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RINGS WHOSE VON NEUMANN REGULAR ELEMENTS ARE FINE

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Abstract. We study rings in which every von Neumann regular element is fine. This condition extends the notion of fine rings and identifies a distinguished subclass of idempotent-fine rings.

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1. Introduction

The rings we consider are nonzero, associative with identity. Hereafter, the term "regular" will be used for von Neumann regular elements (and rings). For a ring R, $U(R)$ denotes the set of all the units of R, $N(R)$ denotes the set of all the nilpotents of R, $reg(R)$ denotes the set of all the regular elements of R, $ureg(R)$ denotes the set of all the unit-regular elements of R and (as in [2]) $\Phi(R)$ denotes the set of all the fine elements of R. The Jacobson radical of a ring R is denoted by $J(R)$. For any subset X of a ring, $X^* := X \setminus \{0\}$. For a ring R, $\mathbb{M}_n(R)$ denotes the ring of all $n \times n$ matrices with entries in R and $\mathbb{T}_n(R)$, its subring, consisting of all the upper triangular matrices. Recall from [2] that a nonzero element of a ring is called *fine* if it is a sum of a unit and a nilpotent. A ring is *fine* if all its nonzero elements are fine. Among other results, in [2] it was proved that

$$\{\text{Artinian simple rings}\} \subset \{\text{fine rings}\} \subset \{\text{simple rings}\}$$

and both inclusions are proper. For any ring R, the right inclusion follows as $RaR = R$ for each $a \in \Phi(R)$. As another main result, it was proved that matrix rings over fine rings are fine. In [3], a ring was called *idempotent-fine* (*IF*, for short), if all its nonzero idempotents are fine. Related to IF rings, another two classes of rings were defined. A nonzero ring R is called *idempotent-simple* if for any nonzero idempotent e of R, $ReR = R$. It was proved in [3], that IF rings are idempotent-simple. Analogously with the fine case, it was proved that

$$\{\text{Artinian idempotent-simple rings}\} \subset \{\text{IF rings}\} \subset \{\text{idempotent-simple rings}\}$$

and both inclusions are proper. Further, following Steger [14], a ring R was called an *ID* ring if every idempotent matrix over R is similar to a diagonal one. Thus, by a result of Song and Guo [13], if every matrix over R is equivalent to a diagonal matrix, then R is an ID ring. Examples of ID rings include: Division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings. Among other results, in [3] it was proved that the idempotent-simple property passes to corners, and passes to matrix rings, whenever the base ring is ID. Moreover, it was proved that over a nonzero commutative ring R and $n \geq 1$, the matrix ring $\mathbb{M}_n(R)$ is idempotent-simple iff R is indecomposable. In this paper, we introduce and investigate a new class of rings that properly generalizes the class of fine rings and is itself properly contained within the class of IF rings. A ring in which all nonzero regular elements are fine will be called a *regular-fine* (*RF*, for short) ring. Similarly, a ring in which every nonzero unit-regular element is fine will be called a *UF* ring. Fine rings are RF and UF by definition. Clearly, a ring R is RF iff $reg^*(R) \subseteq \Phi(R)$ and, is UF iff $ureg^*(R) \subseteq \Phi(R)$. Since 0 is not fine, it must be excepted. Since units are (unit)-regular and also fine (i.e., $U(R) \subseteq reg(R) \cap \Phi(R)$), in order to check whether a ring is RF (or UF), it suffices to deal with the *nonunit* (unit-)regular elements. Since in any unital ring

$$\{\text{idempotents}\} \subseteq \{\text{unit-regular elements}\} \subseteq \{\text{regular elements}\},$$

it follows that

$$\{\text{fine rings}\} \subseteq \{\text{RF rings}\} \subseteq \{\text{UF rings}\} \subseteq \{\text{IF rings}\},$$

that is, RF and UF are two successive restrictions of the class of IF rings, but two generalizations of the fine rings. Examples show that all inclusions above are proper. As units are the only fine elements in any reduced ring, reduced rings with nonunit regular elements are not RF. Since the study of UF rings is analogous to that of RF rings, in this note we restrict our attention to RF rings. Analogously to the IF case, we introduce two additional classes of rings associated with the RF class. A nonzero ring R is called *regular-simple* if for any nonzero regular element a of R, $RaR = R$. Examples of regular-simple rings are simple rings and RF rings (in particular, fine rings). Call *RD* a ring whose regular matrices are diagonalizable. Similarly with the idempotent diagonalization, a diagonal regular matrix must have regular entries. Surprisingly, it is easy to prove that a ring is regular-simple iff it is idempotent-simple. Therefore, all the results obtained in [3] for idempotent-simple rings, hold also for regular-simple rings. Since idempotents are regular, clearly RD implies ID. Whether ID is also sufficient for RD, was not known in [12], where this question was stated as early as 1985. However, it is easily proved that for domains and for commutative rings, RD is equivalent to ID. In general, the question whether ID rings are RD, seems to be still open. In the IF context, the *diagonalization of idempotent matrices* plays a prominent role. Clearly, in the RF context, the *diagonalization of regular matrices* provides analogous results. Among other (many) papers, diagonalization of regular matrices was studied in [5], [6] and [12]. As expected, the results from [3] are accordingly revisited. Moreover, results from [2] and [8] play an important role in our study.

In Section 2, the research goes along the results of [3] and in Section 3, results on matrix rings are provided. In the final section, we revisit the list of open questions on IF rings from [3]. We show that three of the corresponding questions for RF rings have negative answers.

2. General results

We start with some results which provide examples of large classes of RF rings.

Lemma 2.1. *For any ring R, $reg^*(R) = U(R)$ if and only if R is connected.*

Proof. One way, assume $reg^*(R) = U(R)$ and let $e^2 = e \neq 0$. Since $e \in reg^*(R)$, it follows that $e \in U(R)$ and so $e = 1$. Conversely, first notice that $U(R) \subseteq reg^*(R)$ holds in any ring R. Let $axa = a \neq 0$. Since ax, xa are idempotents, these equal 0 or 1. If $ax = 0$ or $xa = 0$, it follows that $a = 0$, a contradiction. Hence $ax = xa = 1$ and so $a \in U(R)$. □

Note that in this case $reg^*(R) = ureg^*(R) = U(R)$.

Corollary 2.2. *Every connected ring is RF. In particular, local rings and domains are RF.*

Proof. Indeed, by the previous lemma, $reg^*(R) = U(R) \subseteq \Phi(R)$. □

In particular, $\mathbb{Z}$ and $\mathbb{Z}(p^n)$ for every prime p and positive integer n, are RF rings.

Example 2.3. A noncommutative RF ring which is not connected is $\mathbb{M}_2(F)$ for any field F (see Section 3).

Corollary 2.4. *Let R be a nonzero ring with $\Phi(R) = U(R)$. The following conditions are equivalent.*

(i) *R is IF;*
(ii) *R is connected;*
(iii) *R is RF.*

Proof. (i) $\Rightarrow$ (ii) If R is IF, then its nonzero idempotents are units, whence R is connected.
(ii) $\Rightarrow$ (iii) is the previous corollary.
(iii) $\Rightarrow$ (i) follows from definitions. □

Remark 2.5. (1) In particular, according to [2, Proposition 2.1(3)], the above conditions are equivalent for nonzero rings such that $N(R) \subseteq J(R)$, this way including local rings, 2-primal rings (i.e., rings such that $N(R)$ is contained in the prime radical of R) and commutative rings.

(2) Left Artinian RF (or IF) rings may not be connected. As an example, take $R = \mathbb{M}_2(k)$ for any field k. Then R is *simple* Artinian and by our Corollary 3.6, it is RF (and IF). However, it is not connected.

In another direction, we have the following result:

Proposition 2.6. *A regular reduced (in particular Abelian ring - i.e., idempotents are central) is RF if and only if it is a division ring.*

Proof. As units are the only fine elements in any reduced ring, if the ring is RF, it must have only units as regular elements, whence it is a division ring. The converse is clear. Abelian regular rings are reduced (see [4, Theorem 3.2]). □

Since Abelian rings may not be 2-primal, this is not a direct consequence of Corollary 2.4.

To validate our new class of rings, we must provide an example of an IF ring that is not an RF ring and an example of RF ring which is not fine. In Section 3 (see Proposition 3.4), we demonstrate that $\mathbb{M}_2(\mathbb{Z})$ is an IF ring but not an RF ring. As for an RF ring that is not fine, by Corollary 2.2, it suffices to give an example of a connected ring that is not fine. Any local ring which is not a division ring will do (e.g., $\mathbb{Z}_4$).

Since RF rings are IF, from [3, Proposition 2], it follows that

Corollary 2.7. *The RF rings are indecomposable. The converse fails.*

Thus, any direct product of two or more RF rings is not an RF ring. The indecomposable ring $\mathbb{T}_n(F)$ (where $n \geq 2$ and F is a field) is not an IF ring (see [3, Corollary 4(5)]), and hence is not RF.

Proposition 2.8. *If R is an RF ring, then $char(R) = 0$ or $char(R) = p^m$ where p is prime and m is a positive integer.*

Proof. If R is an RF ring and $char(R) = t \neq 0$, then the subring $Z_t := \{n \cdot 1_R : n \in \mathbb{Z}\}$ is isomorphic to $\mathbb{Z}_t$ and contained in the center $Z(R)$. If Z_t were decomposable, it would contain a nontrivial idempotent e which is central in R, and yields a nontrivial product decomposition $R \cong Re \times R(1-e)$, contradicting the indecomposability of R. Hence Z_t is indecomposable and so $char(R) = p^m$ where p is prime and m is a positive integer. □

Similarly to the IF case, we have the following result.

Proposition 2.9. *The center of an RF ring is also RF.*

Proof. Let R be any RF ring. Then R is IF and its center is (a commutative) IF ring. Then it is RF by Corollary 2.4. □

However, the RF property does not pass, in general, to subrings. Alternative examples will be provided in the matrix section. A lifting property is needed for the next proposition, analogous with the idempotent lifting modulo nil ideals (see [7, 21.28]). As one might expect, in the case of RF factor rings we require a lifting property for regular elements modulo nil ideals. Fortunately, this property holds and, as follows from references, is even more general. First recall from [11], that a one-sided ideal I of a ring R is said to be *strongly lifting* if whenever $x^2 - x \in I$ for some $x \in R$, there is an idempotent $e \in xR$ such that $e - x \in I$, and that a one-sided ideal is called *π-regular* if some power of each element is regular. Examples of strongly lifting one-sided ideals include π-regular one-sided ideals (see [11, Proposition 7]) and in particular one-sided nil-ideals.

The following lifting result is proved (see [9, Theorem 2.4]).

Theorem 2.10. *If a one-sided ideal I of a ring R is strongly lifting, then regular elements lift modulo I.*

Secondly, we provide an elementary elementwise proof for a result, well-known for rings (see [4, Theorem 3.2]).

Lemma 2.11. *In any Abelian ring, the only regular nilpotent element is zero.*

Proof. Suppose $a = axa$ and $a^n = 0$ for some $n \geq 1$. As ax and xa are idempotents, these are central. Hence we successively deduce the following equalities.

First $a = a^2x$ and $a^2 = a^3x$. Then $a = a^2x = a^3x^2$. Continuing this way, we get $a^m = a^{m+1}x$ and $a = a^2x = \cdots = a^{m+1}x^m$, for any $m \geq 1$. For $m = n-1$, we obtain $a = 0$. □

Next, note that *factor rings of RF rings may not be RF*: $\mathbb{Z}$ is RF, but $\mathbb{Z}_6$ is not IF (commutative with nontrivial idempotents). However, we do have a positive result below.

Proposition 2.12. *Let R be a ring, and I an ideal of R.*

(1) *Suppose that regular elements lift modulo I in R. If R is an RF ring, then R/I is an RF ring. In particular, this holds if I is strongly lifting and so, for nil ideals.*
(2) *Suppose I is a nil ideal. Then if R/I is an RF ring, so is R.*

Proof. (1) Let $\overline{a}$ be a nonzero regular element in R/I. As regular elements lift modulo I, we can assume that a is a (nonzero) regular element of R. So $a = b + u$ where b is nilpotent and u is a unit. Therefore, $\overline{a} = \bar{b} + \bar{u}$ is fine in R/I.

(2) Let a be a nonzero regular element of R. Then $\overline{a} \in R/I$ is a nonzero regular element. To see this, note that the nil ideal I does not contain any nonzero regular nilpotents a, because a being regular means we can write $a = aba$, and then ab is a nonzero idempotent, hence cannot belong to a nil ideal. Write $\overline{a} = \bar{b} + \bar{u}$ where $\bar{b} \in N(R/I)$ and $\bar{u} \in U(R/I)$. As nilpotents lift modulo nil ideals, we can suppose that b is nilpotent and $a = b + (u + j)$ for a unit u and some $j \in I$. Since I is nil, it follows that $u + j$ is a unit of R. So a is fine. □

Corollary 2.13. *Let R be any ring.*

(1) *For a nontrivial bimodule M over R, the trivial extension $R \propto M$ is an RF ring if and only if R is an RF ring.*
(2) *For $n \geq 2$, $R[t]/(t^n)$ is an RF ring if and only if R is an RF ring.*
(3) *Let σ be a ring endomorphism of R with $\sigma(1) = 1$. Then the left skew power series ring $R[[t;\sigma]]$ is an RF ring if and only if R is an RF ring.*
(4) *Let R, S be nontrivial rings and M an (R, S)-bimodule. The formal triangular matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is not an RF ring.*
(5) *For $n \geq 2$, $\mathbb{T}_n(R)$ is not an RF ring.*

Proof. Follows *mutatis mutandis* from the proof in the IF case. □

Recall that a nonzero ring R is called *idempotent-simple* if for any nonzero idempotent e of R, $ReR = R$. Similarly, we call *regular-simple*, a nonzero ring R if for any nonzero regular element a of R, $RaR = R$. Examples of regular-simple rings are simple rings and RF rings (in particular, fine rings).

Proposition 2.14. *Every nonzero RF ring is regular-simple. The converse fails.*

Proof. Any nonzero regular element a of R is fine, so $RaR = R$ by [2, Theorem 2.2(1)]. Hence R is regular-simple. As for the converse, $\mathbb{M}_2(\mathbb{Z})$ is regular-simple but not RF (see Proposition 3.4). □

Surprisingly, we can prove the following equivalence.

Proposition 2.15. *A ring is regular-simple if and only if it is idempotent-simple.*

Proof. Clearly, regular-simple rings are idempotent-simple. For the converse, suppose a ring R is idempotent-simple, and r is a nonzero regular element of R. Say $r = rsr$. Then rs is a nonzero idempotent, so by assumption, $RrsR = R$. But RrR contains $RrsR$, hence $RrR = R$. □

3. RF for matrix rings

Recall that a ring R is called an *ID* ring if every idempotent matrix over R is similar to a diagonal matrix. Likewise, R is an RD ring if every regular matrix over R is diagonalizable. Collecting results from several references, we obtain the following classes of RD rings.

Proposition 3.1. (i) *The ID-domains are RD.*
(ii) *The commutative ID-rings and the connected ID-rings are RD.*
(iii) *The Artinian rings are RD.*

Proof. (i) Indeed, the regular matrices over ID-domains are diagonalizable (see [12, Corollary 3]).
(ii) Indeed, over rings of this type, the regular matrices are diagonalizable.
(iii) Indeed, over any Artinian ring, every regular matrix is diagonalizable (see [6, Proposition 2.1]). □

Since the regular-simple property coincides with the idempotent-simple property, it passes to corners, and to matrix rings, provided the base ring is also a regular-simple ring (see [3, Proposition 9]).

3.1. The 2×2 case.

A ring is said to be *GCD* if greatest common divisors of pairs of elements exist. The elements of a finite subset of a ring are called *coprime* if there exists a linear combination of these elements which equals 1. Suppose R is a commutative domain and let $S := \mathbb{M}_n(R)$. If $A = AXA$, taking determinants, $\det(A)(\det(AX) - 1) = 0$ so $\det(A) = 0$ or else $\det(AX) = 1$ (and also $\det(XA) = 1$). In the later case, AX and XA are units, and since the matrix ring is Dedekind finite, both A, X are units. Therefore, for the determination of the regular matrices, only the $\det(A) = 0$ case remains to be settled. It is easy to give an elementary proof for the following characterization.

Theorem 3.2. *Let R be a commutative domain. A nonzero 2×2 matrix with zero determinant is regular if and only if its nonzero entries are coprime.*

Proof. Set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0_2$ with $ad = bc$ and $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $AXA = A$ amounts to a (nonhomogeneous) system, namely

$$\begin{aligned} a^2x + acy + abz + bct &= a \\ abx + ady + b^2z + bdt &= b \\ acx + c^2y + adz + cdt &= c \\ bcx + cdy + bdz + d^2t &= d. \end{aligned}$$

Since $ad = bc$, the system reduces to

$$\begin{aligned} a(ax + cy + bz + dt) &= a \\ b(ax + cy + bz + dt) &= b \\ c(ax + cy + bz + dt) &= c \\ d(ax + cy + bz + dt) &= d. \end{aligned}$$

If any of a, b, c, d is zero, the corresponding equality holds for any x, y, z, t. Since we have assumed $A \neq 0_2$, at least one entry (say a) is nonzero. Cancelling a in the first equation, we get $ax + cy + bz + dt = 1$, which holds if and only if a, b, c, d are coprime. □

Recall that the ideals of $S := \mathbb{M}_2(\mathbb{Z})$ are of the form $\mathbb{M}_2(n\mathbb{Z})$ for some $n \geq 1$, and that the ideal generated by a matrix $A = [a_{ij}]$ is $SAS = \mathbb{M}_2(\delta\mathbb{Z})$ if $\delta = \gcd(a_{11}, a_{12}, a_{21}, a_{22})$.

Clearly, $SAS = S$ if and only if $\delta = 1$ if and only if the entries of A are coprime. It follows that

Corollary 3.3. *The matrix ring* $\mathbb{M}_2(\mathbb{Z})$ *is regular-simple.*

Summarizing, the *regular* 2×2 *integral matrices* are the zero matrix and the units and, rank 1 matrices (i.e., with zero determinant) with coprime nonzero entries. These are $\pm E_{11}$, $\pm E_{12}$, $\pm E_{21}$, $\pm E_{22}$, the matrices with two nonzero coprime entries, and the matrices with four nonzero (collectively) coprime entries (only one zero entry is not possible, not having zero determinant). If stronger hypothesis on R are assumed, more explicit results can be proved. Let R be a GCD domain. According to [3, Corollary 19], $\mathbb{M}_2(R)$ is IF. Below, we show that the corresponding result for RF, fails.

Proposition 3.4. *The matrix ring* $\mathbb{M}_2(\mathbb{Z})$ *is an IF but not an RF ring.*

Proof. As $\mathbb{Z}$ is a GCD domain, that $\mathbb{M}_2(\mathbb{Z})$ is IF, follows by the above mentioned corollary. To show it is not RF, it suffices to provide a regular matrix which is not fine. Take $A = \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$. That A is not fine follows from [2, Corollary

5.4]. To see it is regular, we use [8, Proposition 4.1]. Explicitly, as 3, 5 are coprime in $\mathbb{Z}$ and $2\cdot 5+(-3)\cdot 3 = 1$, $M = \begin{bmatrix} 2 & 0 \\ -3 & 0 \end{bmatrix}$ is an inner inverse for $A = AMA$. □

Note (see Corollary 3.3) that this is *an example of regular-simple (equivalently, idempotent-simple) ring that is not RF.*

Corollary 3.5. *Matrix rings over RF rings may not be RF.*

Clearly, *every regular fine ring is RF.* We also have

Corollary 3.6. *Matrix rings over division rings are RF.*

Proof. Indeed, division rings are fine. □

Note that, from the corollary above, we cannot conclude that matrix rings over commutative rings which are not fields (for example, $\mathbb{Z}$) are not RF. In such rings, not all matrices are fine; however, this does not rule out the possibility that all regular matrices are fine. However, we do have the following consequence.

Corollary 3.7. *Let R be a commutative regular ring which is not a field. Then $\mathbb{M}_n(R)$ is not RF.*

Proof. Follows as $\mathbb{M}_n(R)$ contains non-trivial central idempotents. □

In another direction, *the RF property does not pass to subrings.* Indeed, by Corollary 3.6, for any field F, $\mathbb{M}_n(F)$ is RF but by Corollary 2.13(5), as $\mathbb{T}_n(F)$ is not RF. As another example, let k be a field and $R = \mathbb{M}_2(k)$ the 2×2 matrix ring over k. Then R is RF and indecomposable (in fact, simple): its center is $k\cdot I_2$, so the only central idempotents are 0_2 and I_2. The subring $S \subseteq R$, consisting of diagonal matrices $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in k \right\}$, is isomorphic to $k \times k$, which is decomposable (it has the nontrivial central idempotents $(1,0)$ and $(0,1)$), so is not RF. To obtain further examples of RF matrix rings, we recall the following well-known result.

Lemma 3.8. *Let R be an Artinian connected ring. Then:*

(i) *The Jacobson radical $J(R)$ is nilpotent (in particular, nil).*
(ii) *$R/J(R) \cong \mathbb{M}_n(D)$ for some division ring D.*

Proof. Since R is Artinian, the Jacobson radical $J(R)$ is nilpotent and the quotient $R/J(R)$ is semisimple. Hence, by the Wedderburn–Artin theorem,

$$R/J(R) \cong \prod_{i=1}^{t} \mathbb{M}_{n_i}(D_i)$$

for some integers n_i and division rings D_i. If $t \geq 2$, the projections onto the factors give nontrivial central idempotents in $R/J(R)$, which lift to central idempotents in R. This contradicts the assumption that R is connected. Therefore $t = 1$ and $R/J(R) \cong \mathbb{M}_n(D)$ for some division ring D. □

Theorem 3.9. *Let R be an Artinian connected ring. Then $\mathbb{M}_n(R)$ is RF.*

Proof. Follows using Lemma 3.8, Proposition 2.12(2) and Corollary 3.6. □

Using this theorem, we can improve [3, Proposition 8].

Corollary 3.10. *Let R be an Artinian idempotent-simple ring. Then R is RF.*

Proof. In the proof of [3, Proposition 8] (which has the same hypothesis), it is shown that R is isomorphic to a matrix ring $\mathbb{M}_n(T)$ over a local ring T. As local rings are connected, the statement follows from the previous theorem. □

Corollary 3.11. *Let R be an Artinian ring. Then R is RF if and only if R is IF.*

Proof. IF rings are idempotent-simple (see [3, Proposition 7]). □

Remark 3.12. It is also worth mentioning another (unsuccessful) attempt to prove the previous theorem. Let R be an Artinian connected ring. First recall from [6, Propositions 2.1 and 2.2] that any regular matrix over an Artinian ring, is equivalent to a diagonal idempotent matrix. Under the additional hypothesis that the ring is connected, such a matrix is in fact equivalent to a diagonal matrix of the form $diag(1, \dots, 1, 0, \dots, 0)$.
Secondly, observe that the statement "Every diagonal idempotent matrix is fine" does not hold in general. For instance, if e is a nontrivial central idempotent in a ring R, then eI_n is not fine in $\mathbb{M}_n(R)$. However, under the above hypotheses on the base ring, every regular matrix is equivalent to a very special diagonal idempotent matrix. Such matrices are known to be fine: by [3, Theorem 12], the diagonal idempotents $E_{11} + \cdots + E_{ii}$, for some $1 \leq i \leq n$, are fine over any ring.
This observation is still not sufficient to guarantee that a regular matrix is fine, as the following example shows.

Example 3.13. Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$. Since $A = A \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} A$, the matrix A is regular. Now take the invertible matrices $U = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Then $UAV = E_{11}$, so A is equivalent to the diagonal idempotent E_{11}.

However, A is not fine. To see this, consider a general nilpotent matrix $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$. The condition $\det(A - T) = \pm 1$ leads to $-3x + 2y + 2z = \pm 1$. Multiplying by y and eliminating z, gives the quadratic Diophantine equations $2x^2 + 3xy - 2y^2 = \pm y$, which have only the solution $(0, 0)$ (use [1] or [10]). However, replacing in the linear Diophantine equation gives $2z = \pm 1$, with no integer solution. Hence A is not fine.

4. Open IF and RF questions

In [3], the authors established general results on idempotent-simple rings rather than focusing specifically on IF rings. As the list of results on idempotent-simple rings already mentioned indicates, a number of questions on IF rings - six in total - remain unresolved in [3]. They are as follows.

Question 1. Is every simple ring, an IF ring ?

Question 2. Is every idempotent-simple ring, an IF ring?

Question 3. Is every corner ring of an IF ring, an IF ring?

Question 4. Is $\mathbb{M}_n(R)$ an IF ring for every commutative indecomposable ring R?

Question 5. Is $\mathbb{M}_n(R)$ an IF ring for every commutative indecomposable reduced ring R?

Question 6. Is the matrix ring over an IF ring, an IF ring?

It natural to ask the same questions, replacing IF by RF.

Note that according to Proposition 3.4, the last three questions for RF have a negative answer, as $\mathbb{Z}$ is RF, indecomposable and reduced, but $\mathbb{M}_2(\mathbb{Z})$ is not RF. The first three corresponding questions remain open.

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