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CORE-EP DECOMPOSITION AND RELATED RELATIONS REVISITED

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Abstract. We study relations that are induced by the core-EP decomposition. We revisit the core-EP preorder and extend the concepts of the core-minus, the c-minus, and the c-star partial orders from the set of all $n \times n$ complex matrices to the set of all core-EP invertible elements in a $*$-ring. Several characterizations of these relations are presented and thus some known results are generalized.

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1. Introduction

Throughout the paper, the term ring means an associative ring with identity 1. Let $M_n(\mathbb{F})$ be the ring of all $n \times n$ matrices over the field $\mathbb{F}$. We denote by $\operatorname{rank}(A)$ the rank of $A \in M_n(\mathbb{F})$ and by $i(A)$ its index, i.e., the smallest nonnegative integer k such that $\operatorname{rank}\left(A^{k+1}\right) = \operatorname{rank}\left(A^k\right)$. There are many generalized inverses that may be defined on $M_n(\mathbb{F})$ and one of the best known is *the core inverse.* It was originally discussed by Rao and Mitra [23] on $M_n(\mathbb{C})$, where $\mathbb{C}$ is the field of complex numbers, but was later independently reintroduced (and named) by Baksalary and Trenkler [1]. Rakić et al. [22] generalized this inverse to $*$-rings, i.e., rings equipped with an involution $*$. Let $\mathcal{R}$ be a $*$-ring. We say that $a \in \mathcal{R}$ has the core inverse $a^{\#\!\!\!\bigcirc}$ if $x = a^{\#\!\!\!\bigcirc}$ is the unique solution to the following equations:

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a, \quad \text{and} \quad ax^2 = x. \tag{1}$$

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Xu et al. [25] proved that $a \in \mathcal{R}$ has the core inverse $a^{\textcircled{\#}}$ if $x = a^{\textcircled{\#}}$ is the unique solution to the following equations:

$$xa^2 = a, \qquad ax^2 = x, \qquad \text{and} \qquad (ax)^* = ax. \tag{2}$$

Note (see [1]) that the core inverse of $A \in M_n(\mathbb{C})$ exists if and only if $i(A) \leq 1$. Prasad et al. [14] extended the notion of the core inverse to the full matrix algebra $M_n(\mathbb{F})$ by introducing a new kind of matrix generalized inverse called *the core-EP inverse* and showed that whenever $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, where $\mathbb{R}$ is the field of real numbers, every matrix $A \in M_n(\mathbb{F})$ has the unique core-EP inverse. As a generalization for both the core inverse in a $*$-ring and the core-EP inverse for complex or real matrices, Gao and Chen [7] introduced a new generalized inverse in a $*$-ring $\mathcal{R}$. Let $a \in \mathcal{R}$. If there exists $x = a^{\textcircled{d}} \in \mathcal{R}$ such that

$$xa^{m+1} = a^m \text{ for some positive integer } m, \quad ax^2 = x, \quad \text{and} \quad (ax)^* = ax, \tag{3}$$

then we say that a is *pseudo-core* (or *core-EP*) *invertible* and call $a^{\textcircled{d}}$ *the pseudo-core* (or *core-EP*) *inverse* of a. It turns out (see [7, Theorem 2.2]) that $a^{\textcircled{d}}$ is unique if it exists and in this case the smallest positive integer m in (3) is called *the pseudo-core index* of a and is denoted by $I(a)$. Note that for $I(a) = 1$, the core-EP inverse becomes the core inverse (compare (2) with (3)). Also, for $A \in M_n(\mathbb{C})$, its pseudo-core index $I(A)$ either equals $i(A)$ if $i(A) > 0$, or is 1 if $i(A) = 0$ (see [7, Theorem 2.3]). It was proved in [7] that $a \in \mathcal{R}$ has the core-EP inverse $a^{\textcircled{d}} \in \mathcal{R}$ if and only if $x = a^{\textcircled{d}}$ satisfies the following equations

$$xax = x \qquad \text{and} \qquad x\mathcal{R} = x^*\mathcal{R} = a^m\mathcal{R} \tag{4}$$

for some positive integer m. We denote by $\mathcal{R}^{\textcircled{\#}}$, $\mathcal{R}^{\textcircled{d}}$ the sets of all core and core-EP invertible elements in $\mathcal{R}$, respectively.

For a semigroup $\mathcal{S}$, we say that $a \in \mathcal{S}$ is *regular* if there exists $x \in \mathcal{S}$ with $axa = a$. If such x exists, it is called *an inner generalized inverse* or $\{1\}$*-inverse* of a, and we write $x = a^-$, i.e., $aa^-a = a$. The set of all regular elements in $\mathcal{S}$ is denoted by $\mathcal{S}^{(1)}$. We say that a has *the group inverse* $a^\sharp$ in $\mathcal{S}$ if $x = a^\sharp$ satisfies the following equations: $axa = a$, $xax = x$, and $ax = xa$. The group inverse, if it exists, is unique (see [10]) and we denote the set of all elements in $\mathcal{S}$ that have the group inverse by $\mathcal{S}^\sharp$.

The aforementioned and some other well-known generalized inverses (such as *the Drazin inverse* and *the Moore-Penrose inverse*) have numerous applications in systems of linear differential and difference equations, cryptography, iterative methods in numerical analysis, Markov chains, combinatorics, and graph theory (see [11,27,28] and the references therein).

Many partial orders can be defined on rings or even on semigroups using generalized inverses. Let us recall some of them that will come to the forefront in the continuation. One of the best known partial orders is *the minus order* $\leq^-$ [9]. For a semigroup $\mathcal{S}$ with $a \in \mathcal{S}^{(1)}$ and $b \in \mathcal{S}$, we write

$$a \leq^- b \quad \text{if} \quad a^- a = a^- b \quad \text{and} \quad aa^- = ba^- \tag{5}$$

for some inner generalized inverse a^- of a. This relation is a partial order on $\mathcal{S}^{(1)}$. Let us mention that the minus order has applications in the theory of linear statistical models (see, e.g., [18]). The next relations are defined on a $*$-ring $\mathcal{R}$ as follows.

(i) *The core order* $\leq^{\textcircled{\#}}$ [1,22]: For $a \in \mathcal{R}^{\textcircled{\#}}$ and $b \in \mathcal{R}$, we write

$$a \leq^{\textcircled{\#}} b \quad \text{if} \quad a^{\textcircled{\#}} a = a^{\textcircled{\#}} b \quad \text{and} \quad aa^{\textcircled{\#}} = ba^{\textcircled{\#}}.$$

(ii) *The core-EP (pre)order* $\leq^{\textcircled{\dagger}}$ [8,26]: For $a \in \mathcal{R}^{\textcircled{\dagger}}$ and $b \in \mathcal{R}$, we write

$$a \leq^{\textcircled{\dagger}} b \quad \text{if} \quad a^{\textcircled{\dagger}} a = a^{\textcircled{\dagger}} b \quad \text{and} \quad aa^{\textcircled{\dagger}} = ba^{\textcircled{\dagger}}.$$

The relation $\leq^{\textcircled{\#}}$ is a partial order on $\mathcal{R}^{\textcircled{\#}}$ (see [22]) while the relation $\leq^{\textcircled{\dagger}}$ is merely a preorder, i.e., it is reflexive and transitive (see [8, Theorem 4.2]), however it is not antisymmetric (see [26, Example 4.1]).

For the case of the square complex matrices, Wang introduced in [26] a matrix decomposition and named it *the core-EP decomposition*. Gao et al. [8] generalized this notion to $*$-rings. Let $\mathcal{R}$ be a $*$-ring. Suppose $a \in \mathcal{R}^{\textcircled{\dagger}}$ and let $I(a) = m$. By [8, Theorem 3.1], we may write $a = a_1 + a_2$, where:

$$a_1^{\sharp} \text{ exists}, \qquad a_2^m = 0, \qquad \text{and} \qquad a_1^* a_2 = a_2 a_1 = 0. \tag{6}$$

Gao et al. named this decomposition *the pseudo-core decomposition* in $\mathcal{R}$ however, since it is a generalization of the core-EP decomposition from $M_n(\mathbb{C})$ to $*$-rings, we call it *the core-EP decomposition* in $\mathcal{R}$. It turns out that this decomposition is unique and that

$$a_1 = aa^{\textcircled{\dagger}} a \qquad \text{and} \qquad a_2 = a - aa^{\textcircled{\dagger}} a, \tag{7}$$

where $a^{\textcircled{\dagger}}$ is the core-EP inverse of a. We call a_2 *the nilpotent part* of the core-EP decomposition of a.

Let us now present two relations on $M_n(\mathbb{C})$ that are induced by the core-EP decomposition. Recall first that every complex matrix has the core-EP inverse, i.e., $M_n(\mathbb{C})^{\textcircled{\dagger}} = M_n(\mathbb{C})$. Note also that every $A \in M_n(\mathbb{C})$ has an inner generalized inverse A^-. Let $A, B \in M_n(\mathbb{C})$ and suppose $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B, respectively, where A_2 and B_2 are the nilpotent parts.

(i) *The core-minus order* $\leq^{cm}$ [26]: We write

$$A \leq^{cm} B \quad \text{if} \quad A_1 \leq^{\#} B_1 \quad \text{and} \quad A_2 \leq^{-} B_2.$$

(ii) *The c-minus order* $\leq^{©}$ [13]: We write

$$A \leq^{©} B \quad \text{if} \quad A_1 \leq^{\#} B_1 \quad \text{and} \quad A \leq^{-} B.$$

Since the core $\leq^{\#}$ and the minus $\leq^{-}$ relations are partial orders and since the core-EP decomposition is unique, it is easy to see that the core-minus and c-minus relations are partial orders on $M_n(\mathbb{C})$ (see [26, Theorem 5.1] and [13, Theorem 4.2]).

The main goal of this paper is to extend notions of the core-minus and c-minus partial orders to $*$-rings, study their properties, and thus generalize some known results. In Section 2, we present some preliminary notions and results. The core-minus and c-minus partial orders are studied in the context of $*$-rings in Sections 3 and 4, respectively. In Section 5, we investigate another relation which is also induced by the core-EP decomposition and by the well known star partial order. We generalize this relation from the context of C^*-algebras to $*$-rings and present its characterizations using 2×2 and 3×3 matrix representations of elements in rings.

2. Preliminaries

Let us now present some tools which will be useful throughout the paper. If for $p \in \mathcal{R}$, $p^2 = p$, then p is said to be an idempotent. A projection $p \in \mathcal{R}$ is a self-adjoint idempotent, i.e., $p = p^2 = p^*$. The equality $1 = e_1 + e_2 + \cdots + e_n$, where $e_1, e_2, \ldots, e_n$ are idempotent elements in $\mathcal{R}$ and $e_i e_j = 0$ for $i \neq j$, is called a decomposition of the identity of $\mathcal{R}$. Let $1 = e_1 + \cdots + e_n$ and $1 = f_1 + \cdots + f_n$ be two decompositions of the identity of $\mathcal{R}$. We have

$$x = 1 \cdot x \cdot 1 = (e_1 + e_2 + \cdots + e_n)x(f_1 + f_2 + \cdots + f_n) = \sum_{i,j=1}^{n} e_i x f_j.$$

Then any $x \in \mathcal{R}$ can be uniquely represented in the following matrix form:

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f}, \tag{8}$$

where $x_{ij} = e_i x f_j \in e_i \mathcal{R} f_j$. If $x = (x_{ij})_{e \times f}$ and $y = (y_{ij})_{e \times f}$, then $x + y = (x_{ij} + y_{ij})_{e \times f}$. Moreover, if $1 = g_1 + \cdots + g_n$ is a third decomposition of the identity of $\mathcal{R}$ and $z = (z_{ij})_{f \times g}$, then, by the orthogonality of the idempotents involved, $xz = \left(\sum_{k=1}^{n} x_{ik} z_{kj}\right)_{e \times g}$. Thus, if we have decompositions of the identity of $\mathcal{R}$, then the usual algebraic operations in $\mathcal{R}$ can be interpreted as simple operations between appropriate $n \times n$ matrices over $\mathcal{R}$. When $n = 2$ and $p, q \in \mathcal{R}$ are idempotent elements, we may write

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{p \times q}.$$

Here $x_{1,1} = pxq$, $x_{1,2} = px(1-q)$, $x_{2,1} = (1-p)xq$, $x_{2,2} = (1-p)x(1-q)$.

By (8), we may write

$$x^* = \begin{bmatrix} x_{11}^* & \cdots & x_{n1}^* \\ \vdots & \ddots & \vdots \\ x_{1n}^* & \cdots & x_{nn}^* \end{bmatrix}_{f^* \times e^*},$$

where this matrix representation of x^* is given relative to the decompositions of the identity $1 = f_1^* + \cdots + f_n^*$ and $1 = e_1^* + \cdots + e_n^*$.

Let $a \in \mathcal{R}^{\textcircled{d}}$. Note that, by (4), $a^{\textcircled{d}} a a^{\textcircled{d}} = a^{\textcircled{d}}$. So, $p = aa^{\textcircled{d}}$ (and $q = a^{\textcircled{d}} a$) is an idempotent. Let $a = a_1 + a_2$ be the core-EP decomposition of a in which a_2 is the nilpotent part. It was proved in [4] that for $p = aa^{\textcircled{d}}$,

$$a_1 = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p \times p} \quad \text{and} \quad a_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}_{p \times p},$$

where t is invertible in the ring $p\mathcal{R}p$, i.e., there exists $t^{-1} \in p\mathcal{R}p$ such that $tt^{-1} = t^{-1}t = p$. So,

$$a = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p \times p}. \tag{9}$$

Note that $t = a^2 a^{\textcircled{d}}$ and $t^{-1} = a^{\textcircled{d}}$ (see [5]).

Recall that when $a \in \mathcal{R}^{\textcircled{\#}}$, $a^{\textcircled{d}} = a^{\textcircled{\#}}$ and that $aa^{\textcircled{\#}}a = a$ (see (1)). So, $a = a_1 + 0$ is the core-EP decomposition of a (i.e., $a_2 = 0$) and hence (9) becomes

$$a = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p \times p}.$$

The following two results were proved in [8].

Lemma 2.1. *Let $a \in \mathcal{R}^{\textcircled{d}}$ and let $a = a_1 + a_2$ be the core-EP decomposition of a, where a_2 is the nilpotent part. Then $a_1^{\textcircled{\#}} = a^{\textcircled{d}}$.*

Lemma 2.2. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. Then $a \leq^{\textcircled{d}} b$ if and only if $a_1 \leq^{\textcircled{\#}} b_1$.*

The next lemma was proved in [17].

Lemma 2.3. *Let $\mathcal{R}$ be a ring and let*

$$b = \begin{bmatrix} t & z_1 & z_2 \\ 0 & t_1 & z_3 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$$

be the matrix representation of $b \in \mathcal{R}$ relative to the decomposition of the identity $1 = e_1 + e_2 + e_3$, where t and t_1 are invertible in the rings $e_1\mathcal{R}e_1$ and $e_2\mathcal{R}e_2$, respectively. Then $b \in \mathcal{R}^\sharp$.

Lemma 2.4. *Let $a \in \mathcal{R}^{\textcircled{\#}}$ and suppose*

$$a = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p}$$

for some projection $p \in \mathcal{R}$, where t is invertible in the ring $p\mathcal{R}p$. If $a \leq^{\textcircled{\#}} b$, then

$$b = \begin{bmatrix} t & s \\ 0 & z \end{bmatrix}_{p\times p},$$

where $z \in (1-p)\mathcal{R}(1-p)$.

Proof. By (2), it follows that

$$a^{\textcircled{\#}} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

Let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times p}.$$

From $a^{\textcircled{\#}}a = a^{\textcircled{\#}}b$, we get

$$\begin{bmatrix} p & t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1}b_1 & t^{-1}b_2 \\ 0 & 0 \end{bmatrix}_{p\times p},$$

and thus $p = t^{-1}b_1$ and $t^{-1}s = t^{-1}b_2$. By multiplying these equations from the left by t, we obtain $tp = pb_1$ and $ps = pb_2$. Note that $b_1, b_2, s \in p\mathcal{R}$ and $t \in \mathcal{R}p$. It follows that $b_1 = t$ and $b_2 = s$. From $aa^{\textcircled{\#}} = ba^{\textcircled{\#}}$, we get

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} b_1t^{-1} & 0 \\ b_3t^{-1} & 0 \end{bmatrix}_{p\times p}.$$

It follows that $b_3t^{-1} = 0$ and so $b_3p = 0$. Since $b_3 \in \mathcal{R}p$, $b_3 = 0$. Denote $b_4 = z$ to get

$$b = \begin{bmatrix} t & s \\ 0 & z \end{bmatrix}_{p\times p},$$

where $z \in (1-p)\mathcal{R}(1-p)$. $\square$

Lemma 2.5. *Let* $a \in \mathcal{R}^{\textcircled{\dagger}}$ *and suppose*

$$a = \begin{bmatrix} t & s \\ 0 & z \end{bmatrix}_{p \times p},$$

where t *is invertible in the ring* $p\mathcal{R}p$ *for some projection* $p \in \mathcal{R}$*. Then*

$$z \in ((1-p)\mathcal{R}(1-p))^{\textcircled{\dagger}}.$$

Proof. Let

$$a^{\textcircled{\dagger}} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{p \times p}.$$

We have

$$aa^{\textcircled{\dagger}} = \begin{bmatrix} t & s \\ 0 & z \end{bmatrix}_{p \times p} \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{p \times p} = \begin{bmatrix} * & * \\ * & zc_4 \end{bmatrix}_{p \times p}$$

and since $\left(aa^{\textcircled{\dagger}}\right)^* = aa^{\textcircled{\dagger}}$, we obtain

$$(zc_4)^* = zc_4. \tag{10}$$

Also, by

$$a^{\textcircled{\dagger}}a^2 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}_{p \times p} \begin{bmatrix} t^2 & ts+sz \\ 0 & z^2 \end{bmatrix}_{p \times p} = \begin{bmatrix} c_1t^2 & c_1(ts+sz)+c_2z^2 \\ c_3t^2 & c_3(ts+sz)+c_4z^2 \end{bmatrix}_{p \times p}$$

and since $a^{\textcircled{\dagger}}a^2 = a$, we get $c_3t^2 = 0$ and $c_3(ts+sz)+c_4z^2 = z$. So, $c_3 = 0$ and hence

$$c_4z^2 = z. \tag{11}$$

Finally, since

$$a\left(a^{\textcircled{\dagger}}\right)^2 = \begin{bmatrix} t & s \\ 0 & z \end{bmatrix}_{p \times p} \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix}_{p \times p} \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix}_{p \times p} = \begin{bmatrix} * & * \\ 0 & zc_4^2 \end{bmatrix}_{p \times p}$$

equals

$$a^{\textcircled{\dagger}} = \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix}_{p \times p},$$

we get

$$zc_4^2 = c_4. \tag{12}$$

By (10)–(12), we conclude that $c_4 = z^{\textcircled{\dagger}}$, i.e., $z \in ((1-p)\mathcal{R}(1-p))^{\textcircled{\dagger}}$. □

3. The core-minus partial order in rings

There are many equivalent definitions and generalizations of Hartwig's minus partial order $\leq^-$ (see [2,3,12,15,16,19,21,24]). For example, Mitsch [19, Theorem 3] generalized the minus partial order to arbitrary semigroups. The definition is as follows: Suppose a, b are two elements of an arbitrary semigroup $\mathcal{S}$. Then we write

$$a \leq^M b \quad \text{if} \quad a = xb = by \quad \text{and} \quad xa = a \tag{13}$$

for some $x, y \in \mathcal{S}^1$, where $\mathcal{S}^1$ denotes $\mathcal{S}$, if $\mathcal{S}$ has the identity, and $\mathcal{S}$ with the identity adjoined otherwise. Mitsch proved that definitions (5) and (13) are equivalent when every element in $\mathcal{S}$ is regular and that (13) is indeed a partial order for any semigroup $\mathcal{S}$. From now on, we will use (13) as the definition of the minus partial order in a $*$-ring $\mathcal{R}$ and write $\leq^-$ instead of $\leq^M$. Let us now generalize the notion of the core-minus order from $M_n(\mathbb{C})$ to $*$-rings.

Definition 3.1. Let $a, b \in \mathcal{R}^{\textcircled{d}}$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. We say that a is below b under the core-minus relation and write

$$a \leq^{cm} b \quad \text{if} \quad a_1 \leq^{\textcircled{\#}} b_1 \quad \text{and} \quad a_2 \leq^- b_2.$$

Since the core $\leq^{\textcircled{\#}}$ and the minus $\leq^-$ relations are partial orders on $\mathcal{R}^{\textcircled{\#}}$ and $\mathcal{R}$, respectively, and since the core-EP decomposition is unique, we obtain the following result.

Theorem 3.2. *The core-minus relation is a partial order on $\mathcal{R}^{\textcircled{d}}$.*

Remark 3.3. Let $a \in \mathcal{R}^{\textcircled{d}}$ with $I(a) = m$. Recall that in case when $m = 1$, then by (6), the nilpotent part a_2 of the core-EP decomposition $a_1 + a_2$ of a equals 0, and so $a = a_1$. Thus, on the set of all pseudo-core index 1 elements in $\mathcal{R}^{\textcircled{d}}$, the core-minus partial order equals the core partial order.

We will present in the continuation a matrix characterization of the core-minus partial order but first let us turn for a while our attention to the core-EP preorder $\leq^{\textcircled{d}}$. This relation was studied in [4] in the context of $*$-rings. With [4, Theorem 2], a matrix characterization of this relation was presented. For $a, b \in \mathcal{R}^{\textcircled{d}}$ with $a \leq^{\textcircled{d}} b$, the authors proved that a and b have a certain matrix form under additional rather technical assumption that $\left(1 - aa^{\textcircled{d}}\right) b \left(1 - aa^{\textcircled{d}}\right) \in \mathcal{R}^{\textcircled{d}}$. With the following result, we show that [4, Theorem 2] holds also without this assumption (see also [20, Corollary 3.5]).

Proposition 3.4. *Let $a, b \in \mathcal{R}^{ⓓ}$ with $p = aa^{ⓓ}$. Then $a \leq^{ⓓ} b$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, with $e_1 = p$ and $e_2^* = e_2$, such that*

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}, \tag{14}$$

where t is invertible in the ring $e_1\mathcal{R}e_1$, t_1 is invertible in the ring $e_2\mathcal{R}e_2$, $n_5^{I(b)} = 0$, and $(n_1 + n_2 + n_3 + n_4)^{I(a)} = 0$.

Proof. For $a, b \in \mathcal{R}^{ⓓ}$, suppose $a \leq^{ⓓ} b$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. By Lemma 2.2, we have $a_1 \leq^{\text{#}} b_1$. Let us write a in the matrix form (9) of the core-EP decomposition. So,

$$a_1 = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p}$$

with $p = aa^{ⓓ}$ and hence

$$a^{\text{#}} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

Note that

$$aa^{ⓓ} = p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = a_1 a_1^{\text{#}}.$$

Since $a_1 \leq^{\text{#}} b_1$, it follows by Lemma 2.4 that

$$b_1 = \begin{bmatrix} t & s \\ 0 & z \end{bmatrix}_{p\times p}$$

for some $z \in (1-p)\mathcal{R}(1-p)$. From $b \in \mathcal{R}^{ⓓ}$, we have by Lemma 2.1 that $b_1^{\text{#}} = b^{ⓓ}$. So, $b_1 \in \mathcal{R}^{\text{#}}$ and therefore by Lemma 2.5 that $z \in ((1-p)\mathcal{R}(1-p))^{\text{#}}$. Using the core-EP decomposition, we can then represent the element z with the matrix form

$$z = \begin{bmatrix} t_1 & z_1 \\ 0 & 0 \end{bmatrix}_{q\times q},$$

where $q = zz^{\text{#}}$ and t_1 is invertible in the ring $q\mathcal{R}q$. Since $z \in (1-p)\mathcal{R}$, we have $pq = pzz^{\text{#}} = 0$ and since p and q are self-adjoint, we also obtain $qp = 0$. It follows that

$$\begin{aligned} b_1 &= t + s + t_1 + z_1 \\ &= pb_1p + pb_1(1-p) + q(1-p)b_1(1-p)q + q(1-p)b_1(1-p)(1-q) \\ &= pb_1p + pb_1(1-p) + qb_1q + qb_1(1-p-q). \end{aligned}$$

Denote $s_1 = pb_1q$ and $s_2 = pb_1(1-p-q)$, and note that $t = pb_1p$, $s = pb_1(1-p) = pb_1q + pb_1(1-p-q) = s_1 + s_2$, $t_1 = qzq = qb_1q$, and $z_1 = qz(1-q) = qb_1(1-p-q)$. Let $e_1 = p$, $e_2 = q$, and $e_3 = 1-p-q$. Since then $e_1 + e_2 + e_3 = 1$ and $e_ie_j = 0$ when $i \neq j$, $i, j \in \{1, 2, 3\}$, we may represent b_1 as

$$b_1 = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & 0 \end{bmatrix}_{e \times e}.$$

Since $e_2 = q$, we have that $e_2^* = e_2$ and t_1 is invertible in the ring $e_2\mathcal{R}e_2$. Let us show that a is as in (14). We have

$$a = a_1 + a_2 = t + s + a_2 = pap + pa(1-p) + (1-p)a(1-p).$$

So, on the one hand, $s = pa(1-p) = paq + pa(1-p-q)$, but on the other hand $s = s_1 + s_2 = pb_1q + pb_1(1-p-q)$. Hence, $paq + pa(1-p-q) = pb_1q + pb_1(1-p-q)$ and therefore

$$s_1 = pb_1q = paq$$

and

$$s_2 = pb_1(1-p-q) = pa(1-p-q).$$

Denote $n_1 = qaq$, $n_2 = qa(1-p-q)$, $n_3 = (1-p-q)aq$, and $n_4 = (1-p-q)a(1-p-q)$. We have

$$\begin{aligned} a_2 &= (1-p)a(1-p) = qaq + qa(1-p-q) + (1-p-q)aq + (1-p-q)a(1-p-q) \\ &= n_1 + n_2 + n_3 + n_4 \end{aligned}$$

and therefore $a = t + s_1 + s_2 + n_1 + n_2 + n_3 + n_4$, i.e.,

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e \times e}.$$

Finally, let us show that b can be represented as in (14). Let

$$b_2 = \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & n_5 \end{bmatrix}_{e \times e}.$$

From

$$0 = b_2 b_1 = \begin{bmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e},$$

we get $0 = d_1 t = d_4 t = d_7 t$ and $0 = d_1 s_1 + d_2 t_1 = d_4 s_1 + d_5 t_1 = d_7 s_1 + d_8 t_1$. It follows that $0 = d_1 = d_4 = d_7$ and thus also $0 = d_2 = d_5 = d_8$. Then, by

$$0 = b_1^* b_2 = \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & t_1^* & 0 \\ s_2^* & z_1^* & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & d_3 \\ 0 & 0 & d_6 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e},$$

we get $d_3 = 0$ which implies $d_6 = 0$. So, $b_2 = n_5$ and thus

$$b = b_1 + b_2 = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}.$$

Note that $(n_1 + n_2 + n_3 + n_4)^{I(a)} = 0$ and $n_5^{I(b)} = 0$ since a_2 and b_2 are the nilpotent parts in the core-EP decompositions of a and b, respectively.

Conversely, if we assume that $a, b \in \mathcal{R}^{\textcircled{d}}$ can be represented as in (14), then we may prove that $a \leq^{\textcircled{d}} b$ in the same way as in the proof of [4, Theorem 2]. □

For $a \in \mathcal{R}$, we denote by a° the right annihilator of a, i.e., the set $a^\circ = \{x \in \mathcal{R} : ax = 0\}$. Similarly, we denote the left annihilator ${}^\circ a$ of a, i.e., the set ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$. With the next result, we present a characterization of the core-minus partial order in $*$-rings and thus generalize [26, Theorem 5.3] (see also [20, Corollary 4.3]).

Theorem 3.5. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$ with $p = aa^{\textcircled{d}}$. Then $a \leq^{cm} b$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, with $e_1 = p$ and $e_2^* = e_2$, such that*

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}, \tag{15}$$

where t is invertible in the ring $e_1\mathcal{R}e_1$, t_1 is invertible in the ring $e_2\mathcal{R}e_2$, $n_4^{I(a)} = 0 = n_5^{I(b)}$, and $n_4 \leq^- n_5$.

Proof. Let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. Suppose first $a \leq^{cm} b$. By

Definition 3.1, $a_1 \leq^{\textcircled{\dagger}} b_1$ and $a_2 \leq^- b_2$. Proposition 3.4 yields that

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e},$$

where t is invertible in the ring $e_1\mathcal{R}e_1$ and t_1 is invertible in the ring $e_2\mathcal{R}e_2$. Also, by the proof of Proposition 3.4, we observe that $a_2 = n_1+n_2+n_3+n_4$ and $b_2 = n_5$. To prove that (15) holds, we will show that $n_1 = n_2 = n_3 = 0$. Since $a_2 \leq^- b_2$, there exist by (13) elements $x, y \in \mathcal{R}$ such that $a_2 = xb_2 = b_2y$. This implies that

$$^\circ n_5 \subseteq {}^\circ(n_1 + n_2 + n_3 + n_4) \quad \text{and} \quad n_5^\circ \subseteq (n_1 + n_2 + n_3 + n_4)^\circ. \tag{16}$$

We have $n_1 + n_2 + n_3 + n_4 = e_2ae_2 + e_2ae_3 + e_3ae_2 + e_3ae_3$ and $n_5 = e_3be_3$. Since $e_2e_3 = 0 = e_3e_2$, we obtain $e_2n_5 = 0 = n_5e_2$, i.e., $e_2 \in {}^\circ n_5 \cap n_5^\circ$. So, by (16), we have $e_2 \in {}^\circ(n_1 + n_2 + n_3 + n_4) \cap (n_1 + n_2 + n_3 + n_4)^\circ$ and therefore

$$0 = e_2(n_1 + n_2 + n_3 + n_4) = e_2ae_2 + e_2ae_3 = n_1 + n_2$$

and

$$0 = (n_1 + n_2 + n_3 + n_4)e_2 = e_2ae_2 + e_3ae_2 = n_1 + n_3.$$

Thus, $n_2 = n_3 = -n_1$. Note that $n_2 \in e_2\mathcal{R}e_3$ and $n_3 \in e_3\mathcal{R}e_2$. Since $e_2\mathcal{R}e_3 \cap e_3\mathcal{R}e_2 = \{0\}$, we may conclude that $0 = n_1 = n_2 = n_3$. So, a and b are of the form (15), where n_4 and n_5 are nilpotent ($n_4^{I(a)} = 0 = n_5^{I(b)}$) with $n_4 \leq^- n_5$.

Conversely, let a and b be of the form (15). By Proposition 3.4, we obtain $a \leq^{\textcircled{d}} b$ and so by Lemma 2.2, $a_1 \leq^{\textcircled{\dagger}} b_1$. Recall that $a^{\textcircled{d}} = t^{-1}$ and hence

$$a_1 = aa^{\textcircled{d}}a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{bmatrix}_{e\times e} \begin{bmatrix} t^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}.$$

Thus, $a_2 = a - aa^{\textcircled{d}}a = n_4$. Let

$$u = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \quad \text{and} \quad v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}.$$

By Lemma 2.3, $u \in \mathcal{R}^{\sharp}$. Also $u^*v = vu = 0$ and, by assumption, $v^{I(b)} = n_5^{I(b)} = 0$. It follows by (6) that $b_1 = u$ and $b_2 = v$. Since, by Lemma 2.1, $a_1^{\textcircled{\dagger}} = a^{\textcircled{d}}$, we have

$$a_1^{\textcircled{\dagger}} a_1 = \begin{bmatrix} t^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = \begin{bmatrix} p & t^{-1}s_1 & t^{-1}s_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e}$$

$$= \begin{bmatrix} t^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = a_1^{\textcircled{\dagger}} b_1$$

and similarly,

$$a_1 a_1^{\textcircled{\dagger}} = \begin{bmatrix} p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = b_1 a_1^{\textcircled{\dagger}}.$$

Thus, $a_1 \leq^{\textcircled{\dagger}} b_1$ and since $n_4 = a_2$ and $n_5 = b_2$, we also have, by assumption, that $a_2 \leq^{-} b_2$. Therefore, $a \leq^{cm} b$. □

Let $a, b \in \mathcal{R}^{\textcircled{d}}$. Clearly, if $a \leq^{cm} b$, then $a \leq^{\textcircled{d}} b$. When does the core-EP preorder imply the core-minus partial order? If $a \leq^{\textcircled{d}} b$, then by Proposition 3.4, a and b have the matrix form (14). If $n_1 = n_2 = n_3 = 0$ in (14), then by Theorem 3.5, $a \leq^{cm} b$. If we state this observation in a slightly different way, we obtain (see (7) and Lemma 2.2) another characterization of the core-minus partial order.

Theorem 3.6. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$. Then $a \leq^{cm} b$ if and only if $a \leq^{\textcircled{d}} b$ and $a - aa^{\textcircled{d}}a \leq^{-} b - bb^{\textcircled{d}}b$.*

We end this section by showing that the core-minus partial order implies the minus partial order on $\mathcal{R}^{\textcircled{d}}$.

Corollary 3.7. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$ and $a \leq^{cm} b$. Then $a \leq^{-} b$.*

Proof. Let $a \leq^{cm} b$ for $a, b \in \mathcal{R}$. By Theorem 3.5, a and b have the matrix forms (15). Since $n_4, n_5 \in e_3\mathcal{R}e_3$ with $n_4 \leq^{-} n_5$, there exist, by (13), $x_1, y_1 \in e_3\mathcal{R}e_3$ with

$$n_4 = x_1 n_5 = n_5 y_1 \quad \text{and} \quad x_1 n_4 = n_4.$$

Let

$$x = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_1 \end{bmatrix}_{e\times e} \quad \text{and} \quad y = \begin{bmatrix} e_1 & t^{-1}s_1 & t^{-1}\left(s_2 + s_1 t_1^{-1} z_1 y_1 - s_2 y_1\right) \\ 0 & 0 & -t_1^{-1} z_1 y_1 \\ 0 & 0 & y_1 \end{bmatrix}_{e\times e}.$$

Then

$$xa = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_1 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & x_1 n_4 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{bmatrix}_{e\times e} = a$$

and

$$xb = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x_1 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & x_1 n_5 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{bmatrix}_{e\times e} = a.$$

Also,

$$\begin{aligned} by &= \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} e_1 & t^{-1}s_1 & t^{-1}\left(s_2 + s_1 t_1^{-1} z_1 y_1 - s_2 y_1\right) \\ 0 & 0 & -t_1^{-1} z_1 y_1 \\ 0 & 0 & y_1 \end{bmatrix}_{e\times e} \\ &= \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_5 y_1 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & 0 & 0 \\ 0 & 0 & n_4 \end{bmatrix}_{e\times e} = a. \end{aligned}$$

It follows, by (13), that $a \leq^- b$. □

4. The c-minus partial order in rings

We will now generalize the notion of the c-minus order to from $M_n(\mathbb{C})$ to $*$-rings.

Definition 4.1. Let $a, b \in \mathcal{R}^{ⓓ}$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. We say that a is below b under the c-minus relation and write

$$a \leq^{ⓒ} b \quad \text{if} \quad a_1 \leq^{\#} b_1 \quad \text{and} \quad a \leq^- b.$$

As in the case of the core-minus relation, the properties of the core $\leq^{\#}$ and the minus $\leq^-$ relations and the uniqueness of the core-EP decomposition yield the following result.

Theorem 4.2. *The c-minus relation is a partial order on $\mathcal{R}^{ⓓ}$.*

Let us present a characterization of the c-minus partial in $*$-rings.

Theorem 4.3. *Let $a, b \in \mathcal{R}^{ⓓ}$ with $p = aa^{ⓓ}$. Then $a \leq^{ⓒ} b$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, with $e_1 = p$ and $e_2^* = e_2$, such that*

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}, \tag{17}$$

where t is invertible in $e_1\mathcal{R}e_1$, t_1 is invertible in $e_2\mathcal{R}e_2$, $(n_1+n_2+n_3+n_4)^{I(a)}=0$, $n_5^{I(b)}=0$, and $n_1+n_2+n_3+n_4 \leq^{-} t_1+z_1+n_5$.

Proof. Let $a=a_1+a_2$ and $b=b_1+b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. Suppose first $a \leq^{\textcircled{c}} b$. Since then $a_1 \leq^{\textcircled{\#}} b_1$, it follows, by Lemma 2.2 and Proposition 3.4, that there exists a decomposition of the identity $1=e_1+e_2+e_3$, with $e_1=p$ and $e_2^*=e_2$, such that

$$a=\begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}, \qquad b=\begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e},$$

where t and t_1 are invertible in the rings $e_1\mathcal{R}e_1$ and $e_2\mathcal{R}e_2$, respectively, $(n_1+n_2+n_3+n_4)^{I(a)}=0$, and $n_5^{I(b)}=0$. By assumption, $a \leq^{-} b$. Thus there exist $x,y\in\mathcal{R}$ such that $a=xb=by$ and $xa=a$. Let

$$x=\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}_{e\times e} \quad \text{and} \quad y=\begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{bmatrix}_{e\times e}.$$

From $xa=a$, we get $x_4t=0=x_7t$, i.e., $0=x_4=x_7$. We thus have,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 0 & x_5 & x_6 \\ 0 & x_8 & x_9 \end{bmatrix}_{e\times e}\begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}=\begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}$$

and hence

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & x_5 & x_6 \\ 0 & x_8 & x_9 \end{bmatrix}_{e\times e}\begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e},$$

i.e., $(x_5+x_6+x_8+x_9)(n_1+n_2+n_3+n_4)=n_1+n_2+n_3+n_4$. Similarly, by $xb=a$, we have $(x_5+x_6+x_8+x_9)(t_1+z_1+n_5)=n_1+n_2+n_3+n_4$, and $by=a$ yields

$$(t_1+z_1+n_5)(y_5+y_6+y_8+y_9)=n_1+n_2+n_3+n_4.$$

Thus, $n_1+n_2+n_3+n_4 \leq^{-} t_1+z_1+n_5$.

Conversely, let a and b be of the matrix form (17). By Proposition 3.4, $a \leq^{\textcircled{d}} b$ which is by Lemma 2.2 equivalent to $a_1 \leq^{\textcircled{\#}} b_1$. The assumption $n_1+n_2+n_3+n_4 \leq^{-} t_1+z_1+n_5$ implies that there exist $u,v\in\mathcal{R}$ such that

$$n_1+n_2+n_3+n_4=u\,(n_1+n_2+n_3+n_4) \tag{18}$$

and

$$n_1+n_2+n_3+n_4=u\,(t_1+z_1+n_5)=(t_1+z_1+n_5)\,v. \tag{19}$$

Without loss of generality we may assume that $u, v \in e_2\mathcal{R}e_2 + e_2\mathcal{R}e_3 + e_3\mathcal{R}e_2 + e_3\mathcal{R}e_3$, i.e.,

$$u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & u_1 & u_2 \\ 0 & u_3 & u_4 \end{bmatrix}_{e\times e} \quad \text{and} \quad v = \begin{bmatrix} 0 & 0 & 0 \\ 0 & v_1 & v_2 \\ 0 & v_3 & v_4 \end{bmatrix}_{e\times e}.$$

Let

$$x = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & u_1 & u_2 \\ 0 & u_3 & u_4 \end{bmatrix}_{e\times e}.$$

Then by (18),

$$xa = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & u_1 & u_2 \\ 0 & u_3 & u_4 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} = a$$

and by (19),

$$xb = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & u_1 & u_2 \\ 0 & u_3 & u_4 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} = a.$$

Let now

$$y = \begin{bmatrix} e_1 & t^{-1}(s_1 - s_1v_1 - s_2v_3) & t^{-1}(s_2 - s_1v_2 - s_2v_4) \\ 0 & v_1 & v_2 \\ 0 & v_3 & v_4 \end{bmatrix}_{e\times e}.$$

By (19), we get

$$by = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} e_1 & t^{-1}(s_1 - s_1v_1 - s_2v_3) & t^{-1}(s_2 - s_1v_2 - s_2v_4) \\ 0 & v_1 & v_2 \\ 0 & v_3 & v_4 \end{bmatrix}_{e\times e}$$
$$= \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} = a.$$

It follows that $a \leq^- b$ and therefore by Definition 4.1, $a \leq^{©} b$. □

Let $a, b \in \mathcal{R}^{ⓓ}$ with $a \leq^{cm} b$. By Definition 3.1, we have $a_1 \leq^{\#} b_1$ and by Corollary 3.7, $a \leq^- b$. Thus, by Definition 4.1, $a \leq^{©} b$. The converse implication is not true in general, i.e., the c-minus partial order does not imply in general the core-minus partial order as [13, Example 4.1] shows. Let now $a \leq^{\textcircled{c}} b$ for some $a, b \in \mathcal{R}^{\textcircled{\dagger}}$. By Theorem 4.3,

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}, \qquad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}.$$

If $n_1 = n_2 = n_3 = 0$, then by Theorem 3.5, $a \leq^{cm} b$. With the last result of this section, we present some other conditions under which the c-minus partial order is equivalent to the core-minus partial order and thus generalize [13, Theorem 4.5]. The proof is similar to the proof of [13, Theorem 4.5] and thus we omit it.

Proposition 4.4. *Let $a, b \in \mathcal{R}^{\textcircled{\dagger}}$ and let $k = \max\{I(a), I(b)\}$. Suppose $a \leq^{\textcircled{c}} b$. Then the following statements are equivalent:*

(i) $a \leq^{cm} b$;
(ii) $bb^{\textcircled{\dagger}}ab^k = ab^k$ *and* $bb^{\textcircled{\dagger}}a = aa^{\textcircled{\dagger}}a$;
(iii) $aa^{\textcircled{\dagger}} = ab^{\textcircled{\dagger}}$ *and* $a^{\textcircled{\dagger}}a = b^{\textcircled{\dagger}}a$;
(iv) $b^{\textcircled{\dagger}}a = a^{\textcircled{\dagger}}b$ *and* $aa^{\textcircled{\dagger}} = ab^{\textcircled{\dagger}}$;
(v) $aa^{\textcircled{\dagger}} = ab^{\textcircled{\dagger}}$ *and* $bb^{\textcircled{\dagger}}a = aa^{\textcircled{\dagger}}a$.

5. The c-star partial order in rings

An involution $*$ on a semigroup $\mathcal{S}$ is called *proper* if $a^*a = a^*b = b^*b$, where $a, b \in \mathcal{S}$, implies $a = b$. If a semigroup $\mathcal{S}$ is equipped with a proper involution, then $\mathcal{S}$ is called *a proper $*$-semigroup.* Drazin introduced in [6] in the setting of proper $*$-semigroups a partial order now known as *the star partial order.* The definition follows. Let $\mathcal{S}$ be a proper $*$-semigroup. For $a, b \in \mathcal{S}$, we write

$$a \leq^* b \quad \text{if} \quad a^*a = a^*b \quad \text{and} \quad aa^* = ba^*.$$

Natural special cases of proper $*$-semigroups are all proper $*$-rings (in particular, $M_n(\mathbb{C})$), with "properness" defined via $aa^* = 0$ implying $a = 0$.

From now on, let $\mathcal{R}$ be a proper $*$-ring. Another partial order that is closely related to the c-minus partial order was introduced and studied in [20] in the context of C^*-algebras. With the following definition, we extend this concept to proper $*$-rings.

Definition 5.1. Let $a, b \in \mathcal{R}^{\textcircled{\dagger}}$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. We say that a is below b under the c-star relation and write

$$a \leq^{\textcircled{\dagger},\textcircled{*}} b \quad \text{if} \quad a_1 \leq^{\textcircled{\#}} b_1 \quad \text{and} \quad a \leq^* b.$$

As in the case of the relations that were studied in previous two sections, the properties of the core $\leq^{\textcircled{\#}}$ and the star $\leq^{*}$ relations and the uniqueness of the core-EP decomposition yield the following result.

Theorem 5.2. *The c-star relation is a partial order on* $\mathcal{R}^{\textcircled{\dagger}}$.

With the next theorem, we extend [20, Corollary 5.4] to rings. First, let us recall a known auxiliary result [4, Lemma 5].

Lemma 5.3. *Let* $a \in \mathcal{R}^{\textcircled{\dagger}}$ *be of the matrix form (9). Then for* $b \in \mathcal{R}^{\textcircled{\dagger}}$, *we have*

$$a \leq^{\textcircled{\dagger}} b \quad \text{if and only if} \quad b = \begin{bmatrix} t & s \\ 0 & b_3 \end{bmatrix}_{p\times p},$$

where $b_3 \in (1-p)\mathcal{R}(1-p)$.

Theorem 5.4. *Let* $a, b \in \mathcal{R}^{\textcircled{\dagger}}$ *with* $p = aa^{\textcircled{\dagger}}$. *Then* $a \leq^{\textcircled{\dagger},\circledast} b$ *if and only if*

$$a = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \qquad \text{and} \qquad b = \begin{bmatrix} t & s \\ 0 & b_3 \end{bmatrix}_{p\times p}, \tag{20}$$

where t *is invertible in the ring* $p\mathcal{R}p$, $a_2^{I(a)} = 0$, $(a_2 - b_3)\, s^* = 0$ *and* $a_2 \leq^* b_3$.

Proof. Let $a \leq^{\textcircled{\dagger},\circledast} b$, where a is represented with the matrix form (9). Since in (9) a_2 is the nilpotent part of the core-EP decomposition of a, we have that $a_2^{I(a)} = 0$. Also, by Definition 5.1, $a_1 \leq^{\textcircled{\#}} b_1$ and hence by Lemma 2.2, $a \leq^{\textcircled{\dagger}} b$. So, Lemma 5.3 yields that

$$b = \begin{bmatrix} t & s \\ 0 & b_3 \end{bmatrix}_{p\times p}.$$

By assumption, $a \leq^* b$ and thus $a^*a = a^*b$ and $aa^* = ba^*$. By the latter equation, we have

$$\begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \begin{bmatrix} t^* & 0 \\ s^* & a_2^* \end{bmatrix}_{p\times p} = \begin{bmatrix} t & s \\ 0 & b_3 \end{bmatrix}_{p\times p} \begin{bmatrix} t^* & 0 \\ s^* & a_2^* \end{bmatrix}_{p\times p}$$

and thus

$$\begin{bmatrix} tt^* + ss^* & sa_2^* \\ a_2 s^* & a_2 a_2^* \end{bmatrix}_{p\times p} = \begin{bmatrix} tt^* + ss^* & sa_2^* \\ b_3 s^* & b_3 a_2^* \end{bmatrix}_{p\times p}.$$

So, $a_2 s^* = b_3 s^*$, i.e., $(a_2 - b_3)\, s^* = 0$, and $a_2 a_2^* = b_3 a_2^*$. Similarly, the equation $a^*a = a^*b$ yields $a_2^* a_2 = a_2^* b_3$. Thus, $a_2 \leq^* b_3$.

Conversely, let a and b be of the matrix form (20) with $(a_2 - b_3)\, s^* = 0$ and $a_2 \leq^* b_3$. Then by Lemma 5.3, $a \leq^{\textcircled{\dagger}} b$, and with respect to the core-EP decomposition of a and b, we thus have by Lemma 2.2, $a_1 \leq^{\textcircled{\#}} b_1$. Since $a_2 s^* = b_3 s^*$ and $a_2 a_2^* = b_3 a_2^*$, we obtain by (20) that $aa^* = ba^*$, and from $a_2^* a_2 = a_2^* b_3$, we similarly get $a^*a = a^*b$. Thus, $a \leq^* b$ and hence $a \leq^{\textcircled{\dagger},\circledast} b$. □

We end the paper with another characterization of the c-star partial order using 3×3 matrix representations of elements in $\mathcal{R}$.

Theorem 5.5. *Let $a, b \in \mathcal{R}^{@}$ with $p = aa^{@}$. Then $a \leq^{@,\circledast} b$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, with $e_1 = p$ and $e_2^* = e_2$, such that*

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e \times e}, \qquad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e \times e}, \tag{21}$$

where t is invertible in the ring $e_1\mathcal{R}e_1$, t_1 is invertible in the ring $e_2\mathcal{R}e_2$, $(n_1 + n_3 - t_1)s_1^* + (n_2 + n_4 - z_1 - n_5)s_2^* = 0$, $n_5^{I(b)} = 0 = (n_1 + n_2 + n_3 + n_4)^{I(a)}$, and $n_1 + n_2 + n_3 + n_4 \leq^* t_1 + z_1 + n_5$.

Proof. Let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. Suppose first $a \leq^{@,\circledast} b$. Then $a_1 \leq^{\textcircled{\#}} b_1$, i.e., $a \leq^{@} b$, and thus by Proposition 3.4, there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, with $e_1 = p$ and $e_2^* = e_2$, such that

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e \times e}, \qquad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e \times e},$$

where $n_5^{I(b)} = 0 = (n_1 + n_2 + n_3 + n_4)^{I(a)}$, t is invertible in the ring $e_1\mathcal{R}e_1$, and t_1 is invertible in the ring $e_2\mathcal{R}e_2$. Also, by assumption, $a \leq^* b$. So, $a^*a = a^*b$ and thus

$$\begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e \times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e \times e} = \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e \times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e \times e}.$$

We have, $s_1^*s_1 + n_1^*n_1 + n_3^*n_3 = s_1^*s_1 + n_1^*t_1$, $s_1^*s_2 + n_1^*n_2 + n_3^*n_4 = s_1^*s_2 + n_1^*z_1 + n_3^*n_5$, $s_2^*s_1 + n_2^*n_1 + n_4^*n_3 = s_2^*s_1 + n_2^*t_1$, and $s_2^*s_2 + n_2^*n_2 + n_4^*n_4 = s_2^*s_2 + n_2^*z_1 + n_4^*n_5$. Hence, $n_1^*n_1 + n_3^*n_3 = n_1^*t_1$, $n_1^*n_2 + n_3^*n_4 = n_1^*z_1 + n_3^*n_5$, $n_2^*n_1 + n_4^*n_3 = n_2^*t_1$, and $n_2^*n_2 + n_4^*n_4 = n_2^*z_1 + n_4^*n_5$. So,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1^* & n_3^* \\ 0 & n_2^* & n_4^* \end{bmatrix}_{e \times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e \times e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1^* & n_3^* \\ 0 & n_2^* & n_4^* \end{bmatrix}_{e \times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e \times e}$$

or equivalently,

$$(n_1 + n_2 + n_3 + n_4)^* (n_1 + n_2 + n_3 + n_4) = (n_1 + n_2 + n_3 + n_4)^* (t_1 + z_1 + n_5).$$

Also, $aa^* = ba^*$, i.e.,

$$\begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e\times e} \tag{22}$$

and thus

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1^* & n_3^* \\ 0 & n_2^* & n_4^* \end{bmatrix}_{e\times e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1^* & n_3^* \\ 0 & n_2^* & n_4^* \end{bmatrix}_{e\times e},$$

i.e.,

$$(n_1+n_2+n_3+n_4)(n_1+n_2+n_3+n_4)^* = (t_1+z_1+n_5)(n_1+n_2+n_3+n_4)^*.$$

It follows that $n_1+n_2+n_3+n_4 \leq^* t_1+z_1+n_5$. By (22), we also get

$$n_1s_1^* + n_2s_2^* = t_1s_1^* + z_1s_2^*$$

and

$$n_3s_1^* + n_4s_2^* = n_5s_2^*.$$

We add these two equations and obtain $n_1s_1^*+n_2s_2^*+n_3s_1^*+n_4s_2^* = t_1s_1^*+z_1s_2^*+n_5s_2^*$, and then by rearranging, we finally get

$$(n_1+n_3-t_1)s_1^* + (n_2+n_4-z_1-n_5)s_2^* = 0.$$

Conversely, let a and b be of the form (21). By Proposition 3.4, we have $a \leq^{@} b$, i.e., $a_1 \leq^{\textcircled{\dagger}} b_1$. Equation $(n_1+n_3-t_1)s_1^* + (n_2+n_4-z_1-n_5)s_2^* = 0$ may be rewritten as

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & n_1-t_1 & 0 \\ 0 & n_3 & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ s_1^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & n_2-z_1 \\ 0 & 0 & n_4-n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_2^* & 0 & 0 \end{bmatrix}_{e\times e} = 0$$

and thus it decomposes into two equations: $(n_1-t_1)s_1^* + (n_2-z_1)s_2^* = 0$ and $n_3s_1^* + (n_4-n_5)s_2^* = 0$. It follows that

$$n_1s_1^* + n_2s_2^* = t_1s_1^* + z_1s_2^* \tag{23}$$

and

$$n_3s_1^* + n_4s_2^* = n_5s_2^*. \tag{24}$$

Also, by assumption, $n_1+n_2+n_3+n_4 \leq^* t_1+z_1+n_5$ and therefore

$$(n_1+n_2+n_3+n_4)^*(n_1+n_2+n_3+n_4) = (n_1+n_2+n_3+n_4)^*(t_1+z_1+n_5) \tag{25}$$

and

$$(n_1 + n_2 + n_3 + n_4)\,(n_1 + n_2 + n_3 + n_4)^* = (t_1 + z_1 + n_5)\,(n_1 + n_2 + n_3 + n_4)^* . \tag{26}$$

If we use equations (23)–(26), we get

$$\begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e\times e} = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e\times e},$$

i.e., $aa^* = ba^*$, and

$$\begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} = \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & n_1^* & n_3^* \\ s_2^* & n_2^* & n_4^* \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e},$$

i.e., $a^*a = a^*b$. It follows that $a \leq^* b$ which together with $a_1 \leq^{\textcircled{\dagger}} b_1$ implies $a \leq^{@,\circledast} b$. □

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