+1$). If $g(i+1) < g(i+2)$, then there are $(n-l)!\sum_{r=1}^{n+1-l}(n+2-l-r) = (n-l)!(n+1-l)(n+2-l)/2$ choices for g, where $r = g(1)$, and there are the same number of choices when $g(l+1) < g(1)$, yielding a total number $(n-l)!(n+1-l)(n+2-l)$ of coset representatives g for which $l(gXg^{-1}) = l(X)$. To obtain the number of minimal length conjugates we must divide by $|N_W(X)|/|X|$, which is $(n-l)!$ when $i \neq j$ and $2(n-l)!$ otherwise. $\square$

Proposition 6.3. *The number of conjugates of X having (maximal) length* $\binom{n+1}{2} - \binom{n-i-j-1}{2}$ *is*

$$2^{-\delta_{ij}} \sum_{s=0}^{i+j-2} \binom{s+n-i-j-1}{s} \cdot \left[2 \sum_{r=\max\{0,s+1-i\}}^{j-1} \binom{s}{r} + (i+j-1-s)\left(\binom{s}{j-1} + \binom{s}{i-1}\right)\right]$$

where δ_{ij} is defined to be 1 if $i = j$ and 0 otherwise.

Proof. Suppose gXg^{-1} has maximal length. By Lemma 6.1 (i) and (ii), we have $g(i+2) < g(i+1)$ and $g(1) < g(l+1)$. Set $\lambda = \max\{g(1), g(i+2)\}$ and $\mu = \min\{g(i+1), g(l+1)\}$. Then $m := \mu - \lambda > 0$. Now by Lemma 6.1 (iii) and (iv), for each $k > l+1$ we have $\lambda < g(k) < \mu$. So there are $\frac{(m+n-l-1)!}{(m-1)!}$ ways to assign the set $\{g(k) : k > l+1\}$. To calculate the number of different g once the $g(k)$ have been decided we need to consider separately the four possibilities for the ordering of $g(1), g(i+1), g(i+2)$ and $g(l+1)$.

Firstly, suppose $g(1) < g(i+2) < g(i+1) < g(l+1)$. We have $i+2 < \mu < l$ because $g(1) < \cdots < g(i+1) = \mu$ and $g(i+2) < \mu$. But also $\lambda \geq 2$, so $\mu \geq 2+m$. Once we have specified $g(i+3), \ldots, g(l)$, g will be uniquely determined (we have already dealt with $g(k)$ for $k > l+1$). The first $\mu - 1 - (i+1)$ of these will be less than $g(i+1)$. So there are

$$\sum_{\mu=\max\{i+2,2+m\}}^{l} \binom{m-1}{\mu-i-2} = \sum_{r=\max\{0,m-i\}}^{l-i-2} \binom{m-1}{r}$$

choices for $g(i+3), \ldots, g(l)$.

Next, let $g(1) < g(i+2) < g(l+1) < g(i+1)$. Then $\lambda = g(i+2)$, $\mu = l+1$ and $2 \leq \lambda \leq l-m$. This time $g(i+3), \ldots, g(l)$ must all lie between λ and μ, so the number of choices for them is $\begin{pmatrix} m-1 \\ j-1 \end{pmatrix}$; there are $l-m-1$ possible λ, giving $(l-m-1)\begin{pmatrix} m-1 \\ j-1 \end{pmatrix}$ choices for $g(i+3),\ldots,g(l)$ (which determines g).

If $g(i+2) < g(1) < g(i+1) < g(l+1)$ then by symmetry there are $(l-m-1)\begin{pmatrix} m-1 \\ i-1 \end{pmatrix}$ choices for $g(2),\ldots,g(i)$.

Finally, let $g(i+2) < g(1) < g(l+1) < g(i+1)$. Calculations similar to those used in the first case show that there are $\sum_{r=\max\{0,m-i\}}^{l-i-2}\begin{pmatrix} m-1 \\ r \end{pmatrix}$ choices for $g(2),\ldots,g(i)$.

Once we have chosen m therefore, the number of choices for g (replacing l by $i+j+1$ in the above formulae where it occurs) is

$$\tfrac{(m+n-i-j-2)!}{(m-1)!}\left[2\sum_{r=\max\{0,m-i\}}^{j-1}\tbinom{m-1}{r} + (i+j-m)\left(\tbinom{m-1}{j-1}+\tbinom{m-1}{i-1}\right)\right].$$

Now m can range from 1 to $i+j-1$, so the total possible number of g for which $l(gXg^{-1})$ is maximal is

$$\sum_{m=1}^{i+j-1}\tfrac{(m+n-i-j-2)!}{(m-1)!}\left[2\sum_{r=\max\{0,m-i\}}^{j-1}\tbinom{m-1}{r} + (i+j-m)\left(\tbinom{m-1}{j-1}+\tbinom{m-1}{i-1}\right)\right].$$

Substituting $s = m-1$ and dividing by $|N_W(X)|/|X|$ gives the result. □

7. Parabolic Subgroups of the Exceptional Groups

In this section we give the length distributions for conjugates of irreducible standard parabolic subgroups of the exceptional Coxeter groups. The calculations were carried out using the computer algebra package MAGMA [2]. For each conjugacy class of parabolic subgroups of the given Coxeter group, the type of an element X of the class is given in the first column, the size of the conjugacy class in the second column and the distribution of lengths in the conjugacy class in the third column, where 'l^k' means k subgroups of length l. So for example in Table 1, we see that the conjugacy class of parabolic subgroups of type D_4 has 45 subgroups, one of length 12, 8 of length 20, 8 of length 28 and 28 of length 36. In other words the length polynomial is $t^{12} + 8t^{20} + 8t^{28} + 28t^{36}$. Note that Table 1 appeared as

Table 1 of [5] but is included here for completeness. Note also that in F_4 (Table 4), there are two conjugacy classes of parabolic subgroups of type B_3. Each has the same length distribution, namely $9^1 \cdot 17^2 \cdot 23^9$ as given in the table.

Type of X	$[W : N_W(X)]$	Lengths of conjugates of X
A_1	36	$1^6 \cdot 3^5 \cdot 5^5 \cdot 7^5 \cdot 9^4 \cdot 11^3 \cdot 13^3 \cdot 15^2 \cdot 17^1 \cdot 19^1 \cdot 21^1$
A_2	120	$3^5 \cdot 6^{10} \cdot 9^{15} \cdot 12^{16} \cdot 15^{15} \cdot 18^{18} \cdot 21^{14} \cdot 24^8 \cdot 27^9 \cdot 30^{10}$
A_3	270	$6^5 \cdot 10^{12} \cdot 12^9 \cdot 14^8 \cdot 16^{16} \cdot 18^{14} \cdot 20^{38} \cdot 22^{14} \cdot 24^{16} \cdot$
		$26^{23} \cdot 28^2 \cdot 30^{46} \cdot 34^{67}$
A_4	216	$10^4 \cdot 15^4 \cdot 20^{28} \cdot 25^{10} \cdot 30^{40} \cdot 35^{130}$
A_5	36	$15^1 \cdot 35^{35}$
D_4	45	$12^1 \cdot 20^8 \cdot 28^8 \cdot 36^{28}$
D_5	27	$20^2 \cdot 36^{25}$

TABLE 1. Parabolic subgroups of E_6

Type of X	$[W : N_W(X)]$	Lengths of conjugates of X
A_1	63	$1^7 \cdot 3^6 \cdot 5^6 \cdot 7^6 \cdot 9^6 \cdot 11^5 \cdot 13^5 \cdot 15^4 \cdot 17^4 \cdot 19^3 \cdot 21^3 \cdot$
		$23^2 \cdot 25^2 \cdot 27^1 \cdot 29^1 \cdot 31^1 \cdot 33^1$
A_2	336	$3^6 \cdot 6^{12} \cdot 9^{18} \cdot 12^{24} \cdot 15^{25} \cdot 18^{30} \cdot 21^{28} \cdot 24^{32} \cdot 27^{27} \cdot 30^{30} \cdot$
		$33^{22} \cdot 36^{24} \cdot 39^{13} \cdot 42^{14} \cdot 45^{15} \cdot 48^{16}$
A_3	1260	$6^6 \cdot 10^{15} \cdot 12^9 \cdot 14^{20} \cdot 16^{16} \cdot 18^{27} \cdot 20^{48} \cdot 22^{19} \cdot 24^{54} \cdot$
		$26^{32} \cdot 28^{61} \cdot 30^{62} \cdot 32^{32} \cdot 34^{124} \cdot 36^{23} \cdot 38^{73} \cdot$
		$40^{66} \cdot 42^{51} \cdot 44^{68} \cdot 46^9 \cdot 48^{128} \cdot 50^3 \cdot 52^{134} \cdot 56^{180}$
A_4	2016	$10^5 \cdot 15^{12} \cdot 20^{39} \cdot 25^{33} \cdot 30^{104} \cdot 35^{171} \cdot 40^{162} \cdot 45^{154} \cdot$
		$50^{221} \cdot 55^{468} \cdot 60^{647}$
A_5	1008	$15^2 \cdot 21^5 \cdot 30^{15} \cdot 35^5 \cdot 36^6 \cdot 41^{20} \cdot 47^1 \cdot 50^{56} \cdot 56^{329} \cdot 62^{539}$
A_6	288	$21^1 \cdot 56^{71} \cdot 63^{216}$
D_4	315	$12^1 \cdot 20^8 \cdot 28^{18} \cdot 36^{43} \cdot 44^{39} \cdot 52^{53} \cdot 60^{153}$
D_5	378	$20^2 \cdot 30^5 \cdot 36^{25} \cdot 46^{17} \cdot 52^{36} \cdot 62^{293}$
D_6	63	$30^1 \cdot 62^{62}$
E_6	28	$36^1 \cdot 63^{27}$

TABLE 2. Parabolic subgroups of E_7

Type of X	$[W : N_W(X)]$	Lengths of conjugates of X
A_1	120	$1^8 \cdot 3^7 \cdot 5^7 \cdot 7^7 \cdot 9^7 \cdot 11^7 \cdot 13^7 \cdot 15^6 \cdot 17^6 \cdot 19^6 \cdot 21^6 \cdot$
		$23^5 \cdot 25^5 \cdot 27^4 \cdot 29^4 \cdot 31^4 \cdot 33^4 \cdot 35^3 \cdot 37^3 \cdot 39^2 \cdot 41^2 \cdot$
		$43^2 \cdot 45^2 \cdot 47^1 \cdot 49^1 \cdot 51^1 \cdot 53^1 \cdot 55^1 \cdot 57^1$
A_2	1120	$3^7 \cdot 6^{14} \cdot 9^{21} \cdot 12^{28} \cdot 15^{35} \cdot 18^{42} \cdot 21^{42} \cdot 24^{48} \cdot 27^{54} \cdot$
		$30^{60} \cdot 33^{55} \cdot 36^{60} \cdot 39^{52} \cdot 42^{56} \cdot 45^{60} \cdot 48^{64} \cdot$
		$51^{51} \cdot 54^{54} \cdot 57^{38} \cdot 60^{40} \cdot 63^{42} \cdot 66^{44} \cdot 69^{23} \cdot$
		$72^{24} \cdot 75^{25} \cdot 78^{26} \cdot 81^{27} \cdot 84^{28}$
A_3	7560	$6^7 \cdot 10^{18} \cdot 12^9 \cdot 14^{26} \cdot 16^{16} \cdot 18^{47} \cdot 20^{48} \cdot 22^{40} \cdot 24^{66} \cdot$
		$26^{42} \cdot 28^{95} \cdot 30^{90} \cdot 32^{88} \cdot 34^{162} \cdot 36^{91} \cdot 38^{160} \cdot 40^{98} \cdot$
		$42^{178} \cdot 44^{101} \cdot 46^{158} \cdot 48^{183} \cdot 50^{153} \cdot 52^{254} \cdot 54^{172} \cdot$
		$56^{340} \cdot 58^{170} \cdot 60^{205} \cdot 62^{319} \cdot 64^{121} \cdot 66^{280} \cdot$
		$68^{67} \cdot 70^{340} \cdot 72^{85} \cdot 74^{303} \cdot 76^{189} \cdot 78^{205} \cdot 80^{253} \cdot$
		$82^{50} \cdot 84^{474} \cdot 86^{30} \cdot 88^{348} \cdot 90^{15} \cdot 92^{407} \cdot$
		$94^5 \cdot 96^{476} \cdot 100^{576}$
A_4	24192	$10^6 \cdot 15^{16} \cdot 20^{59} \cdot 25^{57} \cdot 30^{141} \cdot 35^{264} \cdot 40^{330} \cdot 45^{335} \cdot$
		$50^{469} \cdot 55^{816} \cdot 60^{1264} \cdot 65^{1062} \cdot 70^{1114} \cdot 75^{1331} \cdot 80^{1284} \cdot$
		$85^{1112} \cdot 90^{1894} \cdot 95^{3249} \cdot 100^{2345} \cdot 105^{2926} \cdot 110^{4118}$
A_5	40320	$15^4 \cdot 21^{15} \cdot 27^{16} \cdot 30^{30} \cdot 33^1 \cdot 35^{35} \cdot 36^{58} \cdot 41^{20} \cdot 42^{143} \cdot$
		$45^{30} \cdot 47^{27} \cdot 48^{19} \cdot 50^{56} \cdot 51^{50} \cdot 53^1 \cdot 56^{358} \cdot 57^{91} \cdot 59^1 \cdot$
		$60^{290} \cdot 62^{849} \cdot 63^{61} \cdot 65^{61} \cdot 66^{189} \cdot 68^{252} \cdot 71^{210} \cdot 72^{104} \cdot$
		$74^{35} \cdot 77^{1201} \cdot 78^{19} \cdot 80^{171} \cdot 83^{1531} \cdot 86^{367} \cdot 89^{347} \cdot 92^{1340} \cdot$
		$95^{15} \cdot 98^{5348} \cdot 101^3 \cdot 104^{6990} \cdot 110^{7182} \cdot 116^{12800}$
A_6	34560	$21^3 \cdot 28^6 \cdot 42^{62} \cdot 49^7 \cdot 56^{71} \cdot 63^{313} \cdot 70^{36} \cdot 77^{132} \cdot 84^{1006} \cdot$
		$91^{177} \cdot 98^{1612} \cdot 105^{4605} \cdot 112^{8157} \cdot 119^{18373}$
A_7	8640	$28^1 \cdot 84^{133} \cdot 92^1 \cdot 112^{1786} \cdot 120^{6719}$
D_4	3150	$12^1 \cdot 20^8 \cdot 28^{18} \cdot 36^{63} \cdot 44^{65} \cdot 52^{107} \cdot 60^{223} \cdot 68^{269} \cdot$
		$76^{212} \cdot 84^{265} \cdot 92^{347} \cdot 100^{424} \cdot 108^{1148}$
D_5	7560	$20^2 \cdot 30^5 \cdot 36^{25} \cdot 40^{15} \cdot 46^{17} \cdot 52^{36} \cdot 56^{32} \cdot 62^{293} \cdot$
		$68^{71} \cdot 72^{152} \cdot 78^{265} \cdot 82^5 \cdot 84^1 \cdot 88^{498} \cdot 94^{95} \cdot 98^{124} \cdot$
		$104^{1328} \cdot 114^{4596}$
D_6	3780	$30^1 \cdot 42^6 \cdot 62^{62} \cdot 74^{43} \cdot 86^1 \cdot 94^{72} \cdot 106^{761} \cdot 118^{2834}$
D_7	1080	$42^1 \cdot 106^{134} \cdot 120^{945}$
E_6	1120	$36^1 \cdot 63^{27} \cdot 90^{63} \cdot 117^{1029}$
E_7	120	$63^1 \cdot 119^{119}$

TABLE 3. Parabolic subgroups of E_8

Type of X	$[W : N_W(X)]$	Lengths of conjugates of X
A_1	12	$1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^1 \cdot 11^1 \cdot 13^1 \cdot 15^1$
A_2	16	$3^1 \cdot 6^1 \cdot 9^3 \cdot 12^2 \cdot 15^2 \cdot 18^3 \cdot 21^4$
B_2	18	$4^1 \cdot 8^4 \cdot 12^3 \cdot 16^2 \cdot 20^8$
B_3	12	$9^1 \cdot 17^2 \cdot 23^9$

Table 4. Parabolic subgroups of F_4

Type of X	$[W : N_W(X)]$	Lengths of conjugates of X
A_1	15	$1^3 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 9^2 \cdot 11^1 \cdot 13^1$
A_2	10	$3^1 \cdot 6^1 \cdot 12^5 \cdot 15^3$
I_5	6	$5^1 \cdot 10^2 \cdot 15^3$

Table 5. Parabolic subgroups of H_3

Type of X	$[W : N_W(X)]$	Lengths of conjugates of X
A_1	60	$1^4 \cdot 3^4 \cdot 5^4 \cdot 7^4 \cdot 9^4 \cdot 11^4 \cdot 13^4 \cdot 15^3 \cdot 17^3 \cdot 19^3 \cdot 21^3 \cdot 23^3 \cdot 25^3 \cdot 27^2 \cdot 29^2 \cdot 31^2 \cdot 33^2 \cdot 35^1 \cdot 37^1 \cdot 39^1 \cdot 41^1 \cdot 43^1 \cdot 45^1$
A_2	200	$3^2 \cdot 6^3 \cdot 9^2 \cdot 12^6 \cdot 15^9 \cdot 18^{10} \cdot 21^8 \cdot 24^3 \cdot 27^8 \cdot 30^{17} \cdot 36^{23} \cdot 39^{15} \cdot 42^{22} \cdot 45^{14} \cdot 48^{13} \cdot 51^{10} \cdot 54^{17} \cdot 57^{18}$
A_3	300	$6^1 \cdot 10^1 \cdot 22^7 \cdot 34^{25} \cdot 46^{40} \cdot 50^1 \cdot 52^{45} \cdot 56^{72} \cdot 60^{108}$
H_3	60	$15^1 \cdot 27^3 \cdot 47^{16} \cdot 59^{40}$
I_5	72	$5^1 \cdot 10^2 \cdot 15^6 \cdot 20^4 \cdot 25^5 \cdot 30^3 \cdot 35^7 \cdot 40^{15} \cdot 45^7 \cdot 50^4 \cdot 55^{18}$

Table 6. Parabolic subgroups of H_4

References

[1] H. Barcelo. and E. Ihrig, *Lattices of Parabolic Subgroups in Connection with Hyperplane Arrangements*, J. Algebraic Combin. 9, no.1, 5–24 (1999).

[2] J.J. Cannon and C. Playoust, *An Introduction to Algebraic Programming with* MAGMA [draft], Springer-Verlag (1997).

[3] R.W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, John Wiley, London (1985).

[4] C.K. Fan, *Structure of a Hecke Algebra Quotient*, J. Amer. Math. Soc. 10, no.1, 139–167 (1997).

[5] S. B. Hart and P. J. Rowley, *Lengths of Subsets in Coxeter Groups*, Turk. J. Math 31 (2007), 1 – 15.

[6] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, 29 (1990).

[7] S.B. Perkins and P.J. Rowley, *Coxeter Length*, J. Algebra 273 (1) (2004), 344–358.

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