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A CERTAIN ABELIAN CATEGORY OF WEAKLY COFINITE MODULES

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Abstract. Let R be a commutative Noetherian ring, I an ideal of R and M an arbitrary R-module. We show that if the set $X = \cup_{i\geq 2} \operatorname{Supp}_R(\mathrm{H}^i_I(R))$ is finite, then the category of all weakly cofinite modules with respect to the ideal I of R is an Abelian subcategory of the category of R-modules. In particular, it is true whenever $\mathrm{q}(I, R) \leq 1$.

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1. Introduction

Throughout this paper, let R denote a commutative Noetherian ring (with identity) and I be an ideal of R. Also, let M denote an arbitrary R-module. The ith local cohomology module M with respect to ideal I is defined as

$$\mathrm{H}^i_I(M) \cong \varinjlim_n \operatorname{Ext}^i_R(R/I^n, M).$$

We refer the reader to [10] for more details about the local cohomology.

Hartshorne in [17] defined a module M to be I-*cofinite* if $\operatorname{Supp}_R(M) \subseteq \mathrm{V}(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$. He asked:

Question 1.1. *Whether the category $\mathscr{C}(R, I)_{cof}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-homomorphism of I-cofinite modules, are* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *I-cofinite?*

With respect to this question, Hartshorne with an example showed that this is not true in general. Kawasaki in [19, Theorem 2.1] and the authors of the present paper in [8, Theorem 4.1] with completely different methods proved that if $\operatorname{ara}(I) \leq 1$, then $\mathscr{C}(R, I)_{cof}$ is Abelian. Also it is proved in [13, Theorem 2.2 (ii)], [22, Theorem 2.6] and [21, Theorem 7.4] that the category $\mathscr{C}(R, I)_{cof}$ of I-cofinite modules forms an Abelian subcategory of the category of all R-modules respectively in the cases

$\operatorname{cd}(I,R) \leq 1$, $\dim R/I \leq 1$ and $\dim R \leq 2$. Note that $\operatorname{cd}(I,R)$ (the cohomological dimension of I in R) is defined as:

$$\operatorname{cd}(I,R) := \sup\{i \geq 0 |\ \mathrm{H}^i_I(R) \neq 0\}.$$

Finally, Bahmanpour in [7, Corollary 2.10 (i)] proved that the category $\mathscr{C}(R,I)_{cof}$ of I-cofinite modules is Abelian in the case $\mathrm{q}(I,R) \leq 1$, where

$$\mathrm{q}(I,R) := \sup\{i \geq 0 |\ \mathrm{H}^i_I(R) \text{ is not Artinian}\}.$$

Recall that an R-module M is said to be a weakly Laskerian module, if the set of associated primes of any quotient module M is finite (see [14]). Also, an R-module M is said to be I-weakly cofinite if $\operatorname{Supp} M \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I,M)$ is a weakly Laskerian module for all $i \geq 0$ (see [15]). Recently this notion has been studied by many authors and answered the Hartshorne's question in the class of weakly Laskerian modules (see for example [2,3,18]).

The Question 1.1 and the definition of I-weakly cofinite module as a suitable generalization of I-cofinite module, motivate us to ask the following question.

Question 1.2. *Whether the category $\mathscr{C}(R,I)_{wcof}$ of I-weakly cofinite modules forms an Abelian subcategory of the category of all R-modules? That is, if $f : M \longrightarrow N$ is an R-homomorphism of I-weakly cofinite modules, are* $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *I-weakly cofinite?*

Bahmanpour et al. in [9, Corollary 3.6], showed that if I is a one dimensional ideal of a Noetherian ring, then the category $\mathscr{C}(R,I)_{wcof}$ is Abelian. The first author in [1, Corollary 2.10] proved that if I is an ideal of a Noetherian ring R such that $\operatorname{ara}(I) \leq 1$, then the category $\mathscr{C}(R,I)_{wcof}$ is Abelian too. Hatami and the first author in [18, Corollary 4.7] showed that for a Noetherian ring R with $\dim R \leq 2$, then for any ideal I of R, the category $\mathscr{C}(R,I)_{wcof}$ is Abelian. Pirmohammadi in [23, Theorem 2.7] proved that, if I is an ideal of a Noetherian semi-local ring such that $\dim R/I \leq 2$, then the category $\mathscr{C}(R,I)_{wcof}$ is Abelian. Finally, the first author and Pirmohammadi in [5, Theorem 2.6] showed that the category $\mathscr{C}(R,\mathfrak{a})_{wcof}$ of I-weakly cofinite modules forms an Abelian subcategory of the category of all R-modules in the case $\operatorname{cd}(I,R) \leq 1$. Now, it is natural to ask whether the category $\mathscr{C}(R,\mathfrak{a})_{wcof}$ of I-weakly cofinite modules forms an Abelian subcategory of the category of all R-modules in the case $\mathrm{q}(I,R) \leq 1$. In this paper, we provide an affirmative answer to this question.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and I will be an ideal of R. We denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $\mathrm{V}(I)$. The radical of I, denoted by $\mathrm{Rad}(I)$, is defined to be the set $\{x \in R : x^n \in I$ for some $n \in \mathbb{N}\}$. For any unexplained notation and terminology, we refer the reader to [11] and [20].

2. Main results

We begin this section with some preliminaries which are needed in the proof of main results of this section. The following remark is some elementary properties of the classes of weakly Laskerian and I-weakly cofinite R-modules which we shall use.

Remark 2.1. The following statements hold:

(i) The class of weakly Laskerian modules contains all finitely generated and Artinian modules.

(ii) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. Then M is weakly Laskerian if and only if L and N are both weakly Laskerian (see [14, Lemma 2.3]). Thus any submodule and quotient of a weakly Laskerian module is weakly Laskerian.

(iii) Over a Noetherian ring, a module M is a weakly Laskerian module if and only if there is a finitely generated submodule N of M such that the quotient module M/N has finite support (see [6, Theorem 3.3]).

(iv) Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. If two of R-modules L, M and N are I-weakly cofinite, then the third one is too.

(v) If M is an I-weakly cofinite R-module, then the R-module M/IM is weakly Laskerian (see [15, Theorem 2.10]).

The next lemma is of assistance in the proof of the main theorems in this paper.

Lemma 2.2. *Let R be a Noetherian ring, I be an ideal of R such that* $\mathrm{cd}(I, R) \leq 1$. *Let M be an R-module with* $\mathrm{Supp}_R(M) \subseteq \mathrm{V}(I)$ *such that the R-modules M/IM and* $\mathrm{Tor}_1^R(R/I, M)$ *are finitely generated. Then M is I-cofinite.*

Proof. See [12, Corollary 2.11] and note that $\mathrm{hd}(I, M) \leq \mathrm{cd}(I, R)$ by [16, Theorem 2.5 and Corollary 3.2]. □

Lemma 2.3. *Let R be a Noetherian ring, I an ideal of R and M an R-module. Then the following statements are equivalent:*

(i) *The R-module* $\mathrm{Ext}^i_R(R/I,M)$ *is weakly Laskerian, for all integers* $i \geq 0$.
(ii) *The R-module* $\mathrm{Tor}^R_i(R/I,M)$ *is weakly Laskerian, for all integers* $i \geq 0$.

Proof. See [4, Theorem 2.2]. □

Theorem 2.4. *Let R be a Noetherian ring and I an ideal of R such that the set* $X = \cup_{i\geq 2}\mathrm{Supp}_R(\mathrm{H}^i_I(R))$ *is finite. Let M be an R-module with* $\mathrm{Supp}_R(M) \subseteq \mathrm{V}(I)$. *Then the following statements are equivalent:*

(i) *The R-modules M/IM and* $\mathrm{Tor}^R_1(R/I,M)$ *are weakly Laskerian.*
(ii) *The R-modules* $\mathrm{Tor}^R_i(R/I,M)$ *are weakly Laskerian for all integers* $i \geq 0$.
(iii) *M is an I-weakly cofinite R-module.*

Proof. (iii)⇒(ii) Follows from Lemma 2.3. (ii)⇒(i) is clear. (i)⇒(iii) Let $T_i := \mathrm{Tor}^R_i(R/I,M)$ for all $i \geq 0$. By assumption and Remark 2.1 (iii), there are finitely generated submodules T'_0 and T'_1 of T_0 and T_1 such that the sets $\mathrm{Supp}_R(T_0/T'_0)$ and $\mathrm{Supp}_R(T_1/T'_1)$ are finite. Let

$$A = \mathrm{Supp}_R(T_0/T'_0) \cup \mathrm{Supp}_R(T_1/T'_1) \cup X,$$

then it is easy to see that $A = \mathrm{V}(J)$, where J is the intersection of all the prime ideals in the set A and therefore $\dim R/J \leq 1$. Suppose n,k be integers such that $J = Rx_1 + \cdots + Rx_n + Ry_1 + \cdots + Ry_k$ where $x_1,\ldots,x_n$ are non-nilpotent and $y_1,\ldots,y_k$ nilpotent elements of R. Then $S_j = \{1, x_j, x_j^2, \ldots\}$ is a multiplicatively closed subset of R for all $1 \leq j \leq n$, and $S_j^{-1}(T_0/T'_0) = 0 = S_j^{-1}(T_1/T'_1)$. Thus, for all $1 \leq j \leq n$, we have

$$S_j^{-1}T'_0 = S_j^{-1}T_0 \simeq \mathrm{Tor}_0^{S_j^{-1}R}(S_j^{-1}R/S_j^{-1}I, S_j^{-1}M),$$
$$S_j^{-1}T'_1 = S_j^{-1}T_1 \simeq \mathrm{Tor}_1^{S_j^{-1}R}(S_j^{-1}R/S_j^{-1}I, S_j^{-1}M),$$
$$\mathrm{cd}(S_j^{-1}I, S_j^{-1}R) \leq 1 \text{ and } \mathrm{Supp}_R(S_j^{-1}M) \subseteq \mathrm{V}(S_j^{-1}I).$$

Since $\mathrm{Tor}_0^{S_j^{-1}R}(S_j^{-1}R/S_j^{-1}I, S_j^{-1}M)$ and $\mathrm{Tor}_1^{S_j^{-1}R}(S_j^{-1}R/S_j^{-1}I, S_j^{-1}M)$ are finitely generated $S_j^{-1}R$-modules, therefore $S_j^{-1}M$ is $S_j^{-1}I$-cofinite for all $1 \leq j \leq n$, by Lemma 2.2. Hence there is a finitely generated submodule $T''_{j,i}$ of T_i such that $S_j^{-1}T_i = S_j^{-1}T''_{j,i}$ for all $i \geq 2$.

Put $T'_i = T''_{1,i} + \cdots + T''_{n,i}$ for all $i \geq 2$. Then T'_i is a finitely generated submodule of T_i such that $S_j^{-1}(T_i/T'_i) = 0$ for all $i \geq 2$ and $1 \leq j \leq n$. Hence T_i/T'_i is x_j-torsion for all $1 \leq j \leq n$ and so T_i/T'_i is $(Rx_1 + \cdots + Rx_n)$-torsion. Since $Ry_1 + \cdots + Ry_k \subseteq \mathrm{nil}(R)$, T_i/T'_i is $(Ry_1 + \cdots + Ry_k)$-torsion too. Therefore T_i/T'_i is J-torsion. Thus $\mathrm{Supp}_R(T_i/T'_i) \subseteq \mathrm{V}(J)$ and therefore, the set $\mathrm{Supp}_R(T_i/T'_i)$ is finite. Hence T_i is a weakly Laskerian R-module by Remark 2.1 (iii). □

Now we are ready to state and prove the main result of this paper.

Theorem 2.5. *Let R be a Noetherian ring and I an ideal of R such that the set $X = \cup_{i\geq 2} \operatorname{Supp}_R(\mathrm{H}^i_I(R))$ is finite. Then $\mathscr{C}(R, I)_{wcof}$ is an Abelian category.*

Proof. Let $M, N \in \mathscr{C}(R, I)_{wcof}$ and let $f : M \longrightarrow N$ be an R-homomorphism. It is enough to show that the R-modules $K = \operatorname{Ker} f$ and $C = \operatorname{Coker} f$ are I-weakly cofinite. Put $L = \operatorname{Im} f$, then the exact sequence $M \longrightarrow L \longrightarrow 0$ yields the exact sequence

$$M/IM \longrightarrow L/IL \longrightarrow 0.$$

Since M/IM is weakly Laskerian, L/IL is too by Remark 2.1 (v) and (ii). The exact sequence $0 \longrightarrow L \longrightarrow N \longrightarrow C \longrightarrow 0$ induces the exact sequence

$$\operatorname{Tor}^R_1(R/I, N) \longrightarrow \operatorname{Tor}^R_1(R/I, C) \longrightarrow L/IL \longrightarrow N/IN \longrightarrow C/IC \longrightarrow 0,$$

that implies the R-modules C/IC and $\operatorname{Tor}^R_1(R/I, C)$ are weakly Laskerian by using Remark 2.1 (ii). Note that the R-modules N/IN and $\operatorname{Tor}^R_1(R/I, N)$ are weakly Laskerian by Lemma 2.3. Therefore it follows from Theorem 2.4 that C is I-weakly cofinite. Now, using the exact sequences

$$0 \longrightarrow L \longrightarrow N \longrightarrow C \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow M \longrightarrow L \longrightarrow 0,$$

it follows that K is I-weakly cofinite. Thus $\mathscr{C}(R, I)_{wcof}$ is an Abelian category as required. □

Corollary 2.6. *Let R be a Noetherian ring and I an ideal of R such that* $\mathrm{q}(I, R) \leq 1$*. Then $\mathscr{C}(R, I)_{wcof}$ is an Abelian category.*

Proof. Since $\mathrm{H}^i_I(R) = 0$, for all $i > \operatorname{ara}(I)$ by [10, Corollary 3.3.3], the set $X = \cup_{i\geq 2} \operatorname{Supp}_R(\mathrm{H}^i_I(R))$ is finite. Now the assertion follows by Theorem 2.5. □

Corollary 2.7. (*See* [18, Corollary 4.7]) *Let R be a Noetherian ring and I an ideal of R such that* $\dim R \leq 2$*. Then $\mathscr{C}(R, I)_{wcof}$ is an Abelian category.*

Proof. Since $\mathrm{H}^i_I(R) = 0$, for all $i > 2$ by Grothendieck's vanishing theorem, and $\mathrm{H}^2_I(R)$ is an Artinian module by [21, Proposition 5.1], the set $X = \cup_{i\geq 2} \operatorname{Supp}_R(\mathrm{H}^i_I(R))$ is finite. Now the assertion follows by Theorem 2.5. □

Corollary 2.8. (*See* [5, Theorem 2.6]) *Let R be a Noetherian ring and I an ideal of R such that* $\operatorname{cd}(I, R) \leq 1$*. Then $\mathscr{C}(R, I)_{wcof}$ is an Abelian category.*

Proof. Note that $\mathrm{q}(I, R) \leq \mathrm{cd}(I, R) \leq 1$. Now the assertion follows by Theorem 2.5. □

Corollary 2.9. *Let R be a Noetherian ring and I a proper ideal of R such that $\mathrm{q}(I, R) \leq 1$. Let M be a non-zero I-weakly cofinite R-module. Then, the R-modules $\mathrm{Ext}^i_R(N, M)$ and $\mathrm{Tor}^R_i(N, M)$ are I-weakly cofinite R-modules, for all finitely generated R-modules N and all integers $i \geq 0$.*

Proof. Since N is finitely generated, N has a free resolution of finitely generated free modules. Now the assertion follows using Theorem 2.5 and computing the modules $\mathrm{Ext}^i_R(N, M)$ and $\mathrm{Tor}^R_i(N, M)$, by this free resolution. □

Corollary 2.10. *Let R be a Noetherian ring and I a proper ideal of R such that $\mathrm{q}(I, R) \leq 1$. Let M be a non-zero weakly Laskerian R-module. Then, the R-modules $\mathrm{Ext}^i_R(N, \mathrm{H}^j_I(M))$ and $\mathrm{Tor}^R_i(N, \mathrm{H}^j_I(M))$ are I-weakly cofinite R-modules, for all finitely generated R-modules N and all integers $i, j \geq 0$.*

Proof. The assertion follows by [15, Theorem 3.1] and Corollary 2.9. □

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