

## ON RINGS IN WHICH EVERY RESIDUE FIELD IS INFINITE

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**ABSTRACT.** A commutative ring with identity is called an *IMR-ring* if it remains infinite modulo each of its maximal ideals. In this paper, we investigate the transfer of the IMR property to various ring extensions, including unital extensions, Nagata rings, Anderson rings, finite direct products, Nagata ideal transforms, trivial ring extensions, amalgamated duplications, amalgamated algebras, and pullbacks.

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### 1. Introduction

In ring theory, the concept of factor rings plays a fundamental role. In particular, factor rings provide a powerful means of reducing problems involving infinite structures to more tractable finite settings. Moreover, they serve as essential tools for defining and characterizing a wide variety of algebraic notions. For instance, it is well known that prime and maximal ideals of a ring can be characterized via their corresponding factor rings. Additionally, the notion of a  $G$ -ideal is defined in terms of the associated factor ring being a  $G$ -domain (see Kaplansky's book [23]). Furthermore, many important classes of rings are defined or characterized by properties of their factor rings modulo certain types of ideals, including  $Q$ -Noetherian rings [7],  $Q$ -strongly 0-dimensional rings [24], strong  $S$ -domains [23], pre-Prüfer rings [8],  $G$ -Noetherian rings [1], Prüfer rings [25], Dedekind rings [26], and Krull domains [13].

An especially interesting notion defined through finiteness conditions on factor rings is that of *residually finite rings*. Chew and Lawn [15] defined a ring  $R$  (not necessarily commutative) to be residually finite if  $R/I$  is finite for every nonzero ideal  $I$  of  $R$ . This concept was, in fact, introduced earlier in the context of integral

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domains by Butts and Wade [10], who referred to such domains as having the *finite norm property*. In the setting of commutative rings with identity, restricting this condition to maximal ideals rather than arbitrary nonzero ideals leads to the concept of an *FMR-ring* (finite maximal residue class ring), introduced by Gilmer and Parker in [28]. A commutative ring with identity  $R$  is called an FMR-ring if the residue field  $R/M$  is finite for every maximal ideal  $M$  of  $R$ . This notion has proven particularly useful in the study of nilpotent elements in semigroup rings of characteristic zero. Notably, every finite ring is an FMR-ring, and any FMR-ring that is a field must necessarily be finite. More recently, this class of rings has been further investigated in [27].

In contrast to the previously discussed class of rings, Dobbs introduced and studied in [20] a class of rings  $R$  satisfying property  $\mathcal{P}$ , which holds if  $R/P$  is infinite for every prime ideal  $P$  of  $R$ ; equivalently,  $R/M$  is infinite for every maximal ideal  $M$  of  $R$ . Motivated by the terminology of Gilmer and Parker, we adopt the term *IMR-rings* (*infinite maximal residue class rings*) to refer to this class of rings throughout the paper. As noted in the introduction of [20], rings satisfying property  $\mathcal{P}$ —that is, IMR-rings in our terminology—also appear as a hypothesis in a result of [29], specifically in [29, Theorem 2.13]. To the best of our knowledge, this class of rings has not been extensively studied and thus offers promising opportunities for further investigation.

In this paper, we aim to study the transfer of the IMR property in various contexts of commutative rings. In Section 2, we give a detailed presentation of the fundamental properties of IMR-rings. Among other results, we begin by characterizing IMR-rings in terms of their prime ideals (Lemma 2.4), and we show that the IMR property is preserved under localization (Proposition 2.6). An alternative characterization is then provided using proper ideals, the nilradical, the Jacobson radical, and localizations at prime and maximal ideals (Theorem 2.7).

Section 3 focuses on the transfer of the IMR property to several well-known ring constructions. We first show that this property is preserved under unital ring extensions (Proposition 3.1), which allows us to deduce corresponding results for polynomial rings, power series rings, and rings of integer-valued polynomials (Corollary 3.3 and Proposition 3.6). We then prove that the Nagata ring is always an IMR-ring, regardless of the base ring (Proposition 3.8). The case of Anderson rings is also addressed (Proposition 3.9). Next, we investigate the behavior of finite direct products of IMR-rings (Proposition 3.10) and, in the infinite case, characterize infinite products of fields that are IMR-rings (Theorem 3.11). We also demonstrate that the property of being an IMR-ring is preserved under Nagata ideal

transforms  $T(I) = \bigcup_{n=0}^{\infty} (D : I^n)$  in the case when  $D$  is a non-local integral domain and  $I$  is a proper principal ideal (Proposition 3.13). Furthermore, we provide a complete characterization of IMR-rings arising from trivial ring extensions and amalgamated duplication (Theorems 3.15 and 3.16) and establish some necessary and sufficient conditions for an amalgamated algebra to be an IMR-ring (Proposition 3.17). Finally, we examine the transfer of the IMR property in pullback constructions (Theorems 3.20 and 3.23).

Throughout this paper, all rings are assumed to be commutative with identity, and all modules are unital. As usual, the symbols  $\subset$  and  $\subseteq$  denote proper and non-strict inclusion, respectively.

## 2. Fundamental properties of IMR-rings

We begin this section by presenting the essential definition of our paper.

**Definition 2.1.** A ring  $R$  is called an *IMR-ring* if, for every maximal ideal  $M$  of  $R$ , the residue field  $R/M$  is infinite.

It is clear that IMR-rings are necessarily infinite. We now provide some natural examples and non-examples of IMR-rings.

**Example 2.2.** (1) Infinite fields and local rings with infinite residue field (for example,  $\mathbb{Z}_{(2)} + X\mathbb{Q}[[X]]$  or  $K[[X]]$  with  $K$  an infinite field) are IMR-rings.  
 (2) The polynomial ring  $\mathbb{C}[X]$  is an IMR-ring. Indeed,

$$\text{Max}(\mathbb{C}[X]) = \{M_a := (X - a)\mathbb{C}[X] \mid a \in \mathbb{C}\}$$

and  $\mathbb{C}[X]/M_a \cong \mathbb{C}$  for each  $a \in \mathbb{C}$ .

(3) The ring of integers  $\mathbb{Z}$  and its polynomial ring  $\mathbb{Z}[X]$  are not IMR-rings.

The next result is straightforward, so we omit its proof.

**Proposition 2.3.** *Every homomorphic image of an IMR-ring is again an IMR-ring.*

The following result characterizes IMR-rings in terms of their prime ideals.

**Lemma 2.4.** [20, Proposition 2.2] *Let  $R$  be a ring. Then  $R$  is an IMR-ring if and only if  $R/P$  is infinite for every prime ideal  $P$  of  $R$ .*

**Remark 2.5.** One might wonder whether it is possible to replace prime ideals in Lemma 2.4 with minimal prime ideals. The answer is negative. For instance, consider the ring of integers  $\mathbb{Z}$ . Its only minimal prime ideal is  $(0)$ , and we have

$\mathbb{Z}/(0) \cong \mathbb{Z}$ , which is infinite. However,  $\mathbb{Z}$  is not an IMR-ring. On the other hand, as shown in [20, Proposition 2.6], a ring  $R$  is an IMR-ring if and only if  $R/Q$  is an IMR-ring for every minimal prime ideal  $Q$  of  $R$ .

We next show that the IMR property is stable under localization.

**Proposition 2.6.** *Let  $R$  be an IMR-ring and  $S$  a multiplicative subset of  $R$ . Then  $R_S$  is an IMR-ring.*

**Proof.** Let  $Q$  be a prime ideal of  $R_S$ . Then  $Q = P_S$  for some prime ideal  $P$  of  $R$  disjoint from  $S$ . Since  $R$  is an IMR-ring, it follows from Lemma 2.4 that  $R/P$  is infinite. Let  $\bar{S}$  denote the image of  $S$  in  $R/P$ . Since  $P \cap S = \emptyset$ , it follows that  $\bar{0} \notin \bar{S}$ . Then  $(R/P)_{\bar{S}}$  is also infinite because  $R/P$  injects into  $(R/P)_{\bar{S}}$ . Now observe that  $R_S/Q \cong R_S/P_S \cong (R/P)_{\bar{S}}$ , and thus  $R_S/Q$  is infinite. Hence,  $R_S$  is an IMR-ring.  $\square$

We now extend Lemma 2.4 to arbitrary proper ideals and restrict it to certain prime ideals. Let  $\text{Nil}(R)$  and  $\text{Jac}(R)$  denote, respectively, the nilradical and the Jacobson radical of  $R$ .

**Theorem 2.7.** *For any ring  $R$ , the following statements are equivalent:*

- (1)  $R$  is an IMR-ring.
- (2)  $R/I$  is infinite for every proper ideal  $I$  of  $R$ .
- (3)  $R/\text{Jac}(R)$  is an IMR-ring.
- (4)  $R/\text{Nil}(R)$  is an IMR-ring.
- (5)  $R_P$  is an IMR-ring for every prime ideal  $P$  of  $R$ .
- (6)  $R_M$  is an IMR-ring for every maximal ideal  $M$  of  $R$ .

**Proof.** (1)  $\Rightarrow$  (2): Suppose  $R$  is an IMR-ring and let  $I$  be a proper ideal of  $R$ . Then  $I$  is contained in a maximal ideal  $M$  of  $R$ , and hence there is a surjection  $R/I \twoheadrightarrow R/M$ . Since  $R/M$  is infinite,  $R/I$  must also be infinite.

(2)  $\Rightarrow$  (3): This is immediate because  $\text{Jac}(R)$  is a proper ideal of  $R$ .

(3)  $\Rightarrow$  (4): Since  $\text{Nil}(R) \subseteq \text{Jac}(R)$ , there is a natural surjection  $R/\text{Nil}(R) \twoheadrightarrow R/\text{Jac}(R)$ . If  $R/\text{Jac}(R)$  is infinite, then  $R/\text{Nil}(R)$  must also be infinite.

(4)  $\Rightarrow$  (1): Assume  $R/\text{Nil}(R)$  is an IMR-ring. For any prime ideal  $P$  of  $R$ , we have  $\text{Nil}(R) \subseteq P$ , so  $P/\text{Nil}(R)$  is prime in  $R/\text{Nil}(R)$ . Thus  $R/P \cong (R/\text{Nil}(R))/(P/\text{Nil}(R))$ . Since  $R/\text{Nil}(R)$  is an IMR-ring,  $R/P$  is infinite. By Lemma 2.4,  $R$  is an IMR-ring.

(5)  $\Rightarrow$  (6): This is clear since every maximal ideal is prime.

(6)  $\Rightarrow$  (1): Assume  $R_M$  is an IMR-ring for every maximal ideal  $M$  of  $R$ . Then for each  $M$ , we have  $R/M \cong R_M/MR_M$ , so  $R/M$  is infinite. Thus,  $R$  is an IMR-ring.

(1)  $\Rightarrow$  (5): This follows from Proposition 2.6.  $\square$

### 3. IMR-rings in ring constructions

In this section, we investigate the transfer of the IMR property to several distinguished ring extensions. We begin by showing that the IMR property is preserved under unital ring extensions.

**Proposition 3.1.** *Let  $R \subseteq T$  be an extension of rings with the same identity. If  $R$  is an IMR-ring, then so is  $T$ .*

**Proof.** Assume that  $R$  is an IMR-ring. Let  $M$  be a maximal ideal of  $T$ , and set  $P = M \cap R$ . Then  $P$  is a prime ideal of  $R$ , and hence the integral domain  $R/P$  is infinite. Since  $R/P \subseteq T/M$ , it follows that  $T/M$  is also infinite. Therefore,  $T$  is an IMR-ring.  $\square$

**Remark 3.2.** The converse of Proposition 3.1 does not hold in general. For example, in the ring extension  $\mathbb{Z} \subset \mathbb{Q}$ , the ring  $\mathbb{Z}$  is not an IMR-ring, whereas the field  $\mathbb{Q}$  is.

For any ring  $R$ , the ring extensions  $R \subseteq R[X]$  and  $R \subseteq R[[X]]$  satisfy the condition of Proposition 3.1. Therefore, we obtain the following result.

**Corollary 3.3.** *Let  $R$  be a ring and  $X$  an indeterminate over  $R$ . Then the polynomial ring  $R[X]$  (resp., the power series ring  $R[[X]]$ ) is an IMR-ring if and only if  $R$  is an IMR-ring.*

**Proof.** If  $R[X]$  (resp.,  $R[[X]]$ ) is an IMR-ring, then so is  $R$  by Proposition 2.3 since  $R \cong R[X]/XR[X]$  and  $R \cong R[[X]]/XR[[X]]$ . The converse follows from Proposition 3.1.  $\square$

As a direct consequence of Corollary 3.3, we obtain:

**Corollary 3.4.** *Let  $K$  be a field. Then  $K[X]$  (resp.,  $K[[X]]$ ) is an IMR-ring if and only if  $K$  is infinite.*

**Corollary 3.5.** *Let  $R \subseteq T$  be an extension of rings with the same identity and let  $X$  be an indeterminate over  $T$ . Then  $R + XT[X]$  (resp.,  $R + XT[[X]]$ ) is an IMR-ring if and only if  $R$  is an IMR-ring.*

Let  $D$  be an integral domain with the quotient field  $K$  and  $X$  an indeterminate over  $K$ . Recall that the ring of integer-valued polynomials over  $D$  is defined as

$$\text{Int}(D) := \{f \in K[X] \mid f(D) \subseteq D\}.$$

**Proposition 3.6.** *Let  $D$  be an integral domain and consider the following statements:*

- (1)  $\text{Int}(D)$  is an IMR-ring.
- (2)  $D$  is an IMR-ring.
- (3)  $\text{Int}(D) = D[X]$ .

*Then, we have (1)  $\Leftrightarrow$  (2) and (2)  $\Rightarrow$  (3).*

**Proof.** The implication (1)  $\Rightarrow$  (2) follows because  $D$  is a homomorphic image of  $\text{Int}(D)$  via the evaluation map at  $X = 0$ . For the reverse implication (2)  $\Rightarrow$  (1), we use Proposition 3.1, since  $D \subset \text{Int}(D)$  is an extension of rings with the same identity. Finally, (2)  $\Rightarrow$  (3) follows from [11, Corollary I.3.7].  $\square$

**Remark 3.7.** The implication (3)  $\Rightarrow$  (2) in Proposition 3.6 does not hold in general. For example, let  $V$  be a rank-one non-discrete valuation domain with the finite residue field. Clearly,  $V$  is not an IMR-ring. However, as pointed out after [11, Proposition I.3.16], we have  $\text{Int}(V) = V[X]$ .

Let  $X$  be an indeterminate over a ring  $R$ . For each  $f \in R[X]$ , we denote by  $c(f)$  the *content* of  $f$ , that is, the ideal of  $R$  generated by the coefficients of  $f$ . The set  $S = \{f \in R[X] \mid c(f) = R\}$  is a multiplicatively closed subset of  $R[X]$ . The localization  $R(X) := R[X]_S$  is called the *Nagata ring* of  $R$ .

**Proposition 3.8.** *For any ring  $R$ , the Nagata ring  $R(X)$  is always an IMR-ring.*

**Proof.** Let  $M$  be a maximal ideal of  $R(X)$ . By [22, Theorem 14.1], there exists a maximal ideal  $N$  of  $R$  such that  $M = NR(X)$ . Since  $R(X)/M = R(X)/NR(X) \cong (R/N)(X)$ , and  $(R/N)(X)$  is the quotient field of  $(R/N)[X]$ , it follows that  $R(X)/M$  is infinite. Therefore,  $R(X)$  is an IMR-ring.  $\square$

Let  $X$  be an indeterminate over a ring  $R$ . The set  $A = \{f \in R[X] \mid f(0) = 1\}$  is a multiplicatively closed subset of  $R[X]$ . The localization  $R[X]_A$  is called the *Anderson ring* of  $R$ , which was introduced in [6].

**Proposition 3.9.** *Let  $R$  be a ring. Then  $R[X]_A$  is an IMR-ring if and only if  $R$  is an IMR-ring.*

**Proof.** By [6, Theorem 2.1], we have

$$\text{Max}(R[X]_A) = \{(M + XR[X])_A \mid M \in \text{Max}(R)\}.$$

Moreover, for each  $M \in \text{Max}(R)$ ,  $R[X]_A/(M + XR[X])_A \cong R/M$ , so  $R[X]_A$  is an IMR-ring if and only if  $R$  is an IMR-ring.  $\square$

Now, we investigate the transfer of the IMR property to direct products of rings.

**Proposition 3.10.** [20, Proposition 2.4(b)] *Let  $\{R_k\}_{1 \leq k \leq n}$  be a finite family of rings and set  $R := \prod_{k=1}^n R_k$ . Then  $R$  is an IMR-ring if and only if each  $R_k$  is an IMR-ring.*

At this stage, it is natural to ask what happens in the case of infinite products of rings. While this question remains open in general, we provide an answer for infinite products of fields in what follows.

**Theorem 3.11.** *Let  $X$  be a (finite or infinite) set, and for each  $x \in X$ , let  $K_x$  be a field. Set  $A := \prod_{x \in X} K_x$ . Then  $A$  is an IMR-ring if and only if  $K_x$  is infinite for each  $x \in X$ .*

**Proof.** If  $X$  is finite, the assertion follows immediately. Hence, we assume throughout that  $X$  is infinite.

Assume that  $A$  is an IMR-ring and let  $x \in X$ . Consider the map

$$\varphi_x : A \rightarrow K_x, \quad \varphi_x((a_y)_{y \in X}) = a_x.$$

It is clear that  $\varphi_x$  is a surjective ring homomorphism with

$$\ker(\varphi_x) = \{(a_y)_{y \in X} \in A \mid a_x = 0\}.$$

Thus, each  $K_x$  is a homomorphic image of  $A$ , and since the class of IMR-rings is closed under homomorphic images, it follows that each  $K_x$  is also an IMR-ring. Hence, each  $K_x$  must be infinite.

Conversely, assume that every  $K_x$  is infinite. By [12, Exercise 4.1.30], every maximal ideal  $M$  of  $A$  is of the form

$$M = M_{\mathcal{D}} := \{f \in A \mid \{x \in X \mid f(x) = 0\} \in \mathcal{D}\}$$

for some ultrafilter  $\mathcal{D}$  on  $X$ , and  $A/M \cong \prod_{\mathcal{D}} K_x$  is the ultraproduct of the fields  $K_x$ . Since each  $K_x$  is infinite and  $\mathcal{D}$  contains all cofinite subsets of  $X$ , Łoś's Theorem implies that  $\prod_{\mathcal{D}} K_x$  is an infinite field. Therefore, for every maximal ideal  $M$  of  $A$ , the residue field  $A/M$  is infinite. Consequently,  $A$  is an IMR-ring.  $\square$

**Remark 3.12.** The analogue of Theorem 3.11 for FMR-rings does not hold in general. Let  $\mathbb{P}$  denote the set of prime numbers and let  $\mathcal{D}$  be a non-principal ultrafilter on  $\mathbb{P}$ . For each  $p \in \mathbb{P}$ , let  $\mathbb{F}_p$  denote the finite field with  $p$  elements, and consider the direct product  $A = \prod_{p \in \mathbb{P}} \mathbb{F}_p$ . Define

$$M = M_{\mathbb{P}} := \{f \in A \mid \{p \in \mathbb{P} \mid f(p) = 0\} \in \mathcal{D}\}.$$

Then  $M$  is a maximal ideal of  $A$ , and the quotient ring  $A/M \cong \prod_{\mathcal{D}} \mathbb{F}_p$  is the ultraproduct of the fields  $\mathbb{F}_p$ . By [5, Proposition 3], this ultraproduct is uncountable.

We next prove that the property of being an IMR-ring is preserved under Nagata ideal transforms when the integral domain is assumed not local. First, for an ideal  $I$  of an integral domain  $D$  (with the quotient field  $K$ ), we recall that the *Nagata (ideal) transform of  $I$*  is the overring of  $D$  defined as follows:

$$T(I) = \bigcup_{n=0}^{\infty} (D : I^n) = \bigcup_{n=0}^{\infty} \{x \in K \mid xI^n \subseteq D\}.$$

**Proposition 3.13.** *Let  $D$  be an integral domain that is not local. Then  $D$  is an IMR-ring if and only if so is  $T((x))$  for each nonunit element  $x$  of  $D$ .*

**Proof.** Assume that  $D$  is an IMR-ring, and let  $x$  be a nonunit element of  $D$ . Then, by [9, Lemma 2.2], we have  $T((x)) = D_S$  for some multiplicative subset  $S$  of  $D$ . It then follows from Proposition 2.6 that  $T((x))$  is also an IMR-ring. Conversely, assume that  $T((x))$  is an IMR-ring for each nonunit element  $x \in D$ , and let  $M$  be a maximal ideal of  $D$ . Since  $D$  is not local, there exists a nonunit element  $x \in D$  such that  $x \notin M$ . By [9, Proposition 1.1], there exists a prime ideal  $P$  of  $T((x))$  such that  $P \cap D = M$  and  $T((x))_P = D_M$ . As  $T((x))$  is assumed to be an IMR-ring, it follows from Theorem 2.7 that  $D_M$  is also an IMR-ring. Hence, again by Theorem 2.7,  $D$  is an IMR-ring, as desired.  $\square$

**Remark 3.14.** It is worth noting that the previous result does not hold, in general, for the local case. In fact, for any one-dimensional local domain  $D$  with the quotient field  $K$  and for any nonzero nonunit element  $x \in D$ , we have  $T((x)) = K$ , as shown in [3, Theorem 1.2(a)].

In particular, consider the ring  $D := \mathbb{Z}_{2\mathbb{Z}}$ , the localization of  $\mathbb{Z}$  at the prime ideal  $2\mathbb{Z}$ . Clearly,  $D$  is not an IMR-ring, but for any nonzero nonunit element  $x \in D$ , the ring  $T((x)) = \mathbb{Q}$  is an IMR-ring.

**3.1. Trivial extensions, amalgamated duplication, and amalgamated algebras.** In this subsection, we examine how the IMR property transfers to trivial extensions, amalgamated duplications, and amalgamated algebras. Recall that for a ring  $R$  and an  $R$ -module  $E$ , the *trivial ring extension* of  $R$  by  $E$  (also called *idealization* of  $E$  over  $R$ ) is the ring  $R \ltimes E$  whose underlying group is  $R \times E$  with multiplication

$$(r, e)(s, f) = (rs, rf + se).$$

**Theorem 3.15.** *Let  $R$  be a ring and  $E$  an  $R$ -module. Then  $R \ltimes E$  is an IMR-ring if and only if so is  $R$ .*

**Proof.** Assume that  $R \times E$  is an IMR-ring. Then, by Proposition 2.3,  $R$  is also an IMR-ring because  $(R \times E)/(\{0\} \times E) \cong R$ . Conversely, assume that  $R$  is an IMR-ring and let  $M$  be a maximal ideal of  $R \times E$ . By [2, Theorem 3.2(1)], we have  $M = N \times E$ , where  $N$  is a maximal ideal of  $R$ . Since  $(R \times E)/(N \times E) \cong R/N$ , the quotient is an infinite field, and hence  $R \times E$  is an IMR-ring.  $\square$

Recall that the *amalgamated duplication* of a ring  $R$  along an ideal  $I$  of  $R$  is the subring of  $R \times R$  defined as

$$R \bowtie I = \{(r, r + i) \mid r \in R, i \in I\}.$$

**Theorem 3.16.** *Let  $R$  be a ring and  $I$  an ideal of  $R$ . Then  $R \bowtie I$  is an IMR-ring if and only if so is  $R$ .*

**Proof.** Assume that  $R \bowtie I$  is an IMR-ring. Then, by Proposition 2.3,  $(R \bowtie I)/(\{0\} \times I) \cong R$ , and therefore  $R$  is an IMR-ring. Conversely, assume that  $R$  is an IMR-ring and let  $M$  be a maximal ideal of  $R \bowtie I$ . By [16, Theorem 3.5],  $M$  is of the form

$$\{(i, i + r) \mid i \in I, r \in N\} \quad \text{or} \quad \{(i + r, i) \mid i \in I, r \in N\},$$

for some maximal ideal  $N$  of  $R$ . In either case,  $(R \bowtie I)/M \cong R/N$ . Since  $R$  is an IMR-ring,  $R/N$  is an infinite field. Therefore,  $R \bowtie I$  is an IMR-ring.  $\square$

We now investigate the transfer of the IMR property in the context of amalgamated algebras. Following [17], we recall the definition of amalgamated algebras along an ideal:

Let  $A$  and  $B$  be two rings,  $f : A \rightarrow B$  a ring homomorphism, and  $J$  an ideal of  $B$ . The subring of  $A \times B$  defined by

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

is called the *amalgamation* of  $A$  with  $B$  along  $J$  with respect to  $f$ .

By [19, Corollary 2.7], the maximal spectrum of  $A \bowtie^f J$  is given by

$$\text{Max}(A \bowtie^f J) = \{P'_f \mid P \in \text{Max}(A)\} \cup \{\overline{Q^f} \mid Q \in \text{Max}(B) \setminus V(J)\},$$

where

$$\begin{aligned} P'_f &:= P \bowtie^f J := \{(p, f(p) + j) \mid p \in P, j \in J\}, \\ \overline{Q^f} &:= \{(a, f(a) + j) \mid a \in A, j \in J, f(a) + j \in Q\}, \\ V(J) &:= \{Q \in \text{Spec}(B) \mid J \subseteq Q\}. \end{aligned}$$

**Proposition 3.17.** *Let  $A$  and  $B$  be two rings,  $f : A \rightarrow B$  a ring homomorphism, and  $J$  an ideal of  $B$ . Then:*

- (1) If  $A \bowtie^f J$  is an IMR-ring, then so are  $A$  and  $f(A) + J$ .
- (2) If  $f$  is surjective, then  $A \bowtie^f J$  is an IMR-ring if and only if both  $A$  and  $B$  are IMR-rings.
- (3) If either  $J \subseteq \text{Nil}(B)$  or  $J \subseteq \text{Jac}(B)$ , then  $A \bowtie^f J$  is an IMR-ring if and only if  $A$  is an IMR-ring.
- (4) If  $f^{-1}(J) = \{0\}$ , then  $A \bowtie^f J$  is an IMR-ring if and only if  $f(A) + J$  is an IMR-ring.

**Proof.** (1) If  $A \bowtie^f J$  is an IMR-ring, then by Proposition 2.3, both  $A$  and  $f(A) + J$  are IMR-rings since  $(A \bowtie^f J)/(\{0\} \times J) \cong A$  and  $(A \bowtie^f J)/(f^{-1}(J) \times \{0\}) \cong f(A) + J$ .

(2) Suppose  $f$  is surjective. If  $A \bowtie^f J$  is an IMR-ring, part (1) implies that  $A$  and  $f(A) + J = B$  are IMR-rings. Conversely, assume that  $A$  and  $B$  are IMR-rings. For any  $M \in \text{Max}(A \bowtie^f J)$ :

Case 1: If  $M = \overline{Q^f}$  for some  $Q \in \text{Max}(B) \setminus V(J)$ , then  $(A \bowtie^f J)_{\overline{Q^f}} \cong B_Q$  by [19, Proposition 2.9(1)], and thus  $(A \bowtie^f J)/M \cong B/Q$  is an infinite field.

Case 2: If  $M = P_f'$  for some  $P \in \text{Max}(A)$ , then  $(A \bowtie^f J)/M \cong A/P$ , which is also an infinite field. Therefore,  $A \bowtie^f J$  is an IMR-ring.

(3) If  $J \subseteq \text{Nil}(B)$  or  $J \subseteq \text{Jac}(B)$ , then by [18, Proposition 2.6], every maximal ideal of  $A \bowtie^f J$  is of the form  $P \bowtie^f J$  for some  $P \in \text{Max}(A)$ . Thus,  $(A \bowtie^f J)/(P \bowtie^f J) \cong A/P$ , and the IMR property transfers exactly from  $A$ .

(4) If  $f^{-1}(J) = \{0\}$ , then by [17, Proposition 5.1(3)], we have  $A \bowtie^f J \cong f(A) + J$ . The result follows immediately.  $\square$

Next, we provide an illustrative example.

**Example 3.18.** Let  $K$  be an infinite field,  $A = K[[X]]$ ,  $B = K[[X, Y]]$ , and  $J = (X, Y)$ . Let  $f : A \rightarrow B$  be the natural inclusion. Since  $J \subseteq \text{Jac}(B)$  and  $A$  is an IMR-ring, Proposition 3.17(3) implies that  $A \bowtie^f J$  is also an IMR-ring.

**3.2. Pullback constructions.** This subsection is devoted to the transfer of the IMR property in pullback constructions. We start with the following lemma.

**Lemma 3.19.** *Let  $\varphi : A \rightarrow C$  and  $\psi : B \rightarrow C$  be surjective ring homomorphisms. Consider the pullback ring  $R = \{(a, b) \in A \times B \mid \varphi(a) = \psi(b)\}$ , which fits into the commutative diagram*

$$\begin{array}{ccc}
 R = A \times_C B & \xrightarrow{\pi_B} & B \\
 \pi_A \downarrow & & \downarrow \psi \\
 A & \xrightarrow{\varphi} & C,
 \end{array}$$

where  $\pi_A : R \rightarrow A$  and  $\pi_B : R \rightarrow B$  denote the canonical projection maps.

If  $J$  is a proper ideal of  $R$ , then at least one of the projections  $\pi_A(J) \subsetneq A$  or  $\pi_B(J) \subsetneq B$  is a proper ideal.

**Proof.** Suppose, toward a contradiction, that  $J \subsetneq R$  is a proper ideal such that  $\pi_A(J) = A$  and  $\pi_B(J) = B$ .

Since  $\pi_A(J) = A$ , there exists an element  $(1, b_1) \in J$  for some  $b_1 \in B$ . Because  $(1, b_1) \in R$ , we have  $\varphi(1) = \psi(b_1) = 1_C$ . Similarly, since  $\pi_B(J) = B$ , there exists an element  $(a_2, 1) \in J$  for some  $a_2 \in A$ . Again, as  $(a_2, 1) \in R$ , we have  $\varphi(a_2) = \psi(1) = 1_C$ . Since  $J$  is an ideal of  $R$ , it is closed under multiplication by elements of  $R$ . Hence,  $(1, b_1)(a_2, 1) = (a_2, b_1) \in J$ .

Now consider the differences:

$$\begin{aligned} (a_2, b_1) - (a_2, 1) &= (0, b_1 - 1) \in J, \\ (a_2, b_1) - (1, b_1) &= (a_2 - 1, 0) \in J. \end{aligned}$$

Since  $J$  is an ideal, it is closed under additive inverses, so  $(0, 1 - b_1) \in J$ .

Finally, adding elements of  $J$ , we obtain  $(1, b_1) + (0, 1 - b_1) = (1, 1) \in J$ . But  $(1, 1)$  is the multiplicative identity of  $R$ , and hence  $J = R$ , contradicting the assumption that  $J$  is a proper ideal.

Therefore, at least one of  $\pi_A(J)$  or  $\pi_B(J)$  must be a proper ideal.  $\square$

**Theorem 3.20.** *Under the same notation and hypotheses as in the previous lemma, the ring  $R$  is an IMR-ring if and only if both  $A$  and  $B$  are IMR-rings.*

**Proof.** Assume that  $R$  is an IMR-ring. Let  $I$  be a proper ideal of  $A$  and set  $J := \pi_A^{-1}(I)$ . We first claim that  $J \neq R$ . Indeed, if  $J = R$ , the surjectivity of  $\pi_A$  implies that  $I = \pi_A(\pi_A^{-1}(I)) = \pi_A(J) = \pi_A(R) = A$ , a contradiction. Then,  $J \neq R$ , and hence  $R/J$  is infinite because  $R$  is an IMR-ring.

Now, as  $\ker(\pi_A) \subseteq J$ , the projection  $\pi_A$  induces the following well-defined surjective homomorphism

$$\begin{aligned} \bar{\pi}_A : R/J &\longrightarrow \pi_A(R)/I = A/I, \\ (x, y) + J &\longmapsto x + I. \end{aligned}$$

Since  $J = \pi_A^{-1}(I)$ , we have  $\ker(\bar{\pi}_A) = \{0\}$ . Indeed, if  $(x, y) \in R$  with  $x \in I$ , then  $(x, y) \in J$ , and hence  $(x, y) + J = 0$  in  $R/J$ . Thus  $\bar{\pi}_A$  is injective, and therefore an isomorphism. Consequently,  $A/I \cong R/J$ . Since  $R/J$  is infinite, it follows that  $A/I$  is infinite, and hence  $A$  is an IMR-ring. By symmetry, the same argument applies to  $B$ , showing that it is also an IMR-ring.

Assume that  $A$  and  $B$  are IMR-rings. Let  $J$  be a proper ideal of  $R$ . Then  $J \subsetneq R$ , so the image of  $J$  under at least one of the canonical projections  $\pi_A : R \rightarrow A$  or  $\pi_B : R \rightarrow B$  must be a proper ideal (Lemma 3.19).

Case 1:  $\pi_A(J)$  is a proper ideal of  $A$ . Since  $A$  is IMR,  $A/\pi_A(J)$  is infinite. The map

$$\begin{aligned} \theta : R &\longrightarrow A/\pi_A(J), \\ (a, b) &\longmapsto a + \pi_A(J) \end{aligned}$$

is a surjective ring homomorphism whose kernel contains  $J$ , so  $R/J$  maps onto the infinite ring  $A/\pi_A(J)$ . Hence  $R/J$  is infinite.

Case 2:  $\pi_A(J) = A$  and  $\pi_B(J)$  is proper. Then a similar argument applies: since  $B$  is IMR,  $B/\pi_B(J)$  is infinite. The map

$$\begin{aligned} \phi : R &\longrightarrow B/\pi_B(J), \\ (a, b) &\longmapsto b + \pi_B(J) \end{aligned}$$

is a surjective ring homomorphism with  $\ker(\phi) \supseteq J$ , and so  $R/J$  has an infinite homomorphic image. In both cases,  $R/J$  is infinite. Hence,  $R$  is an IMR-ring.  $\square$

We provide an example illustrating the above result.

**Example 3.21.** Let us define the components:

- $A := \mathbb{Q} \times \mathbb{Q}$  and  $B := \mathbb{Q}$ , and set  $C := \mathbb{Q}$ .
- Define  $\varphi : A \rightarrow C$  by  $\varphi(a_1, a_2) = a_1$ , which is clearly surjective.
- Define  $\psi : B \rightarrow C$  as the identity map.

Since both  $A = \mathbb{Q} \times \mathbb{Q}$  and  $B = \mathbb{Q}$  are IMR-rings, it follows from Theorem 3.20 that the pullback ring  $R := \{(a_1, a_2), b \in A \times B \mid a_1 = b\}$  is an IMR-ring.

In fact, in the above example, we have the identification

$$R := \{(a_1, a_2), b \in A \times B \mid a_1 = b\} \cong \{(b, a_2) \mid b, a_2 \in \mathbb{Q}\} \cong \mathbb{Q} \times \mathbb{Q}.$$

Thus,  $R$  is an IMR-ring by Proposition 3.10.

We now present a truly nontrivial application of Theorem 3.20.

**Example 3.22.** Let  $K$  be an infinite field. Set  $A := K[x]$ ,  $B := K$ ,  $C := K$ , where  $\varphi : A \rightarrow C$  is the evaluation map at  $x = 0$  and  $\psi : B \rightarrow C$  is the identity map. The corresponding pullback ring is

$$R := A \times_C B = \{(f(x), f(0)) \mid f(x) \in K[x]\}.$$

Thus  $R$  naturally embeds as a subring of  $K[x] \times K$ , but it is not isomorphic to a direct product. Since both  $A = K[x]$  and  $B = K$  are IMR-rings, Theorem 3.20 applies, and therefore  $R$  is an IMR-ring.

In the subsequent results, we focus on a special class of pullback constructions, which arise from localizations of integral domains and play a central role in our analysis.

**Theorem 3.23.** *Let  $T$  be an integral domain, let  $M$  be a maximal ideal of  $T$ , and let  $\varphi: T \rightarrow K := T/M$  be the canonical surjection. Let  $D$  be a subring of  $K$ , and set  $R := \varphi^{-1}(D)$ . Then the following statements hold:*

- (1) *If  $R$  is an IMR-ring, then  $D$  is an IMR-ring.*
- (2) *If both  $D$  and  $T$  are IMR-rings, then  $R$  is an IMR-ring.*
- (3) *If the quotient field of  $D$  is  $K$ , then  $R$  is an IMR-ring if and only if both  $D$  and  $T$  are IMR-rings.*

**Proof.** (1) Since  $D \cong R/M$  and  $M$  is an ideal of  $R$ , Proposition 2.3 implies that  $D$  is an IMR-ring.

(2) Assume that  $D$  and  $T$  are IMR-rings, and let  $P$  be a maximal ideal of  $R$ . We distinguish two cases.

Case 1:  $M \subseteq P$ . Then  $P = \varphi^{-1}(\mathfrak{p})$  for some maximal ideal  $\mathfrak{p}$  of  $D$ . In this case, localization yields  $R_P \cong D_{\mathfrak{p}}$ . Since  $D$  is an IMR-ring, every localization of  $D$  is IMR (Theorem 2.7), hence  $R_P$  is IMR.

Case 2:  $M \not\subseteq P$ . In this case, [4, Lemma 2.1(e)] applies. Thus, there exists a prime ideal  $Q$  of  $T$  such that  $Q \cap R = P$  and  $R_P \cong T_Q$ . Since  $T$  is an IMR-ring, every localization  $T_Q$  is IMR (Theorem 2.7), and hence  $R_P$  is IMR.

In all cases,  $R_P$  is an IMR-ring for every maximal ideal  $P$  of  $R$ . Therefore, by Theorem 2.7,  $R$  is an IMR-ring.

(3) Assume that the quotient field of  $D$  is  $K$ .

( $\Rightarrow$ ) Suppose that  $R$  is an IMR-ring. By part (1),  $D$  is an IMR-ring. It remains to show that  $T$  is an IMR-ring. Let  $N$  be a maximal ideal of  $T$ .

Case 1:  $N = M$ . Localizing the pullback diagram at  $M$  yields

$$\begin{array}{ccc} R_M & \longrightarrow & D_{\varphi(M)} \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & K. \end{array}$$

Since  $\text{qf}(D) = K$ , the localization  $D_{\varphi(M)}$  is a field, hence IMR. Because  $R_M$  is IMR and  $R_M = T_M$ , it follows that  $T_M$  is IMR.

Case 2:  $N \neq M$ . Set  $P := N \cap R$ . Then  $P$  is a prime ideal of  $R$  and satisfies  $M \not\subseteq P$ . By [4, Lemma 2.1(e)], there exists a prime ideal  $Q \subseteq T$  such that  $Q \cap R = P$  and  $R_P \cong T_Q$ . Since  $R$  is IMR, so is  $R_P$ , and therefore  $T_N$  is IMR. Thus  $T_N$  is IMR

for every maximal ideal  $N$  of  $T$ , and hence  $T$  is an IMR-ring by Theorem 2.7.

( $\Leftarrow$ ) If both  $D$  and  $T$  are IMR-rings, then  $R$  is an IMR-ring by part (2). This completes the proof.  $\square$

**Corollary 3.24.** *Let  $(T, M)$  be a local integral domain, let  $\varphi: T \rightarrow T/M$  be the canonical surjection, and let  $D$  be an integral domain whose quotient field is  $T/M$ . Set  $R := \varphi^{-1}(D)$ . Then the following hold:*

- (1)  *$R$  is an IMR-ring if and only if both  $D$  and  $T$  are IMR-rings.*
- (2) *If the residue field  $T/M$  is infinite, then  $R$  is an IMR-ring if and only if  $D$  is an IMR-ring.*

**Proof.** By [21, Lemma 2.5], the ideal  $M$  is a prime ideal of  $R$  such that  $R/M \cong D$  and  $R_M = T$ .

(1) Suppose that  $R$  is an IMR-ring. Since the class of IMR-rings is stable under homomorphic images and localizations (see Proposition 2.3 and Theorem 2.7), it follows that both  $D \cong R/M$  and  $T = R_M$  are IMR-rings. Conversely, if  $D$  and  $T$  are IMR-rings, then  $R$  is an IMR-ring by Theorem 3.23.

(2) Since  $(T, M)$  is a local integral domain,  $T$  is an IMR-ring if and only if its residue field  $T/M$  is infinite. Thus, when  $T/M$  is infinite, statement (1) reduces to the equivalence between  $R$  being IMR and  $D$  being IMR.  $\square$

Let  $D$  be an integral domain with the quotient field  $K$ , and let  $X$  be an indeterminate over  $K$ . For an integer  $n \geq 2$ , define  $K[\theta] := K[X]/(X^n)$ , where  $\theta := X + (X^n)$ . Then  $K[\theta]$  is a finite-dimensional  $K$ -vector space of dimension  $n$ , with basis  $\{1, \theta, \theta^2, \dots, \theta^{n-1}\}$ . Each  $f \in K[\theta]$  can be written uniquely as  $f = a_0 + a_1\theta + \dots + a_{n-1}\theta^{n-1}$ , where  $a_i \in K$ .

Define  $R_n := D + \theta K[\theta] := \{f \in K[\theta] \mid f(0) \in D\}$ . Then  $R_n$  is a subring of  $K[\theta]$ , called the *trivial extension of  $D$  of order  $n$* . It is worth noting that  $R_n$  can be realized as a pullback in the following diagram:

$$\begin{array}{ccc} R_n & \longrightarrow & K[\theta] \\ \downarrow & & \downarrow \\ D & \longrightarrow & K, \end{array}$$

where the horizontal maps are natural inclusions and the vertical maps are projections sending  $f$  to its constant term  $f(0)$ .

**Proposition 3.25.** *Let  $D$  be an integral domain with the quotient field  $K$ , and let  $n \geq 2$  be an integer. Define  $R_n := D + \theta K[\theta] \subseteq K[\theta]$ . Then  $R_n$  is an IMR-ring if and only if  $D$  is an IMR-ring.*

**Proof.** By [14, Corollary 2.6], the maximal ideals of  $R_n$  are given by

$$\text{Max}(R_n) = \{M + \theta K[\theta] \mid M \in \text{Max}(D)\}.$$

Moreover, for each maximal ideal  $M$  of  $D$ , we have  $R_n/(M + \theta K[\theta]) \cong D/M$ . Therefore,  $R_n$  is an IMR-ring if and only if  $D$  is an IMR-ring.  $\square$

**Remark 3.26.** (1) If  $D$  is an IMR-ring, then  $R_n$  is also an IMR-ring. Indeed, since  $D \subseteq R_n$  is a unital ring extension, this follows from Proposition 3.1, which states that the IMR property is preserved under unital extensions.

(2) Using the same notation as in the proposition above, the same reasoning shows that  $R_n$  is an FMR-ring if and only if  $D$  is an FMR-ring.

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