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# A NOTE ON GROUP INVARIANT INCIDENCE FUNCTIONS

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Abstract. Partially ordered sets $(X, \preceq)$ and the corresponding incidence algebra $I(X, \mathbb{F})$ are important algebraic structures also playing a crucial role for the enumeration, construction and the classification of many discrete structures. In this paper we consider partially ordered sets $X$ on which some group $G$ acts via the mapping $X \times G \to X, (x, g) \mapsto x^g$ and investigate such incidence functions $\phi : X \times X \to \mathbb{F}$ of the incidence algebra $I(X, \mathbb{F})$ which are invariant under the group action, i. e. which satisfy the condition $\phi(x, y) = \phi(x^g, y^g)$ for all $x, y \in X$ and $g \in G$. Within these considerations we define for such incidence functions $\phi$ the matrices $\phi^\wedge$ respectively $\phi^\vee$ by summation of entries of $\phi$ and we investigate the structure of these matrices and generalize the results known from group actions on posets.

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## 1. Introduction

A partially ordered set, for short *poset*, $(X, \preceq)$ is a set $X$ together with a reflexive, antisymmetric and transitive binary relation $\preceq$. Instead of $x \preceq y$ and $x \neq y$ the notation $x \prec y$ is also used. The poset is said to be *locally finite* if and only if all its *intervals* $[x, y] := \{z \in X \mid x \preceq z \preceq y\}$ are finite. In the following we consider locally finite posets. Let $\mathbb{F}$ be a field. The set $I(X, \mathbb{F})$ consisting of all mappings $\phi : X \times X \to \mathbb{F}$ with the property that $\phi(x, y) = 0$ unless $x \preceq y$ yields an $\mathbb{F}$-algebra with respect to the addition

$$(\phi + \psi)(x, y) := \phi(x, y) + \psi(x, y),$$

the scalar multiplication

$$(f \cdot \phi)(x, y) := f \cdot \phi(x, y), \quad f \in \mathbb{F},$$

and the convolution product

$$(\phi * \psi)(x, y) := \sum_{z \in X} \phi(x, z) \cdot \psi(z, y) = \sum_{z \in [x,y]} \phi(x, z) \cdot \psi(z, y),$$

the so-called *incidence algebra* over $\mathbb{F}$ on $X$. The identity element with respect to the convolution product is defined by the Kronecker function:

$$\delta(x,y) := \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

An important element of the incidence algebra is the well-known Zeta-function which characterizes the poset completely:

$$\zeta(x,y) := \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

An incidence function $\phi$ is invertible with respect to the convolution product if and only if the values $\phi(x,x)$ are non-zero. In that case we can construct the inverse incidence function $\phi^{-1}$ recursively:

$$\phi^{-1}(x,x) = \phi(x,x)^{-1}$$

for all $x \in X$, and

$$\begin{aligned} \phi^{-1}(x,y) &= -\phi(x,x)^{-1} \sum_{z: x \prec z \preceq y} \phi(x,z) \cdot \phi^{-1}(z,y) \\ &= -\phi(y,y)^{-1} \sum_{z: x \preceq z \prec y} \phi^{-1}(x,z) \cdot \phi(z,y) \end{aligned}$$

for all different $x, y \in X$.

Since $\zeta(x,x) = 1$ for all $x \in X$, the Zeta-function is invertible over $\mathbb{F}$ and its inverse is called *Moebius-function* and is denoted by $\mu$.

## 2. Group invariant incidence functions

From now one we assume a (multiplicatively written) group $G$ with neutral element $1_G$ acting on a poset $X$ via the mapping $X \times G \to X, (x,g) \mapsto x^g$ from the right, i. e. this mapping satisfies $(x^g)^h = x^{gh}$ and $x^{1_G} = x$ for all $x \in X$ and $g, h \in G$. In the following we consider such $\phi \in I(X, \mathbb{F})$ satisfying the equation

$$\phi(x,y) = \phi(x^g, y^g)$$

for all $x, y \in X$ and $g \in G$. We call such incidence functions *G-invariant* and we use the symbol $I(X, \mathbb{F})_G$ for the set of all these functions.

A well-known situation occurs if $\zeta \in (X, \mathbb{F})_G$. This is equivalent to

$$x \prec y \iff x^g \prec y^g$$

for all $x, y \in X$ and $g \in G$. In this case we say that $G$ acts as a group of automorphisms on the poset $X$ [3].

Important properties of $I(X,\mathbb{F})_G$ are described in the following lemma:

**Lemma 1.** *Let $G$ be a group acting on a locally finite poset $X$ and $\mathbb{F}$ be a field. Then $I(X,\mathbb{F})_G$ is a subalgebra of $I(X,\mathbb{F})$. In addition $I(X,\mathbb{F})_G$ is a monoid with respect to the convolution product, i. e. $\delta \in I(X,\mathbb{F})_G$. Furthermore, if $\phi \in I(X,\mathbb{F})_G$ is an invertible incidence function in $I(X,\mathbb{F})$ and $\zeta \in I(X,\mathbb{F})_G$, then $\phi^{-1} \in I(X,\mathbb{F})_G$.*

**Proof.** (i) Let $\phi, \psi \in I(X,\mathbb{F})_G$, $f \in \mathbb{F}$ and $g \in G$. We now show that the functions $\phi+\psi$, $f\cdot\phi$ and $\phi*\psi$ are also $G$-invariant. This implies that $I(X,\mathbb{F})_G$ is a subalgebra of $I(X,\mathbb{F})$:

$$\begin{aligned}(\phi+\psi)(x,y) &= \phi(x,y)+\psi(x,y)\\ &= \phi(x^g,y^g)+\psi(x^g,y^g)\\ &= (\phi+\psi)(x^g,y^g),\end{aligned}$$

$$\begin{aligned}(f\cdot\phi)(x,y) &= f\cdot\phi(x,y)\\ &= f\cdot\phi(x^g,y^g)\\ &= (f\cdot\phi)(x^g,y^g),\end{aligned}$$

$$\begin{aligned}(\phi*\psi)(x,y) &= \sum_{z\in X}\phi(x,z)\cdot\psi(z,y)\\ &= \sum_{z\in X}\phi(x^g,z^g)\cdot\psi(z^g,y^g)\\ &= \sum_{z\in X}\phi(x^g,z^g)\cdot\psi(z^g,y^g)\\ &= \sum_{z'\in X}\phi(x^g,z')\cdot\psi(z',y^g)\\ &= (\phi*\psi)(x^g,y^g).\end{aligned}$$

(ii) Furthermore, the equivalence $x^g = y^g \Leftrightarrow x = y$ for all $x, y \in X$ and $g \in G$ implies $\delta(x,y) = \delta(x^g,y^g)$, i. e. $I(X,\mathbb{F})_G$ is a monoid.

(iii) Now, let $\phi \in I(X,\mathbb{F})_G$ be an invertible incidence function and let $\zeta \in I(X,\mathbb{F})_G$. We show that $\phi^{-1}(x,y) = \phi^{-1}(x^g,y^g)$ for all $x, y \in X$ and $g \in G$. First we consider the case that $x \not\preceq y$. Then we also get $x^g \not\preceq y^g$ since $\zeta \in I(X,\mathbb{F})_G$ and hence we have $\phi^{-1}(x,y) = 0 = \phi^{-1}(x^g,y^g)$. Now we consider the second case $x \preceq y$. There exist chains between $x$ and $y$. Let $\ell(x,y)$ denote the length of a maximal chain between $x$ and $y$. We prove $\phi^{-1}(x,y) = \phi^{-1}(x^g,y^g)$ by induction on $n = \ell(x,y)$:

I. $n=0$. First $\ell(x,y)=0$, i.e. $x=y$. Then

$$\phi^{-1}(x,x)=\phi(x,x)^{-1}=\phi(x^g,x^g)^{-1}=\phi^{-1}(x^g,x^g).$$

II. $n-1\to n$. Then

$$\begin{aligned}\phi^{-1}(x,y)&=-\phi(y,y)^{-1}\sum_{z:x\preceq z\prec y}\phi^{-1}(x,z)\cdot\phi(z,y)\\&=-\phi(y^g,y^g)^{-1}\sum_{z:x\preceq z\prec y}\underbrace{\phi^{-1}(x^g,z^g)}_{\ell(x,z)<n}\cdot\phi(z^g,y^g)\\&=-\phi(y^g,y^g)^{-1}\sum_{z:x^g\preceq z^g\prec y^g}\phi^{-1}(x^g,z^g)\cdot\phi(z^g,y^g)\\&=-\phi(y^g,y^g)^{-1}\sum_{z':x^g\preceq z'\prec y^g}\phi^{-1}(x^g,z')\cdot\phi(z',y^g)\\&=\phi^{-1}(x^g,y^g).\end{aligned}$$

□

From now on let $X$ be a finite poset and let $y^G:=\{y^g\mid g\in G\}$ denote the orbit of $y\in X$. Then we define for a $G$-invariant incidence function $\phi\in I(X,\mathbb{F})_G$ the values

$$\phi(x,y^G):=\sum_{z\in y^G}\phi(x,z)$$

and

$$\phi(y^G,x):=\sum_{z\in y^G}\phi(z,x)$$

for $x,y\in X$.

**Lemma 2.** *Let $G$ be a group acting on the finite poset $X$ and $\mathbb{F}$ be a field. Let $\phi\in I(X,\mathbb{F})_G$. Then the equations*

$$\phi(x,y^G)=\phi(x^g,y^G)$$

*and*

$$\phi(y^G,x)=\phi(y^G,x^g)$$

*hold for all $x,y\in X$ and $g\in G$.*

**Proof.** We prove the first equation, the proof of the second one is analogous. Let $x,y\in X$, $g\in G$ and $\phi\in I(X,\mathbb{F})_G$. Then we have
$\phi(x,y^G)=\sum_{z\in y^G}\phi(x,z)=\sum_{z\in y^G}\phi(x^g,z^g)=\sum_{z'\in y^G}\phi(x^g,z')=\phi(x^g,y^G)$.
□

Let $\mathcal{O}_1, \ldots, \mathcal{O}_n$ denote the orbits of $G$ on the poset $X$ and let $x_i \in \mathcal{O}_i$ denote a representative of the $i$th orbit. Now we can define two $n \times n$ matrices $\phi^\wedge = (\phi^\wedge_{ij})$ and $\phi^\vee = (\phi^\vee_{ij})$ with entries

$$\phi^\wedge_{ij} := \phi(x_i, \mathcal{O}_j)$$

and

$$\phi^\vee_{ij} := \phi(\mathcal{O}_i, x_j).$$

The following lemma shows the connection between $\phi^\wedge$ and $\phi^\vee$.

**Lemma 3.** *Let $G$ be a group acting on the finite poset $X$ with corresponding orbits $\mathcal{O}_1, \ldots, \mathcal{O}_n$ and let $\mathbb{F}$ be a field. Let $\phi \in I(X, \mathbb{F})_G$ and let*

$$\Delta := \begin{pmatrix} |\mathcal{O}_1| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_n| \end{pmatrix}.$$

*Then the following equation holds*

$$\phi^\vee \cdot \Delta = \Delta \cdot \phi^\wedge.$$

*Furthermore, if the characterstic of the field $\mathbb{F}$ does not divide the orbit sizes $|\mathcal{O}_1|, \ldots, |\mathcal{O}_n|$, then*

$$\phi^\vee = \Delta \cdot \phi^\wedge \cdot \Delta^{-1}.$$

**Proof.** Let $M = (m_{ij}) = \phi^\vee \cdot \Delta$ and let $N = (n_{ij}) = \Delta \cdot \phi^\wedge$. In the following we show the equality of these two matrices $M = N$:

$$\begin{aligned} m_{ij} &= \phi^\vee_{ij} \cdot |\mathcal{O}_j| = \phi(\mathcal{O}_i, x_j) \cdot |\mathcal{O}_j| \\ &= \sum_{y \in \mathcal{O}_j} \phi(\mathcal{O}_i, x_j) = \sum_{y \in \mathcal{O}_j} \phi(\mathcal{O}_i, y) \\ &= \sum_{y \in \mathcal{O}_j} \sum_{x \in \mathcal{O}_i} \phi(x, y) = \sum_{x \in \mathcal{O}_i} \sum_{y \in \mathcal{O}_j} \phi(x, y) \\ &= \sum_{x \in \mathcal{O}_i} \phi(x, \mathcal{O}_j) = \sum_{x \in \mathcal{O}_i} \phi(x_i, \mathcal{O}_j) \\ &= |\mathcal{O}_i| \cdot \phi(x_i, \mathcal{O}_j) = |\mathcal{O}_i| \cdot \phi^\wedge_{ij} \\ &= n_{ij}. \end{aligned}$$

Multiplying the inverse of $\Delta$ from the right yields the second equation. □

From now on we restrict our investigation to the matrix $\phi^\wedge$ since the results for $\phi^\vee$ are analogous.

**Lemma 4.** *Let $G$ be a group acting on the finite poset $X$ and $\mathbb{F}$ be a field. Then $\delta^\wedge$ is the $n \times n$ unit matrix, where the dimension $n$ is the number of orbits of $G$ on the poset $X$.*

**Proof.** For all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we obtain

$$\delta^\wedge_{ij} = \sum_{y \in \mathcal{O}_j} \delta(x_i, y) = 0$$

and

$$\delta^\wedge_{ii} = \delta(x_i, x_i) + \sum_{y \in \mathcal{O}_i : y \neq x_i} \delta(x_i, y) = 1 + 0 = 1.$$

□

**Theorem 5.** *Let $G$ be a group acting on the finite poset $X$ and $\mathbb{F}$ be a field. Then the equations*

$$(f \cdot \phi)^\wedge = f \cdot \phi^\wedge, \quad (\phi + \psi)^\wedge = \phi^\wedge + \psi^\wedge, \quad (\phi * \psi)^\wedge = \phi^\wedge \cdot \psi^\wedge$$

*hold for all $\phi, \psi \in I(X, \mathbb{F})_G$ and $f \in \mathbb{F}$.*

**Proof.** (i)

$$\begin{aligned}
(f \cdot \phi)^\wedge_{ij} &= (f \cdot \phi)(x_i, \mathcal{O}_j) = \sum_{y \in \mathcal{O}_j} (f \cdot \phi)(x_i, y) \\
&= \sum_{y \in \mathcal{O}_j} f \cdot \phi(x_i, y) = f \cdot \sum_{y \in \mathcal{O}_j} \phi(x_i, y) \\
&= f \cdot \phi(x_i, \mathcal{O}_j) \\
&= f \cdot \phi^\wedge_{ij}
\end{aligned}$$

(ii)

$$\begin{aligned}
(\phi + \psi)^\wedge_{ij} &= (\phi + \psi)(x_i, \mathcal{O}_j) = \sum_{y \in \mathcal{O}_j} (\phi + \psi)(x_i, y) \\
&= \sum_{y \in \mathcal{O}_j} [\phi(x_i, y) + \psi(x_i, y)] = \sum_{y \in \mathcal{O}_j} \phi(x_i, y) + \sum_{y \in \mathcal{O}_j} \psi(x_i, y) \\
&= \phi(x_i, \mathcal{O}_j) + \psi(x_i, \mathcal{O}_j) \\
&= \phi^\wedge_{ij} + \psi^\wedge_{ij}
\end{aligned}$$

(iii)

$$\begin{aligned}
(\phi * \psi)^{\wedge}_{ij} &= (\phi * \psi)(x_i, \mathcal{O}_j) = \sum_{y \in \mathcal{O}_j} (\phi * \psi)(x_i, y) \\
&= \sum_{y \in \mathcal{O}_j} \sum_{z \in X} \phi(x_i, z) \cdot \psi(z, y) = \sum_{z \in X} \sum_{y \in \mathcal{O}_j} \phi(x_i, z) \cdot \psi(z, y) \\
&= \sum_{z \in X} \phi(x_i, z) \sum_{y \in \mathcal{O}_j} \psi(z, y) = \sum_{z \in X} \phi(x_i, z) \cdot \psi(z, \mathcal{O}_j) \\
&= \sum_{k} \sum_{z \in \mathcal{O}_k} \phi(x_i, z) \cdot \psi(z, \mathcal{O}_j) = \sum_{k} \sum_{z \in \mathcal{O}_k} \phi(x_i, z) \cdot \psi(x_k, \mathcal{O}_j) \\
&= \sum_{k} \psi(x_k, \mathcal{O}_j) \sum_{z \in \mathcal{O}_k} \phi(x_i, z) = \sum_{k} \psi(x_k, \mathcal{O}_j) \cdot \phi(x_i, \mathcal{O}_k) \\
&= \sum_{k} \phi(x_i, \mathcal{O}_k) \cdot \psi(x_k, \mathcal{O}_j) \\
&= \sum_{k} \phi^{\wedge}_{ik} \cdot \psi^{\wedge}_{kj}
\end{aligned}$$

□

**Corollary 6.** *Let $G$ be a group acting on the finite poset $X$ and $\mathbb{F}$ be a field. Let $\zeta \in I(X, \mathbb{F})_G$ and let $\phi \in I(X, \mathbb{F})_G$ be an invertible incidence function. Then $\phi^{\wedge}$ is invertible and for its inverse holds the following equation*

$$(\phi^{\wedge})^{-1} = (\phi^{-1})^{\wedge}.$$

**Proof.** Let $\phi \in I(X, \mathbb{F})_G$ be invertible. Since $\zeta$ is $G$-invariant we obtain from Lemma 1 that $\phi^{-1} \in I(X, \mathbb{F})_G$. Hence we can apply Theorem 5 and get

$$\phi^{\wedge} \cdot (\phi^{-1})^{\wedge} = (\phi * \phi^{-1})^{\wedge} = \delta^{\wedge}$$

which means that $(\phi^{\wedge})^{-1} = (\phi^{-1})^{\wedge}$ since $\delta^{\wedge}$ is the unit matrix. □

## 3. Examples

**3.1. Binomial coefficients.** We consider for a natural number $n$ the matrix $B = (b_{ij})$, $0 \leq i, j \leq n$, where $b_{ij} = \binom{j}{i}$ is the number of $i$-subsets which are contained in a set with $j$ elements. The aim is to compute the inverse matrix $B^{-1}$. We take a set $X$ with $n$ elements and consider the action of the symmetric group $S_X := \{\pi : X \to X \mid \pi \text{ bijectively } \}$ on the power set $P(X) := \{S \mid S \subseteq X\}$ via the mapping

$$P(X) \times S_X \to P(X), (S, \pi) \mapsto S^{\pi} := \{x^{\pi} \mid x \in S\}.$$

It is obvious that $S_X$ acts as a group of automorphisms on $P(X)$. If $\binom{X}{k}$ denotes the set of $k$-subsets of $X$, the orbits of this action are exactly the sets $\mathcal{O}_0 = \binom{X}{0}, \mathcal{O}_1 = \binom{X}{1}, \ldots, \mathcal{O}_n = \binom{X}{n}$. As $S_X$-invariant incidence function we take the Zeta-function

$$\zeta(T,K) := \begin{cases} 1 & \text{if } T \subseteq K \\ 0 & \text{otherwise} \end{cases}$$

together with its inverse $\mu(T,K) = (-1)^{|K|-|T|}\zeta(T,K)$. Then we consider the matrix $\zeta^\vee$ whose entries are

$$\zeta^\vee_{ij} = \zeta(\mathcal{O}_i, S_j) = \sum_{S \in \binom{X}{i}} \zeta(S, S_j) = \binom{j}{i}, \text{ where } S_j \in \mathcal{O}_j = \binom{X}{j}$$

i. e. we have $B = \zeta^\vee$. Because of the equation $(\zeta^\vee)^{-1} = \mu^\vee$ we obtain for the inverse of $B$ the matrix $\mu^\vee$ that is given by the following entries:

$$\begin{aligned}\mu^\vee_{ij} = \mu(\mathcal{O}_i, S_j) &= \sum_{S \in \mathcal{O}_i} \mu(S, S_j) = \sum_{S \in \binom{X}{i}} (-1)^{j-i}\zeta(S, S_j) \\ &= (-1)^{j-i} \sum_{S \in \binom{X}{i}} \zeta(S, S_j) = (-1)^{j-i}\binom{j}{i}\end{aligned}$$

Finally we have that the matrix $B^{-1} = (b^{-1}_{ij})$, $b^{-1}_{ij} = (-1)^{j-i}\binom{j}{i}$ is the inverse of $B = (b_{ij})$, $b_{ij} = \binom{j}{i}$.

**3.2. Table of Marks and Burnside matrix.** The table of marks of a group, introduced by Burnside (see [1]), plays an important role for the enumeration, construction and classification of discrete structures as groups, graphs and $t$-designs (see [3,4,5]). Especially the combinatorial chemistry (see [2]) uses the table of marks as a tool for the enumeration of chemical compounds. Now we show here that the table of marks is a matrix $\phi^\wedge$ with a certain group invariant incidence function $\phi$.

Let $G$ be a finite group, and let $L(G) := \{S \mid S \leq G\}$ denote the set of all subgroups of $G$. This set together with the inclusion relation forms a finite poset, the so-called *subgroup lattice* of $G$. The group $G$ acts on $L(G)$ by conjugation

$$L(G) \times G \to L(G), (g, S) \mapsto g^{-1}Sg := \{g^{-1}sg \mid s \in S\}$$

such that $G$ acts on $L(G)$ as a group of automorphisms, i. e. the equivalence

$$S < T \iff g^{-1}Sg < g^{-1}Tg$$

holds for all $S, T \in L(G)$ and $g \in G$. The orbits of this action are the conjugacy classes of subgroups

$$\widetilde{S} := \{g^{-1}Sg \mid g \in G\}.$$

Now if $G$ acts on a set $X$ and if $N_G(x) := \{g \in G \mid x^g = x\}$ denotes the stabilizer of an element $x \in X$, the conjugacy class of $N_G(x)$ is

$$\widetilde{N_G(x)} = \{g^{-1}N_G(x)g \mid g \in G\} = \{N_G(y) \mid y \in x^G\}$$

where $x^G := \{x^g \mid g \in G\}$ is the orbit of $x$, i.e. the elements of an orbit have as their stabilizers a complete conjugacy class of subgroups of $G$. We say $\widetilde{N_G(x)}$ is the type of the orbit $x^G$. For a given subgroup $S \in L(G)$ we define

$$\Omega(G,X)_{\widetilde{S}} := \{x^G \mid N_G(x) \in \widetilde{S}\}$$

to be the set of orbits of $G$ on $X$ of type $\widetilde{S}$. The task is now to determine the cardinality of this set. In order to determine this number we consider the set of *S-invariants*:

$$X_S := \{x \in X \mid \forall g \in S : x^g = x\}.$$

The cardinality of $X_S$ is called the *mark* of $S$ on $X$ and we get the following well-known connection (see [3]):

$$|X_S| = \sum_{T \in L(G)} \zeta(S,T)\frac{|T\backslash G|}{|\widetilde{T}|}|\Omega(G,X)_{\widetilde{T}}|$$

If we substitute

$$\phi(S,T) := \zeta(S,T)\frac{|T\backslash G|}{|\widetilde{T}|}$$

we obtain a mapping $\phi$ which is obviously an element of $I(L(G),\mathbb{Q})_G$. Moreover, $\phi$ is an invertible function. Therefore, if $\widetilde{S_1},\ldots,\widetilde{S_n}$ denote the orbits of $G$ on $L(G)$, we obtain the equation

$$\begin{pmatrix} |X_{S_1}| \\ \vdots \\ |X_{S_n}| \end{pmatrix} = \phi^\wedge \cdot \begin{pmatrix} |\Omega(G,X)_{\widetilde{S_1}}| \\ \vdots \\ |\Omega(G,X)_{\widetilde{S_n}}| \end{pmatrix},$$

respectively after multiplication with $(\phi^{-1})^\wedge$ from the left

$$\begin{pmatrix} |\Omega(G,X)_{\widetilde{S_1}}| \\ \vdots \\ |\Omega(G,X)_{\widetilde{S_n}}| \end{pmatrix} = (\phi^{-1})^\wedge \cdot \begin{pmatrix} |X_{S_1}| \\ \vdots \\ |X_{S_n}| \end{pmatrix}.$$

The matrix

$$M(G) := \phi^\wedge$$

is known as the *table of marks* of $G$ and its inverse

$$B(G) := (\phi^{-1})^\wedge$$

is called the *Burnside matrix* of $G$.

**3.3. Plesken matrices.** The Plesken matrices [6] provide another application of group invariant incidence functions. If a group $G$ acts on a finite poset $X$ as a group of automorphisms, i. e. $x \prec y \Leftrightarrow x^g \prec y^g$ and if $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are the corresponding orbits with representative $x_i \in \mathcal{O}_i$, then Plesken defined the matrices $A^\wedge = (a_{ij}^\wedge)$ and $A^\vee = (a_{ij}^\vee)$ by

$$a_{ij}^\wedge := |\{y \in \mathcal{O}_j \mid x_i \preceq y\}|$$

and

$$a_{ij}^\vee := |\{y \in \mathcal{O}_i \mid y \preceq x_j\}|.$$

These matrices play an important role for the determination of the number of solutions of equations of the form $x \wedge y = z$, respectively $x \vee y = z$. There is the following correspondence to the group invariant incidence functions:

**Corollary 7.** *Let $G$ be a group acting on a finite poset $X$ as a group of automorphisms. Then $A^\wedge = \zeta^\wedge$ and $A^\vee = \zeta^\vee$.*

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