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SUMMABILITY AND DIRECT SUM OF UNISERIAL MODULES

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Abstract. Given the significance of abelian p-groups in module theory and their connection to algebraic structures, this paper focuses on constructing the $QTAG$-modules using the notion of torsion abelian groups and investigating their algebraic counterparts. These include examining specific types of submodules, such as isotype submodules, high submodules, and h-pure submodules, as well as exploring the concept of subsocles. In addition, we analyze the relationship between the concept of summability and the direct sum of uniserial modules within the context of these $QTAG$-modules.

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1. Introduction and backgrounds

Abelian groups and the theory of modules, two fundamental areas of algebra, play complementary roles. An abelian group focuses on torsion elements and the order of elements, providing a general framework for studying the concept of torsion abelian groups. Module theory, on the other hand, explores a variety of operations and forms the foundation for torsion modules over a ring where the scalar multiplication by elements of the ring induces a similar finite-order condition. These fields often intersect in advanced areas of algebraic notions, such as summability, divisibility, purity, and others, where structures of groups and categories of modules naturally coexist.

During the second half of the twentieth century, many researchers made significant contributions to the development of torsion abelian groups-like-modules, further highlighting the interplay between these two areas of algebra. Many of the most significant algebraic objects combine the properties of these two realms of algebra. Examples include height, decomposition length, exponent, socle, and cardinality. The concept of torsion abelian group-like-module (TAG-module, for short) was first introduced by S. Singh in [23] using the following two conditions related to uniserial modules, where the rings under consideration are assumed to have unity.

(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.
(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided the composition length $d(U_1/N) \leq d(U_2/\phi(N))$.

However, the true inception of the theory occurred in 1987 when S. Singh [24] published a paper of paramount importance and only used condition (i) to introduce the concept of a quasi-torsion abelian group-like-module ($QTAG$-module, for short). In recent years, a variety of authors have explored these modules through different perspectives on torsion abelian groups and their connections to various algebraic structures, resulting in significant findings (see, for example, [17,22]). These findings have not only advanced the theoretical understanding of $QTAG$-modules but also opened new avenues for further exploration in module theory. Unsurprisingly, a lot of these advancements are similar to the earlier developments made in the theory of torsion abelian groups. The present work is a natural generalization of the research conducted in [18] and contributes to existing knowledge on the structure of $QTAG$-modules. It is important to note that various results regarding TAG-modules are also applicable to $QTAG$-modules [19]. For more in-depth discussions of the subject matter covered here, the reader may consult references [7,11,12]. Another valuable source on the subject under discussion is [5] (and also refer to [4,6]). For further interesting generalizations of the topic discussed here, the reader can refer to sources [1,3]. All other notions and notation not explicitly explained herein are well known and mainly follow those from [9] and [10].

Throughout the present paper, unless specified something else, let us assume that all rings R into consideration are associative with unity and modules M are unital $QTAG$-modules, written additively, as is the custom when studying them. A module M over a ring R is known as the uniserial module if its submodules are totally ordered by inclusion, that is, for any two submodules N and L of M, either $N \subseteq L$ or $L \subseteq N$. Likewise, we will state that M is uniform if two of its non-zero submodules intersect in a non-zero fashion. In particular, an element u in a module M is called the uniform element if uR is a non-zero uniform (hence uniserial) module. Typically, the decomposition length of any module M with a unique decomposition series is denoted by $d(M)$. In addition, the exponent of a uniform element u of M, denoted by the symbol $e(u)$, is equal to $d(uR)$. As usual, for such a module M, we specify the height of u in M as $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR$ and v uniform$\}$. Likewise, for $t \geq 0$, $H_t(M) = \{u \in M \mid H_M(u) \geq t\}$ represents the submodule of M that is generated by elements that have at least t heights. As is customary, for any submodule N of M, we denote $H^t(M) = \{u \in M \mid d(uR/(uR \cap N)) \geq t\}$ is the submodule of M generated by the elements of exponents at most t. For a module M, the letter M^1 will always in the sequel denote the submodule of M which contains elements of infinite height.

The module M is called h-divisible if $M = M^1 = \cap_{t=0}^{\infty} H_t(M)$, and h-reduced if it does not contain any h-divisible submodule. A submodule N of M is h-pure in M if $N \cap H_t(M) = H_t(N)$ for every integer $t \geq 0$. In particular, if $t = 1$, N is called h-neat in M. A submodule N of M is said to be a basic submodule of M if N is an h-pure submodule of M, N is a direct sum of uniserial modules and M/N is h-divisible. A submodule N of M is said to be high if it is a complement of M^1 i.e., $M = N \oplus M^1$. The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$ and a submodule N of $Soc(M)$ is called a subsocle of M.

If α is an ordinal and M is a $QTAG$-module, then the infinite height $H_\alpha(M)$ is a submodule of M, consisting of elements of height at least α. This submodule is also known as the α-th Ulm submodule. By analogy, a submodule N of M is said to be α-pure if for all ordinal γ, there exists an ordinal α (depending on N) such that $H_\gamma(M) \cap N = H_\gamma(N)$, and a submodule N of M is said to be an isotype in M if it is α-pure for every ordinal α. A submodule N of M is an α-high submodule of M if N is maximal among the submodules of M that intersect $H_\alpha(M)$ trivially. The cardinality of the minimal generating set of M is denoted by the symbol $g(M)$. For all ordinals α, one can define $f_M(\alpha)$, the α-th Ulm invariant of M as follows: $f_M(\alpha) = g\left(Soc\left(H_\alpha(M)\right)/Soc\left(H_{\alpha+1}(M)\right)\right)$.

2. Summable modules

The study of summable modules plays a crucial role in the theory of $QTAG$-modules, which can be characterized in various ways. We require only for information a few more comprehensive observations (see [20]): A subsocle S of the $QTAG$-module M is said to be summable if there is a direct decomposition $S = \oplus S_\gamma$ where $S_\gamma - \{0\} \subseteq H_\gamma(M) - H_{\gamma+1}(M)$, that is, the non-zero elements of S_γ have height precisely γ. An h-reduced $QTAG$-module M is said to be summable if $Soc(M)$ is summable. In this section, we study summable modules in some detail. Our goal here is to provide a complete description of some important assertions regarding summability, utilizing specific submodules, and to apply these insights to analyze the direct sum of uniserial modules.

The following theorem is one of the fundamental results about summable subsocles. The proof we present here is primarily a simplification of the proof given in [15] (see [2] also), which establishes that a summable module cannot have a length exceeding ω_1, the first uncountable ordinal.

Theorem 2.1. *Let S be a subsocle of an h-reduced QTAG-module M such that $Soc(M) = S + Soc(H_{\omega_1}(M))$, where ω_1 is the first uncountable ordinal. Then $H_{\omega_1}(M) = 0$ only if S is summable.*

Proof. Assume that $Soc(M) = S + Soc(H_{\omega_1}(M))$ where S is summable, and assume the contrary that $H_{\omega_1}(M) \neq 0$. Let $S = \oplus_{\gamma<\omega_1} S_\gamma$ where $S_\gamma - \{0\} \subseteq H_\gamma(M) - H_{\gamma+1}(M)$ and let $\phi_\gamma : S \longrightarrow S_\gamma$ be the projection map. Since $H_{\omega_1}(M) \neq 0$, there exists an element u in M such that $H_M(u) = \omega_1$. Now, for each i, there is a sequence $x_0, x_1, \ldots, x_i, \ldots$ in M such that

(i) $y_i = u$ where $d(x_iR/y_iR) = 1$,
(ii) x_i has countable height λ_i,
(iii) $\lambda_{i+1} > \lambda_i$ and
(iv) $\phi_\gamma(x_{i+1} - x_i) = 0$ for $\gamma \geq \lambda_{i+1}$.

Next, suppose x_ω be an element of M such that $y_\omega = u$ and $d(x_\omega R/y_\omega R) = 1$; so, in particular, x_ω has a countable height $\lambda \geq \sup_{i<\omega}\lambda_i$. If we let $v = x_\omega - x_0$, then $v \in Soc(M)$. But there is no loss of generality in assuming that $v \in S$ and guarantees that $\phi_\gamma(v) = \phi_\gamma(x_{i+1} - x_i)$ for all $\lambda_i \leq \gamma \leq \lambda_{i+1}$. In particular, $\phi_{\lambda_i}(v) \neq 0$ for each i. This contradicts the fact that $\phi_\gamma(v) = 0$ for all but a finite number of γ if $v \in S = \oplus_{\gamma<\omega_1} S_\gamma$, and the proof is complete. □

We continue with another straightforward observation.

Lemma 2.2. *Suppose M is a QTAG-module and γ is an ordinal. If N is a γ-high submodule of M, then N is summable precisely when M is summable.*

Proof. Assuming first that length $(M) = \beta$; in fact, assume $Soc(M) = \oplus_{\gamma<\beta} S_\gamma$ where $S_\gamma - \{0\} \subseteq H_\gamma(M) - H_{\gamma+1}(M)$. Observe that if K is an h-neat submodule of M such that $Soc(K) = \oplus_{\alpha<\gamma} S_\alpha$, then K is a γ-high submodule of M. This follows since the heights computed in K are the same as computed in M, and K is summable. This is in marked contrast to the canonical projection $M \longrightarrow M/H_\gamma(M)$ maps γ-high submodules isomorphically that preserve height. However, we know that the socles of any two γ-high submodules have the same image under this mapping. Henceforth, a simple technical argument applies to get that a γ-high submodule is summable which gives the desired summability of N. □

In light of the previous constructions, we obtain the following.

Proposition 2.3. *Let N be an isotype submodule of a QTAG-module M, where N has a countable length β. Then N is summable only if M is summable.*

Proof. By hypothesis, N embeds as a β-high submodule of M, then one derives by Lemma 2.2 that there is no loss in generality in assuming M has length β. Indeed, $Soc(M) = \oplus_{\gamma<\beta} S_\gamma$ where $S_\gamma - \{0\} \subseteq H_\gamma(M) - H_{\gamma+1}(M)$ and β is countable. Let $\gamma_1, \gamma_2, \dots, \gamma_k, \dots$ be an enumeration of the ordinal less than β and we set $A_k = S_{\gamma_1} + S_{\gamma_2} + \cdots + S_{\gamma_k}$. Then $A_1, A_2, \dots$ are height-finite and $Soc(M) = \cup_{k<\omega} A_k$. But since N is isotype in M, $B_k = A_k \cap N$ is height-finite in N. Furthermore, since $Soc(N) = \cup_{k=1}^{\infty} B_k$, it must be that N is summable, as expected. □

The next proposition is a generalization of an observation in [14].

Proposition 2.4. *Let M_1 and M_2 be QTAG-modules such that M_1 has length γ and M_1 contains an h-neat submodule N with $Soc(M_1) \subseteq Soc(N) + H_\alpha(M_1)$ for all $\alpha < \gamma$ and $M_1/N \cong D/M_2$ where D is the minimal h-divisible module containing M_2. Then there exists a QTAG-module M_3 such that $M_3/H_\gamma(M_3) \cong M_1$, $H_\gamma(M_3) = M_2$ and N is γ-high in M_3.*

Proof. Suppose M_3 is a subdirect sum of M_1 and D with kernels N and M_2, then M_3 is a submodule having $M_1 \oplus D$ as a direct sum such that $M_3 + M_1 = M_3 + D = M_1 \oplus D$, $M_3 \cap M_1 = N$ and $M_3 \cap D = M_2$. Observing that M_3 is h-neat in $M_1 \oplus D$. Suppose $x \in M_1 + D$, $x' \in M_3$; if $d(xR/x'R) = 1$. Then $x = y + z$ with $y \in M_1$ and $z \in M_3$. It follows that $y' = x' - z' \in H_1(M_1) \cap M_3 = H_1(M_1) \cap N = H_1(N)$ where $d(xR/x'R) = d(yR/y'R) = d(zR/z'R) = 1$. This, in turn, implies that $x' = H_1((z - a)R) \in H_1(M_3)$ where $d(xR/x'R) = 1$ and $a \in S$.

Since $Soc(M_1) \subseteq Soc(N) + H_\alpha(M_1)$ for all $\alpha < \gamma$, we obtain that $Soc(N) \subseteq Soc(M_3)$. And since D is h-divisible, we have $Soc(M_1{+}D) \subseteq Soc(M_3){+}H_\alpha(M_1{+}D)$ for all $\alpha < \gamma$. Therefore, by hypothesis of the proposition, we get that $M_3 \cap H_\alpha(M_1 + D) = H_\alpha(M)$ for all $\alpha < \gamma$. Of course, $H_\gamma(M_3) = M_3 \cap D = M_2$ and thus,

$$M_3/H_\gamma(M_3) = M_3/M_2 = M_3/(M_3 \cap D) \cong (M_3 + D)/D = (M_1 + D)/D \cong M_1.$$

Finally, we therefore only need to show that N is maximal in M with the property that $N \cap M_2 = 0$. Toward this end, note that $(N + zR) \cap M_2 \neq 0$ whenever $z \in M_3 - N$. For each $z \in M_3 - N$, choose $z = y + b$ such that $y \in M_1$ and

$b \in D$. Since D is a minimal h-divisible module containing M_2, there exists a positive integer k such that $0 \neq kb \in M_2$. Then $ky = kz - kb$, $M_3 \cap M_1 = N$ and thus kb is a non-zero element of $(N + zR) \cap M_2$. We are finished. □

Some remarks concerning Proposition 2.4 appear to be of significance.

Analysis. First, M_1 and M_2 need not necessarily be $QTAG$-modules. Next, we only need $H_1(M_1) \cap N = H_1(N)$ instead of N being h-neat in M_1. Furthermore, in addition to these weak hypotheses, the same proof goes through if D is only considered to be an h-divisible essential extension of M_2. It is well to note that, given any $QTAG$-module M_1 of length $\gamma \geq \omega$, there is an h-neat submodule N of M such that $Soc(M_1) \subseteq Soc(N) + H_\alpha(M_1)$ for all $\alpha < \gamma$ and $M_1/N \cong M_3$. Thus, it can be inferred from these observations that, given any $QTAG$-module M_2 and any ordinal γ, there is a $QTAG$-module M_3 such that $H_\gamma(M_3) = M_2$ and $M_3/H_\gamma(M_3)$ is a $QTAG$-module. In particular, $M_3/H_\gamma(M_3)$ can be determined to be a direct sum of countably generated h-reduced $QTAG$-modules, provided that $\gamma \leq \omega$.

It is worth noting that the proof of Lemma 2.1 in [14] establishes the following, which leads to a converse of Proposition 2.4.

Proposition 2.5. *Suppose M is a $QTAG$-module and N is a submodule of $H_\omega(M)$. If D is a minimal h-divisible module and L is generated by the elements of M and D such that $M \cap D = N$, then*

(i) *M is an h-pure submodule of L;*
(ii) *$L = K + D$ such that $K \cong M/N$;*
(iii) *$K \cap M$ is a high submodule of M;*
(iv) *$L = K + M$; and*
(v) *M is a subdirect sum of K and D.*

It is directly determined by the results presented in [17] that if K is an h-neat submodule of M with M/K is h-divisible, then K is γ-pure in M if and only if $Soc(M) \subseteq Soc(K) + H_\alpha(M)$ for all $\alpha < \gamma$. A straightforward instance is found in Proposition 2.5, M is γ-pure in L if $N \subseteq H_\gamma(M)$. In that case, $K \cap M$ is a γ-pure submodule of K.

So, we shall verify the validity of the following theorem.

Theorem 2.6. *Let M_1 and M_2 be $QTAG$-modules. If M_1 is summable and M_2 is h-divisible, then $H_{\omega_1}(Ext(M_2, M_1)) = 0$, where ω_1 is the first uncountable ordinal.*

Proof. Suppose on contrary that $H_{\omega_1}(Ext(M_2, M_1)) \neq 0$. Then there is an h-reduced module M_3 containing M_1 as a ω_1-pure submodule such that $M_3/M_1 \cong$

M_2. Hence, it immediately follows that $Soc(M_3) \subseteq Soc(M_1) + H_\gamma(M_3)$ for all $\gamma < \omega_1$. Consequently, by Proposition 2.4, we can find an h-reduced module M_4 containing M_1 as a ω_1-high submodule such that $H_{\omega_1}(M_4) \cong M_2$. Therefore, we have $Soc(M_4) = Soc(M_1) + H_{\omega_1}(M_4)$ and the summability of M_1 implies that of M_4 contrary to the fact that M_4 has length $\omega_1 + 1$. The proof is over. □

If N is a submodule of M, a direct decomposition $N = \oplus N_\alpha$ is called essential if $H_M(x_1+x_2+\cdots+x_k) = \min[H_M(x_1), H_M(x_2), \ldots, H_M(x_k)]$, where $H_M(x)$ denotes the height of x in M, whenever $x_i \in N_{\alpha_i}$ with $\alpha_i \neq \alpha_j$ for $i \neq j$. If $Soc(M) = \oplus_{\gamma<\alpha} S_\gamma$ where $S_\gamma - \{0\} \subseteq H_\gamma(M) - H_{\gamma+1}(M)$, then the decomposition $\oplus_{\gamma<\alpha} S_\gamma$ is essential. If S is a subsocle of M such that $S = \oplus_{i\in I} P_i$ is an essential decomposition and each P_i is summable, it is fairly easy to verify that S is also summable. Since countably generated subsocles are summable, we have the following immediate step.

Lemma 2.7. *Let S be a subsocle of a QTAG-module M and π be a projection of S into itself such that $P = \pi(P) + (1-\pi)(P)$ is essential whenever $\pi(P) \subseteq P$. Then $\pi(S)$ is summable only if S is summable.*

Proof. First, if S is countably generated, there is nothing to prove. We next assume that S is uncountable and let $S = \oplus_{i\in I} P_i$ be an essential decomposition such that $g(P_i) \leq \aleph_0$. Now, we construct a well ordered ascending sequence $\{I_\gamma\}_{\gamma<\alpha}$ of subsets of I such that

(i) $g(I_{\gamma+1} - I_\gamma) \leq \aleph_0$ for all γ,
(ii) $I_\gamma = \cup_{\lambda<\gamma} I_\lambda$ if γ is a limit ordinal and
(iii) $\pi(\oplus_{i\in I} P_i) \subseteq \oplus_{I_\gamma} P_i$ for all γ.

Then, according to the hypothesis of the lemma, the decomposition $\oplus_{I_\gamma} P_i = X_\gamma + Y_\gamma$ is essential, where $X_\gamma = \pi(\oplus_{I_\gamma} P_i)$ and $Y_\gamma = (1-\pi)(\oplus_{I_\gamma} P_i)$. This, in turn, implies that $\oplus_{I_{\gamma+1}} P_i = X_\gamma + Y_\gamma + \oplus_{I_{\gamma+1}-I} P_i$ is also an essential decomposition. Thus, we find that $X_{\gamma+1} = X_\gamma + Z_\gamma$ is essential, where $Z_\gamma = X_{\gamma+1} \cap (Y_\gamma \oplus_{I_{\gamma+1}-I_\gamma} P_i)$.

In the remaining case when $I_0 = \phi$, we get $Z_0 = \phi(\oplus_{I_1} P_i)$. Therefore, it follows that $\pi(S) = \oplus_{\gamma<\alpha} Z_\gamma$ where $g(Z_\gamma) \leq \aleph_0$ and the decomposition of $\pi(P)$ is essential. Hence $\pi(P)$ is summable. We are done. □

As an immediate consequence, we yield the following.

Corollary 2.8. *Let S_1, S_2 and S_3 be subsocles of a QTAG-module M such that $S_1 = S_2 + S_3$ is an essential decomposition. Then S_2 and S_3 are summable, provided S_1 is summable.*

Proof. Just take π to be the projection of S_1 into itself such that $\pi(S_1) = S_2$ and $S_3 = \ker\pi$, and the result follows from Lemma 2.7. □

3. Role of direct sum of uniserial modules

The class of $QTAG$-modules over a ring R may not be closed under direct sums of uniserial modules. It is established in [23] that a $QTAG$-module M can be expressed as a direct sum of uniserial modules if and only if M is the union of an ascending chain of bounded submodules. This statement indicates that M is a direct sum of uniserial modules if and only if $Soc(M) = \oplus_{t\in\omega} S_t$ and $H_M(u) = t$ for every $u \in S_t$.

Likewise, a theorem from [8] addresses the challenge of detecting finite direct sums of uniserial modules. Recently, in [13,16,21], new advancements related to this topic have been made concerning other significant types of $QTAG$-modules, and their relationships with the direct sum of uniserial modules, yielding significant results. These advancements not only enhance our understanding of the structure of $QTAG$-modules but also open up a new discussion for exploring their applications in the classification of modules and their representations. Moreover, they have a sufficiently rich structure to warrant investigation, without being complicated enough to defy analysis. The purpose of this section is to investigate the various assertions of these modules and their interrelations.

Our next result generalizes the fact that a summable module of length ω is a direct sum of uniserial modules.

Theorem 3.1. *Let M be a $QTAG$-module of countable length α. If $M/H_\beta(M)$ is a direct sum of uniserial modules for all $\beta < \alpha$, then M is a direct sum of uniserial modules precisely when M is summable.*

Proof. The result is trivial if α is not a limit ordinal. Hence, we may assume that α is a limit ordinal. Then we fix the decompositions of $M/H_\beta(M)$ for all $\beta < \alpha$ and let $Soc(M) = \oplus_{i\in I} P_i$ be an essential decomposition such that $g(P_i) \leq \aleph_0$ for each i. Now, we construct a well ordered ascending sequence $\{N_\gamma\}_{\gamma<\alpha}$ of submodules of M such that

(i) $N_\gamma = \cup_{\lambda<\gamma} N_\lambda$ if γ is a limit ordinal;
(ii) N_γ is α-pure in M;
(iii) $Soc(N_\gamma) = \oplus_{i\in I_\gamma} P_i$ for some subset I_γ of I;
(iv) $g(N_{\gamma+1}/N_\gamma) \leq \aleph_0$.

Therefore, we find at once that $N_{\gamma+1} = N_\gamma + K_\gamma$, and thus $M = \oplus_{\gamma<\alpha} K_\gamma$ where $g(K_\gamma) \leq \aleph_0$. Since N_γ is α-pure in $N_{\gamma+1}$, what suffices to show is that

$N_{\gamma+1}/N_\gamma$ is α-projective. In fact, we observe that $N_{\gamma+1}/N_\gamma$ is countably generated. It is just necessary that it has length less than or equal to α. We may select $\oplus_{i\in I_{\gamma+1}-I_\gamma}P_i = \oplus_{\beta<\alpha}S_\beta$ where $S_\beta - \{0\} \subseteq H_\beta(M) - H_{\beta+1}(M)$. As N_γ is α-pure in $N_{\gamma+1}$, we routinely observe that $Soc(H_\beta(N_{\gamma+1}/N_\gamma)) = Soc(N_\gamma + H_\beta(N_{\gamma+1}))/N_\gamma$ for all $\beta<\alpha$. But $Soc(H_\beta(N_{\gamma+1})) = Soc(H_\beta(N_\gamma)) + \oplus_{\beta<\delta<\alpha}S_\delta$ and it plainly follows that $Soc(H_\beta(N_{\gamma+1}/N_\gamma)) = (N_\gamma \oplus_{\beta<\delta<\alpha} S_\delta)/N_\gamma = (N_\gamma+S_\beta)/N_\gamma + Soc(H_{\beta+1}(N_{\gamma+1}/N_\gamma))$. Therefore, we have a direct decomposition $Soc(N_{\gamma+1}/N_\gamma) = \oplus_{\beta<\alpha}(N_\gamma+S_\beta)/N_\gamma$ where $(N_\gamma+S_\beta)/N_\gamma - \{0\} \subseteq H_\beta(N_{\gamma+1}/N_\gamma) - H_{\beta+1}(N_{\gamma+1}/N_\gamma)$. Finally, because $Soc(H_\alpha(N_{\gamma+1}/N_\gamma)) = 0$, we obtain the desired assertion. □

The following theorem may be viewed as a generalization of the existence theorem for basic submodules.

Theorem 3.2. *Suppose M is a QTAG-module and α is an ordinal such that $\omega \le \alpha < \omega_1$. If α is a limit ordinal, then the next two points are equivalent.*

(i) *$M/H_\beta(M)$ is a direct sum of uniserial modules for all $\beta<\alpha$.*

(ii) *M contains an α-pure submodule N such that M/N is h-divisible and N is a direct sum of uniserial modules of length less than or equal to α.*

Proof. (ii) ⇒ (i) If α is a limit ordinal and N is an α-pure submodule of M such that M/N is h-divisible, one sees that $M = N + H_\beta(M)$. Consequently, $N/H_\beta(N) = N/H_\beta(M)$ where $N \cong M/H_\beta(M)$ for all $\beta<\alpha$. Moreover, since N is a direct sum of uniserial modules, it is obvious that $M/H_\beta(M)$ must be a direct sum of uniserial modules for all $\beta<\alpha$.

(i) ⇒ (ii) We have two cases to consider.

Case 1: α is a limit ordinal. In this case, let us consider an increasing sequence $\beta_1, \beta_2, \ldots, \beta_k, \ldots$ of ordinals having α as its limit. Let P_0 be the socle of a β_1-high submodule of M and, for $k\ge 1$, let P_k be such that $Soc(H_{\beta_k}(M)) = P_k + Soc(H_{\beta_{k+1}}(M))$ and $Q = \oplus_{k<\omega}P_k$. Since $Soc(M) = P_0 + \cdots + P_{k-1} + Soc(H_{\beta_k}(M))$ for each $k\ge 1$, we deduce that $Soc(M) = Q + Soc(H_\beta(M))$ for all $\beta<\alpha$. Choose N to be an h-neat submodule of M such that $Soc(N) = Q$. Since $Soc(M) = Soc(N) + Soc(H_\beta(M))$, it is clear that N is α-pure in M for all $\beta<\alpha$. In addition, since $\alpha\ge\omega$, it follows plainly that M/N is h-divisible. Note that $N/H_\beta(N) \cong M/H_\beta(M)$ is a direct sum of uniserial modules for all $\beta<\alpha$; we will be done if we can show that N is summable. However, since each P_k is the submodule of an $H_{\beta_{k+1}}(M)$-high submodule of M and, since each $H_{\beta_{k+1}}(M)$-high submodule of M is summable, we can infer that each P_k is summable. As $Q = \oplus_{k<\omega}P_k$ is an essential decomposition, we obtain that $Q = Soc(N)$ is summable, and hence N

itself is summable. Thus, by Theorem 3.1, N has to be a direct sum of uniserial modules, as desired.
Case 2: $\alpha - 1$ exists. Set $\gamma = \alpha - 1$ for some countable ordinal γ. Let K be a basic submodule of $H_\gamma(M)$ and choose N such that N/K is maximal in M/K with respect to intersecting $H_\gamma(M/K)$ trivially. Then, one may see that $M/K = H_\gamma(M)/K + N/K$. It is easy to verify that $N \cap H_1(M) = H_1(N)$ and $Soc(M) = Soc(N) + Soc(H_\gamma(M))$. Thus N is α-pure in M and $K = H_\gamma(M) \cap N = H_\gamma(N)$. Therefore, since both $H_\gamma(N)$ and $N/H_\gamma(N) = N/K \cong M/H_\gamma(M)$ are the direct sums of uniserial modules, we get that N is a direct sum of uniserial modules, as required. □

The lemma presented is of particular interest.

Lemma 3.3. *Suppose M is a QTAG-module and α is a limit ordinal. If N is an α-pure submodule of M having length $\gamma < \alpha$, then M/N has length α.*

Proof. Since N is α-pure in M, it therefore follows that

$$Soc(H_\beta(M/N)) = (N + Soc(H_\beta(M)))/N$$

for all $\beta < \alpha$. Again, since $H_\gamma(M) \cap N = 0$ for some $\gamma < \alpha$, we see that $H_\beta(M/N) \neq 0$ for all $\beta < \alpha$ and $\cap_{\beta<\alpha} Soc(H_\beta(M/N)) = 0$. We are done. □

This leads to the following very satisfactory result.

Theorem 3.4. *Let α be a countable limit ordinal, and M a QTAG-module such that $M/H_\beta(M)$ is a direct sum of uniserial modules for all $\beta < \alpha$. If N is a countably generated α-pure submodule of M of length $\gamma < \alpha$, then N is a direct summand of M.*

Proof. Since $N \cap H_\gamma(M) = 0$ for $\gamma < \alpha$, Theorem 3.2 ensures that there is an α-pure submodule K of M such that $N \subseteq K$, M/K is h-divisible and K is a direct sum of uniserial modules of length less than or equal to α. Also, since K is a direct sum of uniserial modules, we get a direct decomposition $K = L + U$ where $N \subseteq L$ and L are countably generated. Now, we may assume that L has length α, and so by Lemma 3.3, L/N has length α. Thus, L/N is α-projective and consequently, N is a direct summand of L. Therefore, we obtain a direct decomposition $K = N + V$. Finally, since $M = K + H_\gamma(M)$ and $N \cap H_\gamma(M) = 0$, we can find the decomposition $M = N + V + H_\gamma(M)$, as needed. □

This brings us to one of the main results of this section.

Theorem 3.5. *Suppose N is an isotype submodule of a QTAG-module M such that N has countable length α. If M is a direct sum of uniserial modules, then N is also a direct sum of uniserial modules.*

Proof. Since N is an isotype submodule of M having countable length α, we routinely observe that $(N + H_\alpha(M))/H_\alpha(M)$ is isotype in $M/H_\alpha(M)$, and therefore α is also the length of M. The proof is now by induction on α. Assume that the theorem is established for all length less than α. To that goal, we have two cases to consider.

Case 1: α is a limit ordinal. According to induction hypotheses, $N/H_\gamma(N)$ is a direct sum of uniserial modules for all $\gamma < \alpha$. Therefore, we fix direct decompositions of the modules $N/H_\gamma(N)$ and let $M = \oplus_{i\in I} M_i$. Now, we construct a well ordered ascending sequence $\{I_\beta\}_{\beta<\alpha}$ of subsets such that

(i) $N \cap \oplus_{i\in I_\beta} M_i$ is isotype in N.
(ii) $[(N \cap \oplus_{i\in I_\beta} M_i) + H_\gamma(N)]/H_\gamma(N)$ is a direct summand of $N/H_\gamma(N)$ for all $\gamma < \alpha$.
(iii) $I_\beta = \cup_{\lambda<\beta} I_\lambda$ if β is a limit ordinal.
(iv) $g(I_{\beta+1} - I_\beta) \leq \aleph_0$.

Therefore, $N \cap \oplus_{i\in I_\beta} M_i$ is γ-pure in N for all $\gamma < \alpha$ and, hence, α-pure in N. However, $(N \cap \oplus_{i\in I_{\beta+1}} M_i)/(N \cap \oplus_{i\in I_\beta} M_i)$ is a countably generated module of length less than or equal to α and it follows, therefore, that α-projective. Clearly, it is, in fact, isomorphic to

$$(N \cap \oplus_{i\in I_{\beta+1}} M_i \oplus_{i\in I_\beta} M_i)/\oplus_{i\in I_\beta} M_i \subseteq \oplus_{i\in I_{\beta+1}} M_i/\oplus_{i\in I_\beta} M_i \cong \oplus_{i\in I_{\beta+1}-I_\beta} M_i,$$

so that $N \cap \oplus_{i\in I_{\beta+1}} M_i = K_\beta + N \cap \oplus_{i\in I_\beta} M_i$ where $N = \oplus_{\beta<\alpha} K_\beta$.

Case 2: $\alpha - 1$ exists, then by induction hypothesis,

$$N/H_{\alpha-1}(N) \cong (N + H_{\alpha-1}(M))/H_{\alpha-1}(M)$$

is a direct sum of uniserial modules. Note that since $H_{\alpha-1}(N)$ is a direct sum of uniserial modules, it follows that N is a direct sum of uniserial modules. □

4. Concluding discussion and left-open problems

In this project, we study the concept of *QTAG*-modules as a natural extension of abelian p-groups, relaxing certain constraints to explore broader algebraic structures. We investigated the fundamental properties of the $QTAG$-modules, established their connections with summability, and examined their structural components, including submodules and subsocles. Following this, we analyze the relationships between these concepts and their summability utilizing the direct sum of uniserial modules, which provides a comprehensive framework for understanding these $QTAG$-modules. According to our findings, our proposed study is promising and applicable to examine the summability of various types of submodules or other algebraic structures.

We close the work with certain challenging problems which are worthwhile for a further study.

Problem 4.1. *If S_1 and S_2 are summable subsocles, is $S_1 \oplus S_2$ as well?*

Problem 4.2. *If S is a subsocle of a $QTAG$-module M and both M and M/S are summable, does it follow that S is also summable?*

Problem 4.3. *Describe the summable modules that are far from*

(a) *high submodules of the direct sum of uniserial modules;*
(b) *isotype submodules of the direct sum of uniserial modules;*
(c) *h-pure submodules of the direct sum of uniserial modules.*

Problem 4.4. *Determine under what additional circumstances on the $QTAG$-module structure summable modules are direct sums of uniserial modules.*

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