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G-OUTER INVERSES AND PARTIAL ORDERS IN RINGS

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Abstract. The notions of G-outer inverse and its one-sided versions for rectangular complex matrices are generalized to elements of a ring based on the (b,c)-inverse. Various properties and characterizations of G-outer inverse and its one-sided kinds are developed. We consider partial orders applying G-outer inverse and its one-sided types in a ring. Consequently, one-sided G-Drazin inverses are introduced for elements of a ring as well as partial orders induced by them.

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1. Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. If $a \in \mathcal{R}$, we have the next image ideals $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$, $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$; and the kernel ideals $a^\circ = \{x \in \mathcal{R} : ax = 0\}$, ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$. The symbol $a_{\mathcal{B}}^{-1}$ denotes the inverse of a in a subring $\mathcal{B}$ of $\mathcal{R}$.

An element $a \in \mathcal{R}$ is regular if there is $x \in \mathcal{R}$ such that $axa = a$. In this case, x is an inner inverse of a. The set of all regular elements of $\mathcal{R}$ is denoted by $\mathcal{R}^-$ and the set of all inner inverses of $a \in \mathcal{R}^-$ will be marked as $a\{1\}$. For $a \in \mathcal{R}$, if $xax = x$ holds for some $x \in \mathcal{R}\backslash\{0\}$, then x is an outer inverse of a.

An element $a \in \mathcal{R}$ has a Drazin inverse $x = a^d \in \mathcal{R}$ if

$$ax = xa, \quad x = ax^2, \quad a^k = a^{k+1}x,$$

for some non-negative integer k. If there is a Drazin inverse a^d, then a is Drazin invertible and the smallest non-negative integer k in equality $a^k = a^{k+1}x$ is called the Drazin index $\mathrm{ind}(a)$ of a. The set of all Drazin invertible elements of $\mathcal{R}$ is denoted by $\mathcal{R}^d$.

The special type of outer inverse was defined by Drazin [3] and called the (b,c)-inverse. Let $b, c \in \mathcal{R}$. An element $a \in \mathcal{R}$ is (b,c)-invertible if there exists $x \in \mathcal{R}$

such that

$$x \in (b\mathcal{R}x) \cap (x\mathcal{R}c), \quad xab = b \quad \text{and} \quad cax = c.$$

The (b, c)-inverse x of a satisfies $xax = x$, it is unique (if exists) and denoted by $a^{||(b,c)}$ [3]. We will use $\mathcal{R}^{||(b,c)}$ to denote the set of all (b, c)-invertible elements of $\mathcal{R}$. Notice that $b\mathcal{R} = a^{||(b,c)}\mathcal{R}$ and $c^\circ = (a^{||(b,c)})^\circ$. Some interesting results about the (b, c)-inverse were given in [13,14,15,16,17].

In the rest of this paper, we will use the term minus partial order. For $a, b \in \mathcal{R}$, we say that a is below to b under the minus partial order (denoted by $a \leq^- b$) [6] if $a^-a = a^-b$ and $aa^- = ba^-$, for some $a^- \in a\{1\}$. This relation is a partial order on $\mathcal{R}^-$.

The notation of the G-Drazin inverse was defined for square matrices in [15] and extended for elements of a ring in [2]. For $a \in \mathcal{R}^d \cap \mathcal{R}^-$ and $\mathrm{ind}(a) = k$, an element $z \in \mathcal{R}$ is a left G-Drazin inverse of a if

$$aza = a, \quad za^{k+1} = a^k \quad \text{and} \quad a^{k+1}z = a^k.$$

It is known that the G-Drazin inverse is not unique (when it exists) in general.

Let $\mathbb{C}^{m\times n}$ be the set of all $m\times n$ complex matrices. We mark A^*, $\mathrm{rank}(A)$, $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively, conjugate transpose, rank, null space and range (column space) of $A \in \mathbb{C}^{m\times n}$. The uniquely determined outer inverse of A with prescribed range T and null space S is a matrix $X \in \mathbb{C}^{n\times m}$ (represented by $A^{(2)}_{T,S}$) for which

$$XAX = X, \quad \mathcal{R}(X) = T, \quad \mathcal{N}(X) = S,$$

where T is a subspace of $\mathbb{C}^n$ dimension $s \leq r = \mathrm{rank}(A)$, and S is a subspace of $\mathbb{C}^m$ dimension $m - s$. Note that $AT \oplus S = \mathbb{C}^m$ and only if $A^{(2)}_{T,S}$ exists. For a given $A \in \mathbb{C}^{m\times n}$, appropriately chosen T, S, the notation

$$\mathbb{C}^{m\times n}_{T,S} = \{A \in \mathbb{C}^{m\times n} : AT \oplus S = \mathbb{C}^m\} \subseteq \mathbb{C}^{m\times n}$$

will be used.

The concept of G-outer inverse was introduced as a generalization of the G-Drazin inverse using outer inverse with prescribed range and null space in [9]. A matrix $Z \in \mathbb{C}^{n\times m}$ is G-outer (T, S)-inverse of $A \in \mathbb{C}^{m\times n}_{T,S}$ if

$$AZA = A, \quad ZAA^{(2)}_{T,S} = A^{(2)}_{T,S} \quad \text{and} \quad A^{(2)}_{T,S}AZ = A^{(2)}_{T,S}.$$

The definitions of both left and right versions of G-outer inverse were given in [11]. A matrix $Z \in \mathbb{C}^{n\times m}$ is a left G-outer (T, S)-inverse of $A \in \mathbb{C}^{m\times n}_{T,S}$ if it satisfies

$$AZA = A \quad \text{and} \quad ZAA^{(2)}_{T,S} = A^{(2)}_{T,S}.$$

A matrix $Z \in \mathbb{C}^{n\times m}$ is a right G-outer (T,S)-inverse of $A \in \mathbb{C}^{m\times n}_{T,S}$ if it satisfies

$$AZA = A \quad \text{and} \quad A^{(2)}_{T,S}AZ = A^{(2)}_{T,S}.$$

G-Drazin inverses, G-outer inverses and its one-sided versions are significant in solving some matrix equations and studying partial orders [1,4,5,8,10]. A zeroing neural network (ZNN) approach for calculating time-varying G-outer inverse can be found in [12]. Recent results related to the G-Drazin inverse were stated in [7].

In this paper, we generalize the notions of G-outer inverse for rectangular complex matrices to elements of a ring, using the (b,c)-inverse. One-sided G-outer inverses in a ring are defined and characterized as weaker versions of G-outer inverse. By means of G-outer inverse and its one-sided kinds, we study partial orders in a ring. As special cases of one-sided G-outer inverses, one-sided G-Drazin inverses are introduced and investigated for elements of a ring. Also, we consider partial orders induced by one-sided G-Drazin inverses. Thus, we extend some results for complex matrices in a more general setting.

The paper is organized as follows. In Section 2, we introduce the G-outer inverse in a ring and discuss its properties. Section 3 is devoted to the definition and investigation of one-sided G-outer inverses in a ring. In Section 4, we define the left and right G-outer partial orders, as well as the G-outer partial order, and provide characterizations of these relations. Section 5 presents left and right G-Drazin inverses for elements of a ring, together with their characterizations. Finally, in Section 6, we define and examine the left and right G-Drazin partial orders.

2. G-outer inverse in a ring

In this section, we define a G-outer (b,c)-inverse for elements in a ring and establish its properties. Thus, we extend the concept of G-outer (T,S)-inverse in settings of a ring.

Definition 2.1. Let $b,c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^{-}$. An element $z \in \mathcal{R}$ is a G-outer (b,c)-inverse of a if

$$aza = a \quad \text{and} \quad a^{||(b,c)}az = zaa^{||(b,c)}.$$

The G-outer (b,c)-inverse of a is not unique. We introduce a very useful notation. The set of all G-outer (b,c)-inverses of a is denoted by $a\{go,b,c\}$. The next inclusion applies $a\{go,b,c\} \subseteq a\{1\}$.

We characterize the G-outer (b,c)-inverse in the following manner.

Theorem 2.2. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^{-}$. For $z \in \mathcal{R}$, the following conditions are equivalent:*

(i) $z \in a\{go, b, c\}$;

(ii) $aza = a \quad$ *and* $\quad a^{||(b,c)}az = a^{||(b,c)} = zaa^{||(b,c)}$;

(iii) $aza = a, \quad aa^{||(b,c)}az = aa^{||(b,c)} \quad$ *and* $\quad zaa^{||(b,c)}a = a^{||(b,c)}a$;

(iv) $aza = a, \quad caz = c \quad$ *and* $\quad zab = b$;

(v) $aza = a, \quad (az)^{\circ} \subseteq c^{\circ} \quad$ *and* $\quad b\mathcal{R} \subseteq za\mathcal{R}$.

Proof. (i) $\Rightarrow$ (ii): For $z \in \mathcal{R}$ which is a G-outer (b, c)-inverse of a, by Definition 2.1, we have $aza = a$ and $a^{||(b,c)}az = zaa^{||(b,c)}$. We get

$$a^{||(b,c)}azaa^{||(b,c)} = a^{||(b,c)}aa^{||(b,c)} = a^{||(b,c)}$$

and further

$$a^{||(b,c)} = a^{||(b,c)}azaa^{||(b,c)} = zaa^{||(b,c)}aa^{||(b,c)} = zaa^{||(b,c)}.$$

In a similar way, we prove that $a^{||(b,c)} = a^{||(b,c)}az$.

(ii) $\Rightarrow$ (iii): This implication is clear.

(iii) $\Rightarrow$ (iv): Using the fact $caa^{||(b,c)} = c$ and the assumption $aa^{||(b,c)} = aa^{||(b,c)}az$, we conclude that $caz = caa^{||(b,c)}az = caa^{||(b,c)} = c$. Since $a^{||(b,c)}ab = b$, multiplying $a^{||(b,c)}a = zaa^{||(b,c)}a$ by b from the right side, we get $b = zab$.

(iv) $\Rightarrow$ (v): It is evident.

(v) $\Rightarrow$ (i): The hypothesis $aza = a$ gives $azaz = az$. Now, $1 - az \in (az)^{\circ} \subseteq c^{\circ} = (a^{||(b,c)})^{\circ}$ and so $a^{||(b,c)}az = a^{||(b,c)}$. From $a^{||(b,c)} \in b\mathcal{R} \subseteq za\mathcal{R} = (1 - za)^{\circ}$, we deduce that $a^{||(b,c)} = zaa^{||(b,c)}$ and thus $a^{||(b,c)}az = zaa^{||(b,c)}$. □

Based on idempotents, we obtain the next characterizations of G-outer (b, c)-invertibility.

Theorem 2.3. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)}$. The following conditions are equivalent:*

(i) $a\{go, b, c\} \neq \emptyset$;

(ii) *there are idempotents $p, q \in \mathcal{R}$, such that*

$$a^{||(b,c)}p = qa^{||(b,c)}, \quad p\mathcal{R} = a\mathcal{R} \quad \text{and} \quad \mathcal{R}q = \mathcal{R}a.$$

Additionally, for arbitrary $a^{-} \in a\{1\}$, it is valid $qa^{-}p \in a\{go, b, c\}$ and

$$q \cdot a\{1\} \cdot p \subseteq a\{go, b, c\}.$$

Proof. (i) $\Rightarrow$ (ii): Assume that $a \in \mathcal{R}^{-}$ and $z \in a\{go, b, c\}$. Let $p = az$ and $q = za$. Since $z \in a\{1\}$, we get $p^2 = azaz = az = p$ and $q^2 = zaza = za = q$. So, p and

q are idempotents. Notice that $p\mathcal{R} = az\mathcal{R} \subseteq a\mathcal{R}$ and $a\mathcal{R} = aza\mathcal{R} = pa\mathcal{R} \subseteq p\mathcal{R}$, which imply $p\mathcal{R} = a\mathcal{R}$. It also applies $\mathcal{R}q = \mathcal{R}za = \mathcal{R}a$ and $a^{||(b,c)}p = a^{||(b,c)}az = zaa^{||(b,c)} = qa^{||(b,c)}$. Thus, (ii) holds.

(ii) $\Rightarrow$ (i): Let $p, q \in \mathcal{R}$ be idempotents such that $p\mathcal{R} = a\mathcal{R}$, $\mathcal{R}q = \mathcal{R}a$ and $a^{||(b,c)}p = qa^{||(b,c)}$. We prove the existence of a G-outer (b, c)-inverse of a. Because $p\mathcal{R} = a\mathcal{R}$, it is valid $p = ar$ and $a = pu = p^2u = pa$, for some $r, u \in \mathcal{R}$. Furthermore, $a = pa = ara$, which yields that a is regular and $p = ar = aa^-ar = aa^-p$, where $a^- \in a\{1\}$ is arbitrary. From $\mathcal{R}q = \mathcal{R}a$, we have $q = qa^-a$ and $a = aq$. If $z = qa^-p$, then $aza = aqa^-pa = (aq)a^-(pa) = aa^-a = a$ and $a^{||(b,c)}az = a^{||(b,c)}(aq)a^-p = a^{||(b,c)}(aa^-p) = a^{||(b,c)}p = qa^{||(b,c)} = qa^-aa^{||(b,c)} = (qa^-p)aa^{||(b,c)} = zaa^{||(b,c)}$, by which we proved that z is the G-outer (b, c)-inverse of a. $\square$

Note that the product zaz' is a G-outer (b, c)-inverse of a in the case that z and z' are G-outer (b, c)-inverses of a.

Theorem 2.4. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$. Then*

$$a\{go, b, c\} \cdot a \cdot a\{go, b, c\} \subseteq a\{go, b, c\}.$$

Proof. We can verify this result similarly as [9, Theorem 2.4]. $\square$

3. One-sided G-outer inverses in a ring

Removing the second or the third equation in the definition of the G-outer (b, c)-inverse, we introduce one-sided G-outer (b, c)-inverses as weak kinds of generalized inverses. In this way, we extend definitions of left and right G-outer (T, S)-inverses to elements of a ring.

Definition 3.1. Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$. An element $z \in \mathcal{R}$ is

(a) a left G-outer (b, c)-inverse of a if

$$aza = a \quad \text{and} \quad a^{||(b,c)} = zaa^{||(b,c)};$$

(b) a right G-outer (b, c)-inverse of a if

$$aza = a \quad \text{and} \quad a^{||(b,c)} = a^{||(b,c)}az.$$

Let us introduce the following useful tags. We denote the set of all left (or right) G-outer (b, c)-inverses of a with $a\{l, go, b, c\}$ (or $a\{r, go, b, c\}$). The following relations apply $a\{l, go, b, c\} \subseteq a\{1\}$ and $a\{r, go, b, c\} \subseteq a\{1\}$.

By Theorem 2.2, we have the next characterizations of left and right G-outer (b, c)-inverses.

Corollary 3.2. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$. For $z \in \mathcal{R}$, the following conditions are equivalent:*

(i) $z \in a\{l, go, b, c\}$;

(ii) $aza = a$ *and* $zaa^{||(b,c)}a = a^{||(b,c)}a$;

(iii) $aza = a$ *and* $zab = b$;

(iv) $aza = a$ *and* $b\mathcal{R} \subseteq za\mathcal{R}$.

Corollary 3.3. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$. For $z \in \mathcal{R}$, the following conditions are equivalent:*

(i) $z \in a\{r, go, b, c\}$;

(ii) $aza = a$ *and* $aa^{||(b,c)}az = aa^{||(b,c)}$;

(iii) $aza = a$ *and* $caz = c$;

(iv) $aza = a$ *and* $(az)^\circ \subseteq c^\circ$.

We can characterize left and right G-outer (b, c)-invertibility by idempotents as in Theorem 2.3.

Corollary 3.4. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)}$. The following conditions are equivalent:*

(i) $a\{l, go, b, c\} \neq \emptyset$;

(ii) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad b\mathcal{R} \subseteq q\mathcal{R}.$$

Additionally, for arbitrary $a^- \in a\{1\}$, it is valid $qa^-p \in a\{l, go, b, c\}$ and

$$q \cdot a\{1\} \cdot p \subseteq a\{l, go, b, c\}.$$

Corollary 3.5. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)}$. The following conditions are equivalent:*

(i) $a\{r, go, b, c\} \neq \emptyset$;

(ii) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad \mathcal{R}p \subseteq \mathcal{R}c.$$

Additionally, for arbitrary $a^- \in a\{1\}$, it is valid $qa^-p \in a\{r, go, b, c\}$ and

$$q \cdot a\{1\} \cdot p \subseteq a\{r, go, b, c\}.$$

If z and z' are arbitrary left (or right) G-outer (b, c)-inverses of a, we can show that zaz' is also left (or right) G-outer (b, c)-inverse of a.

Corollary 3.6. *Let $b, c \in \mathcal{R}$ and $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^{-}$. Then*

$$a\{l, go, b, c\} \cdot a \cdot a\{l, go, b, c\} \subseteq a\{l, go, b, c\}$$

and

$$a\{r, go, b, c\} \cdot a \cdot a\{r, go, b, c\} \subseteq a\{r, go, b, c\}.$$

4. G-outer partial orders

Applying the G-outer (b, c)-inverse and its one-sided versions, we define new binary relations in a ring.

Definition 4.1. Let $b, c \in \mathcal{R}$ and $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^{-}$. Then

(i) a_1 is below a_2 under the left G-outer (b, c)-relation ($a_1 \leq^{l,go,b,c} a_2$) if there exist $z_1, z_2 \in a_1\{l, go, b, c\}$ such that

$$a_1 z_1 = a_2 z_1 \quad \text{and} \quad z_2 a_1 = z_2 a_2;$$

(ii) a_1 is below a_2 under the right G-outer (b, c)-relation ($a_1 \leq^{r,go,b,c} a_2$) if there exist $z_1, z_2 \in a_1\{r, go, b, c\}$ such that

$$a_1 z_1 = a_2 z_1 \quad \text{and} \quad z_2 a_1 = z_2 a_2;$$

(iii) a_1 is below a_2 under the G-outer (b, c)-relation ($a_1 \leq^{go,b,c} a_2$) if there exist $z_1, z_2 \in a_1\{go, b, c\}$ such that

$$a_1 z_1 = a_2 z_1 \quad \text{and} \quad z_2 a_1 = z_2 a_2.$$

In the first theorem of this section, we characterize the left G-outer (b, c)-relation.

Theorem 4.2. *Let $b, c \in \mathcal{R}$ and $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^{-}$. The following conditions are equivalent:*

(i) $a_1 \leq^{l,go,b,c} a_2$;

(ii) *there is $z_3 \in a_1\{l, go, b, c\}$ such that*

$$a_1 z_3 = a_2 z_3 \quad \text{and} \quad z_3 a_1 = z_3 a_2;$$

(iii) *there is $z_3 \in a_1\{l, go, b, c\}$ such that*

$$a_1 z_3 a_2 = a_1 = a_2 z_3 a_1;$$

(iv) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a_1\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a_1, \quad b\mathcal{R} \subseteq q\mathcal{R} \quad \text{and} \quad pa_2 = a_1 = a_2 q.$$

Proof. (i) $\Rightarrow$ (ii): If $a_1 \leq^{l,go,b,c} a_2$, there exist $z_1, z_2 \in a_1\{l, go, b, c\}$ satisfying $a_1z_1 = a_2z_1$ and $z_2a_1 = z_2a_2$. By Corollary 3.6, $z_3 = z_1a_1z_2$ is a left G-outer (b, c)-inverse of a_1. Now, $a_1z_3 = (a_1z_1)a_1z_2 = a_2(z_1a_1z_2) = a_2z_3$ and also $z_3a_1 = (z_1a_1)z_2a_1 = (z_1a_2z_2)a_2 = z_3a_2$.

(ii) $\Rightarrow$ (iii): Because $a_1z_3 = a_2z_3$ and $z_3a_1 = z_3a_2$, where $z_3 \in a_1\{l, go, b, c\}$, we deduce that $a_1z_3a_2 = a_1z_3a_1 = a_1$ and similarly $a_2z_3a_1 = a_1$.

(iii) $\Rightarrow$ (i): Let's assume that z_3 is a left G-outer (b, c)-inverse of a_1 such that $a_1z_3a_2 = a_1 = a_2z_3a_1$. Then, by Corollary 3.6, $z' = z_3a_1z_3$ is also a left G-outer (b, c)-inverse of a_1. We show that

$$a_1z' = (a_1z_3a_1)z_3 = a_1z_3 = a_2(z_3a_1z_3) = a_2z'.$$

Analogously, we verify that $z'a_1 = z'a_2$ and so it is $a_1 \leq^{l,go,b,c} a_2$.

(ii) $\Rightarrow$ (iv): Suppose that $z_3 \in a_1\{l, go, b, c\}$ satisfies $a_1z_3 = a_2z_3$ and $z_3a_1 = z_3a_2$. For $p = a_1z_3$ and $q = z_3a_1$, according to Corollary 3.4, we have $p\mathcal{R} = a_1\mathcal{R}$, $\mathcal{R}a_1 = \mathcal{R}q$ and $b\mathcal{R} \subset q\mathcal{R}$. Also, $pa_2 = (a_1z_3)a_2 = a_1(z_3a_2) = a_1z_3a_1 = a_1$ and $a_1q = a_1(z_3a_1) = a_1$.

(iv) $\Rightarrow$ (ii): If a_1^- is an inner inverse of a_1, by Corollary 3.4, $z_3 = qa_1^-p$ is a left G-outer (b, c)-inverse of a_1. We now check $a_1z_3 = (a_1q)a_1^-p = a_1a_1^-p = a_2(qa_1^-p) = a_2z_3$ and similarly $z_3a_1 = z_3a_2$. $\square$

If $(\mathcal{R}, \cdot)$ is a ring, then $(\mathcal{R}, \star)$ is another ring if we define $a \star b = b \cdot a$. By this fact and Theorem 4.2, we can prove Theorem 4.3 about characterizations of the relation $\leq^{r,go,b,c}$.

Theorem 4.3. *Let $b, c \in \mathcal{R}$ and $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$. The following conditions are equivalent:*

(i) $a_1 \leq^{r,go,b,c} a_2$;

(ii) *there is $z_3 \in a_1\{r, go, b, c\}$ such that*

$$a_1z_3 = a_2z_3 \quad \text{and} \quad z_3a_1 = z_3a_2;$$

(iii) *there is $z_3 \in a_1\{r, go, b, c\}$ such that*

$$a_1z_3a_2 = a_1 = a_2z_3a_1;$$

(iv) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a_1\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a_1, \quad \mathcal{R}p \subseteq \mathcal{R}c \quad \text{and} \quad pa_2 = a_1 = a_2q.$$

Consequently, by Theorem 4.2 and Theorem 4.3, we characterize the relation $\leq^{go,b,c}$.

Corollary 4.4. *Let $b, c \in \mathcal{R}$ and $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$. The following conditions are equivalent:*

(i) $a_1 \leq^{go,b,c} a_2$;

(ii) *there is $z_3 \in a_1\{go, b, c\}$ such that*

$$a_1 z_3 = a_2 z_3 \quad \text{and} \quad z_3 a_1 = z_3 a_2;$$

(iii) *there is $z_3 \in a_1\{go, b, c\}$ such that*

$$a_1 z_3 a_2 = a_1 = a_2 z_3 a_1;$$

(iv) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a_1\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a_1, \quad a^{||(b,c)}p = qa^{||(b,c)} \quad \text{and} \quad pa_2 = a_1 = a_2 q.$$

By Theorem 4.2, Theorem 4.3 and Corollary 4.4, if $a_1 \leq^{l,go,b,c} a_2$ or $a_1 \leq^{r,go,b,c} a_2$ or $a_1 \leq^{go,b,c} a_2$, then $a_1 \leq^- a_2$. The relations defined in this section imply the following inclusions.

Theorem 4.5. *Let $b, c \in \mathcal{R}$ and $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$.*

(i) *If $a_1 \leq^{l,go,b,c} a_2$, then $a_2\{l, go, b, c\} \subseteq a_1\{l, go, b, c\}$.*

(ii) *If $a_1 \leq^{r,go,b,c} a_2$, then $a_2\{r, go, b, c\} \subseteq a_1\{r, go, b, c\}$.*

(iii) *If $a_1 \leq^{go,b,c} a_2$, then $a_2\{go, b, c\} \subseteq a_1\{go, b, c\}$.*

Proof. If $a_1 \leq^{l,go,b,c} a_2$, then according to Theorem 4.2, there is $z_3 \in a_1\{l, go, b, c\}$ satisfying $a_1 z_3 = a_2 z_3$ and $z_3 a_1 = z_3 a_2$. Let f be a left G-outer (b, c)-inverse of a_2. We have equalities $a_2 f a_2 = a_2$ and $f a_2 a_2^{||(b,c)} = a_2^{||(b,c)}$. Then

$$a_1 f a_1 = a_1(z_3 a_1) f (a_1 z_3) a_1 = a_1 z_3 (a_2 f a_2) z_3 a_1 = a_1 z_3 a_2 z_3 a_1 = a_1 z_3 a_1 = a_1.$$

Since $a_1^{||(b,c)}\mathcal{R} = b\mathcal{R} = a_2^{||(b,c)}\mathcal{R}$, for some $u \in \mathcal{R}$, we show that

$$a_1^{||(b,c)} = a_2^{||(b,c)} u = a_2^{||(b,c)} a_2 (a_2^{||(b,c)} u) = a_2^{||(b,c)} a_2 a_1^{||(b,c)}.$$

Also, we have that it is

$$\begin{aligned} f a_1 a_1^{||(b,c)} &= f(a_1 z_3) a_1 a_1^{||(b,c)} = f a_2 (z_3 a_1 a_1^{||(b,c)}) = f a_2 a_1^{||(b,c)} \\ &= (f a_2 a_2^{||(b,c)}) a_2 a_1^{||(b,c)} = a_2^{||(b,c)} a_2 a_1^{||(b,c)} = a_1^{||(b,c)}. \end{aligned}$$

In this way, we conclude that f is the left G-outer (b, c)-inverse of a_1 and we proved the inclusion in (i). Analogously, we prove parts (ii) and (iii). □

Now we prove that binary relations $\leq^{l,go,b,c}$, $\leq^{r,go,b,c}$ and $\leq^{go,b,c}$ are partial orders on the set $\mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$.

Theorem 4.6. *Let $b, c \in \mathcal{R}$. The left G-outer (b,c)-relation, right G-outer (b,c)-relation and G-outer (b,c)-relation are partial orders on the set $\mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$.*

Proof. Evidently, the relation $\leq^{l,go,b,c}$ is reflexive. To show the antisymmetry, suppose that, for $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$, it is valid $a_1 \leq^{l,go,b,c} a_2$ and $a_2 \leq^{l,go,b,c} a_1$. It follows that the relations $a_1 \leq^- a_2$ and $a_2 \leq^- a_1$ hold. Since we have shown that the relation $\leq^-$ is antisymmetric, we conclude that $a_1 = a_2$. So, the relation $\leq^{l,go,b,c}$ is antisymmetric.

Now, we verify the transitivity of the relation $\leq^{l,go,b,c}$. Let $a_1, a_2, a_3 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ satisfy $a_1 \leq^{l,go,b,c} a_2$ and $a_2 \leq^{l,go,b,c} a_3$. By Theorem 4.2, there are left G-outer (b,c)-inverses e and f of a_1 and a_2, respectively, such that

$$a_1 e = a_2 e, \quad e a_1 = e a_2, \quad a_2 f = a_3 f \quad \text{and} \quad f a_2 = f a_3.$$

Then, by Theorem 4.5, f is a left G-outer (b,c)-inverse of a_1. Thus,

$$a_1 = a_1 e a_1 = a_1 e a_2 = a_1 e a_2 (f a_2) = a_1 e a_2 (f a_3) = a_1 f a_3$$

and

$$a_1 = a_1 e a_1 = a_2 e a_1 = (a_2 f) a_2 e a_1 = a_3 f (a_2 e a_1) = a_3 f a_1.$$

According to Theorem 4.2(iii), $a_1 \leq^{l,go,b,c} a_3$.

The rest follows in a similar manner. □

5. Left and right G-Drazin inverses

As special versions of left and right G-outer (b,c)-inverses, we define left and right G-Drazin inverses for elements of a ring.

Definition 5.1. Let $a \in \mathcal{R}^d \cap \mathcal{R}^-$ and $\mathrm{ind}(a) = k$. An element $z \in \mathcal{R}$ is:

(i) a left G-Drazin inverse of a if

$$aza = a \quad \textit{and} \quad z a^{k+1} = a^k;$$

(ii) a right G-Drazin inverse of a if

$$aza = a \quad \textit{and} \quad a^{k+1} z = a^k.$$

If $c \in \mathcal{R}$ is both the left G-Drazin inverse and right G-Drazin inverse of $a \in \mathcal{R}$, then it is G-Drazin inverse of a. We denote the sets of all left and right G-Drazin inverses of a with $a\{l, GD\}$ and $a\{r, GD\}$, respectively.

The following results are consequences of results of Section 3 and we can omit their proofs.

Corollary 5.2. *Let $a \in \mathcal{R}^d \cap \mathcal{R}^-$ and $\mathrm{ind}(a) = k$. For $z \in \mathcal{R}$, the following conditions are equivalent:*

(i) $z \in a\{l, GD\}$;

(ii) $aza = a$ *and* $zaa^D = a^D$;

(iii) $aza = a$ *and* $zaa^D a = a^D a$;

(iv) $aza = a$ *and* $a^k\mathcal{R} \subseteq za\mathcal{R}$.

Corollary 5.3. *Let $a \in \mathcal{R}^d \cap \mathcal{R}^-$ and $\mathrm{ind}(a) = k$. For $z \in \mathcal{R}$, the following conditions are equivalent:*

(i) $z \in a\{r, GD\}$;

(ii) $aza = a$ *and* $a^D az = aa^D$;

(iii) $aza = a$ *and* $aa^D az = aa^D$;

(iv) $aza = a$ *and* $(az)^\circ \subseteq (a^k)^\circ$.

Let us recall that, relative to an idempotent $p \in \mathcal{R}$, $a \in \mathcal{R}$ has a matrix form:

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

It is known that $a \in \mathcal{R}^d$ can be represented by

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \tag{1}$$

where a_1 is invertible in $p\mathcal{R}p$ and $a_2^k = 0$.

We show that a left G-Drazin invertible element has the following matrix form.

Theorem 5.4. *Let $a \in \mathcal{R}^d \cap \mathcal{R}^-$, $\mathrm{ind}(a) = k$ and $p = aa^d$. If a is represented by* (1)*, then the following conditions are equivalent:*

(i) $z \in a\{l, GD\}$*;*

(ii) $z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p$, $z_3 a_2 = 0$ *and* $z_2 \in a_2\{1\}$.

Proof. (i) $\Rightarrow$ (ii): Let's assume $z = \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p$. We first use the equation $za^{k+1} = a^k$ to get

$$\begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p \begin{bmatrix} a_1^{k+1} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} z_1 a_1^{k+1} & 0 \\ z_4 a_1^{k+1} & 0 \end{bmatrix}_p = \begin{bmatrix} a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p.$$

This implies $z_1 a_1^{k+1} = a_1^k$ and $z_4 a_1^{k+1} = 0$. For the inverse $a_{1,p\mathcal{R}p}^{-1}$ of a_1 in $p\mathcal{R}p$, by $z_1, z_4 \in \mathcal{R}p$, we conclude $z_1 = z_1 p = z_1 a_1 a_{1,p\mathcal{R}p}^{-1} = z_1 a_1^{k+1}(a_{1,p\mathcal{R}p}^{-1})^{k+1} =$

$a_1^k(a_{1,p\mathcal{R}p}^{-1})^{k+1} = a_{1,p\mathcal{R}p}^{-1}$ and $z_4 = z_4 p = z_4 a_1^{k+1}(a_1^{-1})^{k+1} = 0$. From $aza = a$ and

$$aza = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p a = \begin{bmatrix} p & a_1 z_3 \\ 0 & a_2 z_2 \end{bmatrix}_p \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & a_1 z_3 a_2 \\ 0 & a_2 z_2 a_2 \end{bmatrix}_p,$$

we obtain $z_3 a_2 = p z_3 a_2 = a_{1,p\mathcal{R}p}^{-1}(a_1 z_3 a_2) = 0$ and $z_2 \in a_2 \{1\}$.

(ii) ⇒ (i): By elementary computations, we show that it is $z \in a\{l, GD\}$. □

Analogously as Theorem 5.4, we get the following theorem.

Theorem 5.5. *Let $a \in \mathcal{R}^d \cap \mathcal{R}^-$, $\operatorname{ind}(a) = k$ and $p = a^d a$. If a is represented by* (1)*, then the following conditions are equivalent:*

(i) $z \in a\{r, GD\}$;

(ii) $z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ z_4 & z_2 \end{bmatrix}_p$, $a_2 z_4 = 0$ *and* $z_2 \in a_2 \{1\}$.

Also, by results of Section 3, we have the following characterizations of left and right G-Drazin invertibility.

Corollary 5.6. *Let $a \in \mathcal{R}^d$ and* $\operatorname{ind}(a) = k$.

(a) *The following conditions are equivalent:*

(i) $a\{l, GD\} \neq \emptyset$;

(ii) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad a^k\mathcal{R} \subseteq q\mathcal{R}.$$

Additionally, for arbitrary $a^- \in a\{1\}$, it is valid $qa^-p \in a\{l, GD\}$ and $q \cdot a\{1\} \cdot p \subseteq a\{l, GD\}$.

(b) *The following conditions are equivalent:*

(i) $a\{r, GD\} \neq \emptyset$;

(ii) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad \mathcal{R}p \subseteq \mathcal{R}a^k.$$

Additionally, for arbitrary $a^- \in a\{1\}$, it is valid $qa^-p \in a\{r, GD\}$ and $q \cdot a\{1\} \cdot p \subseteq a\{r, GD\}$.

Corollary 5.7. *Let $a \in \mathcal{R}^d \cap \mathcal{R}^-$. Then $a\{l, GD\} \cdot a \cdot a\{l, GD\} \subseteq a\{l, GD\}$ and $a\{r, GD\} \cdot a \cdot a\{r, GD\} \subseteq a\{r, GD\}$.*

6. Left and right G-Drazin partial orders

Using left and right G-Drazin inverses, we introduce left and right G-Drazin binary relations for elements of a ring.

Definition 6.1. Let $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$. Then

(i) a is below b under the left G-Drazin relation ($a \leq^{l,GD} b$) if there exist $z_1, z_2 \in a\{l, GD\}$ such that

$$az_1 = bz_1 \quad and \quad z_2 a = z_2 b;$$

(ii) a is below b under the right G-Drazin relation ($a \leq^{r,GD} b$) if there exist $z_1, z_2 \in a\{r, GD\}$ such that

$$az_1 = bz_1 \quad and \quad z_2 a = z_2 b.$$

Theorem 6.2. *Let $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$ and $\operatorname{ind}(a) = k$. If a is represented by (1), then the following conditions are equivalent:*

(i) $a \leq^{l,GD} b$;

(ii) *there is $z \in a\{l, GD\}$ such that $az = bz$ and $za = zb$;*

(iii) *there is $z \in a\{l, GD\}$ such that $azb = a = bza$;*

(iv) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a, \quad \mathcal{R}a^k \subseteq \mathcal{R}q \quad and \quad pb = a = bq.$$

(v) *there is $z \in \mathcal{R}$ such that for $p = aa^d$,*

$$z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p \quad and \quad b = \begin{bmatrix} a_1 & -a_1 z_3 b_2 \\ 0 & b_2 \end{bmatrix}_p,$$

where $z_3 a_2 = 0$, $z_2 \in a_2\{1\}$, $a_2 z_2 = b_2 z_2$ and $z_2 a_2 = z_2 b_2$.

Proof. (i) $\Leftrightarrow$ (ii)-(iv): It follows by Theorem 4.2.

(ii) $\Rightarrow$ (v): Assume that there is $z \in a\{l, GD\}$ such that $az = bz$ and $za = zb$. Using Theorem 5.4, we know that $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p$ and $z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p$, where a_1 is invertible in $p\mathcal{R}p$, $a_2^k = 0$, $z_3 a_2 = 0$, and $z_2 \in a_2\{1\}$.

Let $b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p$. From $az = bz$,

$$\begin{bmatrix} p & a_1 z_3 \\ 0 & a_2 z_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1 a_{1,p\mathcal{R}p}^{-1} & b_1 z_3 + b_3 z_2 \\ b_4 a_{1,p\mathcal{R}p}^{-1} & b_4 z_3 + b_2 z_2 \end{bmatrix}_p.$$

We conclude $b_1 = b_1 p = (b_1 a_{1,p\mathcal{R}p}^{-1})a_1 = pa_1 = a_1$, $b_4 = b_4 p = (b_4 a_{1,p\mathcal{R}p}^{-1})a_1 = 0$ and $a_2 z_2 = b_2 z_2$. The equality $za = zb$ gives

$$\begin{bmatrix} p & z_3 a_2 \\ 0 & z_2 a_2 \end{bmatrix}_p = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p \begin{bmatrix} a_1 & b_3 \\ 0 & b_2 \end{bmatrix}_p = \begin{bmatrix} p & a_{1,p\mathcal{R}p}^{-1} b_3 + z_3 b_2 \\ 0 & z_2 b_2 \end{bmatrix}_p .$$

We have now $z_2 a_2 = z_2 b_2$ and, by $z_3 a_2 = 0$, $a_{1,p\mathcal{R}p}^{-1} b_3 + z_3 b_2 = 0$, which implies $b_3 = pb_3 = a_1(a_{1,p\mathcal{R}p}^{-1} b_3) = -a_1 z_3 b_2$. So, $b = \begin{bmatrix} a_1 & -a_1 z_3 b_2 \\ 0 & b_2 \end{bmatrix}_p$ and the implication is true.

(v) $\Rightarrow$ (ii): Clearly, by direct computations. □

Analogously, the following theorem can be shown.

Theorem 6.3. *Let $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$ and $\mathrm{ind}(a) = k$. If a is represented by (1), then the following conditions are equivalent:*

(i) *$a \leq^{r,GD} b$;*

(ii) *there is $z \in a\{r, GD\}$ such that $az = bz$ and $za = zb$;*

(iii) *there is $z \in a\{r, GD\}$ such that $azb = a = bza$;*

(iv) *there are idempotents $p, q \in \mathcal{R}$ such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a, \quad p\mathcal{R} \subseteq a^k\mathcal{R} \quad \text{and} \quad pb = a = bq.$$

(v) *there is $z \in \mathcal{R}$ such that for $p = a^d a$,*

$$z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ z_4 & z_2 \end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ -b_2 z_4 a_1 & b_2 \end{bmatrix}_p ,$$

where $a_2 z_4 = 0, z_2 \in a_2\{1\}$, $a_2 z_2 = b_2 z_2$ and $z_2 a_2 = z_2 b_2$.

We know that if $a \leq^{l,GD} b$, it is valid $a \leq^- b$. Under the given conditions, we prove the converse.

Theorem 6.4. *Let $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$ and $\mathrm{ind}(a) = k$. The following conditions are equivalent:*

(i) *$a \leq^{l,GD} b$;*

(ii) *$a \leq^- b$ and $ba^d = aa^d$;*

(iii) *$a \leq^- b$ and $ba^k = a^{k+1}$;*

(iv) *$a \leq^- b$ and $ba^k\mathcal{R} \subseteq a^k\mathcal{R}$.*

Proof. (i) $\Rightarrow$ (ii): Let $a \leq^{l,GD} b$. By definition, it is valid $a \leq^- b$. Using Theorem 6.2, there is $z \in \mathcal{R}$ such that a, b and z are represented as in (v). Since $b = \begin{bmatrix} a_1 & -a_1 z_3 b_2 \\ 0 & b_2 \end{bmatrix}_p$ and $a^d = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p$, we get the following:

$$ba^d = \begin{bmatrix} a_1 & -a_1z_3b_2 \\ 0 & b_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = aa^d.$$

(ii) $\Rightarrow$ (iii): The assumption $ba^d = aa^d$ implies

$$ba^k = ba^d a^{k+1} = aa^d a^{k+1} = a^{k+1}.$$

(iii) $\Rightarrow$ (iv): This is obvious.

(iv) $\Rightarrow$ (i): For $p = aa^d$, we can write

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p,$$

where a_1 is invertible in $p\mathcal{R}p$ and $a_2^k = 0$. Since $ba^k\mathcal{R} \subseteq a^k\mathcal{R}$, we have

$$ba^k = a^k r = aa^d(a^k r) = aa^d ba^k,$$

for some $r \in \mathcal{R}$. Then

$$\begin{bmatrix} b_1a_1^k & 0 \\ b_4a_1^k & 0 \end{bmatrix}_p = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} b_1a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p.$$

We deduce that $b_4a_1^k = 0$, which implies

$$b_4 = b_4p = b_4a_1^k(a_{1,p\mathcal{R}p}^{-1})^k = 0.$$

From the assumption $a \leq^- b$, there is $z \in a\{1\}$ satisfying $az = bz$ and $za = zb$. Let $z = \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p$. Then $aza = a$ yields

$$\begin{bmatrix} a_1z_1a_1 & a_1z_3a_2 \\ a_2z_4a_1 & a_2z_2a_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p$$

which implies $z_1 = a_{1,p\mathcal{R}p}^{-1}$, $z_3a_2 = 0$, $a_2z_4 = 0$ and $z_2 \in a_2\{1\}$. Using equality $az = bz$, we have:

$$\begin{bmatrix} p & a_1z_3 \\ 0 & a_2z_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1 & b_3 \\ 0 & b_2 \end{bmatrix}_p \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1z_1 + b_3z_4 & b_1z_3 + b_3z_2 \\ b_2z_4 & b_2z_2 \end{bmatrix}_p.$$

We concluded that $b_2z_4 = 0, \quad a_2z_2 = b_2z_2,$

$$p = b_1z_1 + b_3z_4 \quad \text{and} \quad a_1z_3 = b_1z_3 + b_3z_2. \tag{2}$$

Also, by $za = zb$, we get

$$\begin{bmatrix} p & z_3a_2 \\ z_4a_1 & z_2a_2 \end{bmatrix}_p = \begin{bmatrix} z_1b_1 & z_1b_3 + z_3b_2 \\ z_4b_1 & z_4b_3 + z_2b_2 \end{bmatrix}_p,$$

so we conclude that $a_1 = b_1$. Using (2), $p = b_1z_1 + b_3z_4 = p + b_3z_4$, which implies $b_3z_4 = 0$. Below, $a_1z_3 = b_1z_3 + b_3z_2 = a_1z_3 + b_3z_2$, which implies $b_3z_2 = 0$.

Also valid $z_3a_2 = a^{-1}_{1,p\mathcal{R}p}b_3+z_3b_2$ and $z_2a_2 = z_4b_3+z_2b_2$. Thus, $0 = a^{-1}_{1,p\mathcal{R}p}b_3+z_3b_2$ implies $b_3 = -a_1z_3b_2$.

Let's define now $z^{'} = \begin{bmatrix} a^{-1}_{1,p\mathcal{R}p} & z_3 \\ 0 & z_2a_2z_2 \end{bmatrix}_p$. Since $z_2a_2z_2 \in a_2\{1\}$, by Theorem 5.4, $z' \in a\{l, GD\}$.

Now we check the accuracy of the equalities $az^{'} = bz^{'}$ and $z^{'}a = z^{'}b$. Equalities $b_3z_2 = 0$ and $a_2z_2 = b_2z_2$ imply

$$az^{'} = \begin{bmatrix} p & a_1z_3 \\ 0 & a_2z_2a_2z_2 \end{bmatrix}_p = \begin{bmatrix} p & a_1z_3 + b_3z_2a_2z_2 \\ 0 & b_2z_2a_2z_2 \end{bmatrix}_p = bz^{'}.$$

Analogously, by $z_3a_2 = 0 = a^{-1}_{1,p\mathcal{R}p}b_3+z_3b_2$ and $z_2a_2z_2b_2 = z_2a_2(z_2a_2-z_4b_3) = z_2a_2$, we show $z^{'}a = z^{'}b$. Finally, it was shown to be valid $a \leq^{l,GD} b$. □

We similarly consider the following theorem related to the relation $\leq^{r,GD}$.

Theorem 6.5. *Let $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$ and* ind$(a) = k$. *The following conditions are equivalent:*

(i) $a \leq^{r,GD} b$;

(ii) $a \leq^- b$ *and* $a^db = a^da$;

(iii) $a \leq^- b$ *and* $a^kb = a^{k+1}$;

(iv) $a \leq^- b$ *and* $\mathcal{R}a^kb \subseteq \mathcal{R}a^k$.

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