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FINITE GROUP WHOSE MAXIMAL SUBGROUPS HAVE SMALL CODEGREE SUMS

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Abstract. The purpose of this paper is to give the structure of finite groups whose maximal subgroups H satisfy $\mathrm{T^c}(H) < 24$ where $\mathrm{T^c}(K)$ is the sum of codegrees of a group K.

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1. Introduction

All groups in this note are finite. Let $\mathrm{Irr}(G)$ be the set of all complex irreducible characters of a group G and let $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ be the set of character degrees for a group G. The codegree of a character $\chi \in \mathrm{Irr}(G)$ is defined as

$$\chi^c(1) = |G : \ker \chi|/\chi(1)$$

and denote by

$$\mathrm{cd}_c(G) = \{\chi^c(1) : \chi \in \mathrm{Irr}(G)\}$$

be the character codegree set of a group G (see [12]). Liu and Shang in [11] have determined the groups by using the maximal codegree of a group. Wang et al. [14] have defined average codegree of a group G as

$$\mathrm{acod}(G) = \frac{1}{|\mathrm{Irr}(G)|} \sum_{\chi \in \mathrm{Irr}(G)} \chi^c(1)$$

and proved that if G is non-solvable, then $\mathrm{acod}(G) \geq 68/5$ and that if G is non-supersolvable, then $\mathrm{acod}(G) \geq 11/4$. Aziziheris and Eivazzadeh in [1] have shown that for a nonsolvable group G with

$$\mathrm{T}^c(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi^c(1)$$

which is the sum of irreducible character codegrees of a group G. Lewis and Yan

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in [8] have shown that if G is a nonsolvable group, then $\mathrm{T}^c(G) \geq 68$, with equality if and only if $G \cong A_5$.

In this paper, the following problem is considered.

Problem: What occurs to the group structure if its maximal subgroups H satisfy $\mathrm{T}^c(H) < 24$.

We call a group G **a** T^c**-group** if for each maximal subgroup M of G, $\mathrm{T}^c(M) < 24$.

Let $A \circ B$ denote the central product of two groups A and B and let E_r^n or E_{r^n} be an elementary abelian r-group of order r^n. We will prove the following result.

Theorem 1.1. *Let G be a T^c-group. Then G is isomorphic to $\mathrm{PSL}_2(q)$ with $q = 5, 7$ or is solvable, in particular, G is isomorphic to one of the following:*

(1) E_2^k *with* $k \leq 4$, C_4, D_8, Q_8, $M_2(2,1,1)$, D_{16}, SD_{16}, Q_{16}, $C_2 \times D_8$, $C_2 \times Q_8$, $D_8 \circ C_4 \cong Q_8 \circ C_4$,

(2) *a p-group of order at most p^2, $p = 3, 5$,*

(3) $C_p \times C_q$ *with* $p, q \in \{2,3,5\}$ *and* $p \neq q$,

(4) D_{2p} *with* $p \in \{3,5\}$, $C_2 \times S_3$, A_4, $E_{2^3} : C_3$, $A_4 \times C_2$, $\mathrm{SL}_2(3)$, S_4 *or* $C_5 : C_4$.

The paper is organized as follows. In Section 2, some elementary results are given, for example, $\mathrm{T}^c(D_{2n})$ and $\mathrm{T}^c(E_q : C_p)$ where $E_q : C_p$ is a Frobenius group; in Section 3, we first show the structure of non-solvable T^c-groups and then, that of solvable T^c-groups. The notion and symbols are standard; see [3] and [5].

2. Some needed results

In this section, some basic results are given, for instance $\mathrm{T}^c(D_{2n})$ where D_{2n} is a dihedral group of order $2n$.

Proposition 2.1. *Let p be a prime and let $G = E_q : C_p$ be a Frobenius group with kernel E_q and complement C_p, respectively. Then* $\mathrm{T}^c(G) = 1 + (p-1)p + \frac{(q-1)q}{p}$.

Proof. We see that $G' = E_q$ and C_p acts fixed-point-freely on E_q, so $\mathrm{cd}(E_q : C_p) = \{1, p\}$. It follows that

$$\begin{aligned}\mathrm{T}^c(E_q : C_p) &= 1 + (|G/G'| - 1) \cdot \frac{|G : E_q|}{1} + \frac{|E_q| - 1}{|C_p|} \cdot \frac{|G : 1|}{p} \\ &= 1 + (p-1)p + \frac{(q-1)q}{p}.\end{aligned}$$

The proof is complete. □

Remark 2.2. Note that, if p is not a prime, Proposition 2.1 is not true, for instance $\mathrm{T}^c(E_7 : C_6) = 28$ by GAP [2].

Lemma 2.3. *Let G be a finite group with $\mathrm{T^c}(G) < 24$. Then G is solvable.*

Proof. Assume that the result is not true with minimal order $|G|$, then G is non-solvable but its maximal subgroups are solvable. Thus we can assume that G is a minimal non-abelian simple group and so, for all $\chi \in \mathrm{Irr}(G)\backslash\{1_G\}$, $\ker\chi = \{1\}$, that is, every non-principal irreducible character of G is faithful. Now simplicity of G shows

$$\begin{aligned}
\mathrm{T^c}(G) &= 1 + \sum_{\chi\in\mathrm{Irr}(G)\backslash\{1_G\}} \chi^c(1) \\
&= 1 + \sum_{\chi\in\mathrm{Irr}(G)\backslash\{1_G\}} \frac{|G:\ker\chi|}{\chi(1)} \\
&= 1 + \sum_{\chi\in\mathrm{Irr}(G)\backslash\{1_G\}} \frac{|G|}{\chi(1)} \\
&\geq 1 + |G|\frac{k}{\sqrt{\frac{\sum_{i=2}^{k}\chi_i^2}{k}}} \quad (\text{let } k = |\mathrm{Irr}(G) - 1_G|) \\
&= 1 + |G|\frac{k\sqrt{k}}{\sqrt{|G|-1}} \\
&\geq 1 + \frac{8|G|}{\sqrt{|G|-1}}
\end{aligned}$$

where the first inequality is obtained by using $\frac{n}{\sum_{i}^{n}\frac{1}{x_i}} \leq \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{n}}$ and the last inequality is obtained because the minimal order non-abelian simple group is A_5 that has 5 conjugacy classes. We have by hypothesis that

$$24 > 1 + \tfrac{8|G|}{\sqrt{|G|-1}} \geq \tfrac{8(\sqrt{|G|-1})^2}{\sqrt{|G|-1}} = 8\sqrt{|G|-1},$$

so $|G| \leq 10$, a contradiction as a nonsolvable group has order at least 60.

Thus the minimal contra-example does not exist, so G is solvable. □

Note that Lemma 2.3 also can be obtained from [8, Theorem 1.1] which says: a nonsolvable G satisfies $\mathrm{T^c}(G) \geq 68$ with equality if and only if $G \cong \mathrm{PSL}_2(5)$.

Proposition 2.4. *Let $G = D_{2n}$ be a dihedral group of order $2n$. Assume that the p_i's, $i = 1, 2, \cdots, s$, are odd different primes and the m_i's, $i = 1, 2, \cdots, s$, and r are non-negative integers, then*

$$\mathrm{T^c}(G) = \begin{cases} \frac{5}{2} + \frac{1}{2}\prod_{i=1}^{s}(p_i\frac{p_i^{2m_i}-1}{p_i+1}+1), & \textit{if } n = p_1^{m_1}p_2^{m_2}\cdots p_s^{m_s}, \\ \frac{7}{2} + \left(\frac{2^{2r}-1}{3}+\frac{1}{2}\right)\prod_{i=1}^{s}(p_i\frac{p_i^{2m_i}-1}{p_i+1}+1), & \textit{if } n = 2^r p_1^{m_1}\cdots p_s^{m_s}. \end{cases}$$

in particular, $\mathrm{T^c}(G) \geq \mathrm{T^c}(D_{2p}) = \frac{5}{2} + \frac{1}{2}[p(p-1)+1]$ *with* $p \geq 3$ *a prime.*

Table 1: Character table of D_{2n} with n odd [6, p. 182]

g_i	1	$a^r (1 \leq r \leq (n-1)/2)$	b
$\lvert \mathrm{C}_{D_{2n}}(g_i)\rvert$	$2n$	n	2
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $(1 \leq j \leq (n-1)/2)$	2	$\varepsilon^{jr}+\varepsilon^{-jr}$	0

In Table 1, $\varepsilon = e^{2\pi \mathrm{i}/n}$.

Proof. In light of n, we consider three cases as follows.

(1) Let $n = p^s$ with $p \geq 3$ and $s \geq 1$. By Table 1, $|\ker\chi_1| = |G|$, $|\ker\chi_2| = 1 + 2\cdot\frac{n-1}{2} = n$. Note that the number of integers from the interval $[1, (p^s-1)/2]$ that are divisible by p^k, $0 \leq k \leq s$, is equal to $\frac{p^{s-k}-1}{2}$, so the number of integers j such that $(p^s, j) = p^k$ and $1 \leq j \leq (p^s-1)/2$ is $\frac{p^{s-k}-p^{s-k-1}}{2}$, and if $(p^s, j) = p^k$, then

$$|\ker\psi_j| = 1 + 2\frac{p^{s-(s-k)}-1}{2} = p^k = (p^s, j).$$

Hence we have

$$\mathrm{T^c}(G) = 1 + 2 + \sum_{j=1}^{(p^s-1)/2}\frac{2p^s}{2(p^s,j)} = 3 + \sum_{k=0}^{s-1}\frac{p^s}{p^k}\frac{p^{s-k}-p^{s-k-1}}{2} = 3 + p\frac{p^{2s}-1}{2(p+1)}.$$

(2) Let n be an odd number and k be a divisor of n. The number of $j \in \mathbb{Z}$ dividing k with $1 \leq j \leq \frac{n-1}{2}$ is $\frac{\frac{n}{k}-1}{2}$. Assume that $(j, n) = k$. Then the number of integers $r \in [1, \frac{n-1}{2}]$ with $n \mid jr$ is equal to $\frac{k-1}{2}$. So by Table 1, $|\ker\chi_1| = |G|$, $|\ker\chi_2| = n$ and $|\ker\psi_j| = 1 + 2\cdot\frac{k-1}{2} = k = (n, j)$. This implies that

$$\mathrm{T^c}(G) = 1 + 2 + \sum_{j=1}^{(n-1)/2}\frac{2n}{2(n,j)} = 3 + \sum_{j=1}^{(n-1)/2}\frac{n}{(n,j)} = \frac{5}{2} + \frac{1}{2}\sum_{j=0}^{n-1}\frac{n}{(n,j)}.$$

Say $n = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$, where the p_i's are odd different primes and the m_i's are positive integers. Then

$$\begin{aligned}\mathrm{T}^c(G) &= \frac{5}{2} + \frac{1}{2}\sum_{j=0}^{n-1} \frac{n}{(n,j)} \\ &= \frac{5}{2} + \frac{1}{2}\sum_{j_1=0}^{p_1^{m_1}-1}\sum_{j_2=0}^{p_2^{m_2}-1}\cdots\sum_{j_s=0}^{p_s^{m_s}-1} \frac{p_1^{m_1}}{(p_1^{m_1},j_1)}\frac{p_2^{m_2}}{(p_2^{m_2},j_2)}\cdots\frac{p_s^{m_s}}{(p_s^{m_s},j_s)} \\ &= \frac{5}{2} + \frac{1}{2}\prod_{i=1}^{s}(p_i\frac{p_i^{2m_i}-1}{p_i+1}+1).\end{aligned}$$

Table 2: Character table of D_{2n}, $n = 2m$ [6, p. 183]

g_i	1	a^m	$a^r (1 \le r \le m-1)$	b	ab
$\lvert\mathrm{C}_{D_{2n}}(g_i)\rvert$	$2n$	$2n$	n	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_4	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j $(1 \le j \le m-1)$	2	$2(-1)^j$	$\varepsilon^{jr} + \varepsilon^{-jr}$	0	0

In Table 2, $\varepsilon = e^{2\pi i/n}$.

(3) Let n be an even number and write $n = 2m$. By Table 2, $|\ker\chi_1| = 2n$, $|\ker\chi_2| = 1 + 1 + (m-1)\cdot 2 = 2m = n$. In view of m, the kernels of χ_3 and χ_4 are determined by both m and r.

If m is odd, then the number of the even integers from the interval $[1, m-1]$ equals to $(m-1)/2$, and so, $|\ker\chi_3| = 1 + \frac{(m-1)}{2}\cdot 2 + m = 2m = n = |\ker\chi_4|$. If m is even, then similarly, $|\ker\chi_3| = 1 + 1 + [(m-1)/2]\cdot 2 + m = 2m = n = |\ker\chi_4|$. This implies that $|\ker\chi_3| = |\ker\chi_4| = n$.

Now we will do with the $\ker\psi_j$ for $j \in [1, m-1]$. Let k be a divisor of $2m$. Then for $1 \le i \le m-1$, the number of integers $i \in \mathbb{Z}$ divisible by k is equal to $\frac{m}{k} - 1$ if $k \mid m$ and $\frac{m}{k} - \frac{1}{2}$ if $k \nmid m$. Let $(j, 2m) = k$. If $2 \mid k$, then there are $\frac{k}{2} - 1$ r's with $r \in [1, m-1]$ and $2m|jr$. If $2 \nmid k$, then there are $\frac{k-1}{2}$ r's with $r \in [1, m-1]$ and $2m \mid jr$. Hence if $2 \mid (2m, j) = k$, then

$$|\ker\psi_j| = 1 + 1 + 2(\tfrac{k}{2} - 1) = k = (2m, j).$$

If $2 \nmid (2m, j) = k$, then

$$|\ker\psi_j| = 1 + 2\tfrac{k-1}{2} = k = (2m, j).$$

Therefore,

$$|\ker\psi_j| = (2m, j),$$

and

$$\mathrm{T^c}(G) = 1 + \frac{2n}{n} + 2\frac{2n}{n} + \sum_{j=1}^{m-1}\frac{4m}{2(2m,j)} = 5 + \sum_{j=1}^{m-1}\frac{2m}{(2m,j)} = \frac{7}{2} + \frac{1}{2}\sum_{j=0}^{n-1}\frac{n}{(n,j)}.$$

Let $n = 2^r p_1^{m_1}\cdots p_s^{m_s}$. Then

$$\begin{aligned}\mathrm{T^c}(G) &= \frac{7}{2} + \frac{1}{2}\sum_{j=0}^{n-1}\frac{n}{(n,j)}\\ &= \frac{7}{2} + \frac{1}{2}\sum_{j_0=0}^{2^r-1}\sum_{j_1=0}^{p_1^{m_1}-1}\cdots\sum_{j_s=0}^{p_s^{m_s}-1}\frac{2^r}{(2^r,j_0)}\frac{p_1^{m_1}}{(p_1^{m_1},j_1)}\cdots\frac{p_s^{m_s}}{(p_s^{m_s},j_s)}\\ &= \frac{7}{2} + \left(\frac{2^{2r}-1}{3}+\frac{1}{2}\right)\prod_{i=1}^{s}(p_i\frac{p_i^{2m_i}-1}{p_i+1}+1).\end{aligned}$$

This completes the proof. □

The following result will be used frequently.

Lemma 2.5. *Let A and B be groups. Then the following hold:*

(1) $\mathrm{T^c}(A) \le \mathrm{T^c}(A\times B)$ *with equality if and only if* $B = 1$ *and* $\mathrm{T^c}(A\times B) \le \mathrm{T^c}(A)\mathrm{T^c}(B)$ *with equality if and only if* $(|A|,|B|) = 1$.

(2) *If A is normal in B, then* $\mathrm{T^c}(B/A) \le \mathrm{T^c}(B)$.

Proof. (1) As for each $\chi \in \mathrm{Irr}(A)$, χ can be viewed as $\chi\times 1_B$ where 1_G is the principal character of a group G, so $\chi\times 1_B \in \mathrm{Irr}(A\times B)$. Now the first inequality is gotten and the equality is obtained if $B = 1$. The second inequality is obtained by [9].

(2) It follows from [5, Lemma 2.22], that a character of B/A can be viewed as a character of B, so we get the desired result. □

Remark 2.6. Note in Lemma 2.5, that for a normal subgroup N of a group G, $\mathrm{T^c}(N)$ possibly is not less than $\mathrm{T^c}(G)$, for instance, by Lemma 2.1, $\mathrm{T^c}(E_7 : C_3) = 21$, and $\mathrm{T^c}(E_7) = 1 + (7-1)\cdot 7 = 43$, so $\mathrm{T^c}(E_7) \not< \mathrm{T^c}(E_7 : C_3)$, a contradiction.

Lemma 2.7. [9, Lemma 2.4] *Let $G = C_m$ be a cyclic group of order m with $m = p_1^{s_1}p_2^{s_2}\cdots p_t^{s_t}$, $p_1 < p_2 < \cdots < p_t$ and $s_1, s_2, \cdots, s_t$ positive integers, then* $\mathrm{T^c}(G) = \prod_{i=1}^{t}\frac{p_i^{2s_i+1}+1}{p_i+1}$.

3. The proof of Theorem 1.1

In this section, we mainly give the proof of Theorem 1.1. First the structure of non-solvable $\mathrm{T^c}$-groups is given and then that of solvable $\mathrm{T^c}$-groups is determined.

Let $\max G$ be the set of maximal subgroups of a group G subject to the order divisibility.

3.1. Nonsolvable $\mathrm{T^c}$-groups. We deal with nonsolvable $\mathrm{T^c}$-groups by the following two cases: a non-abelian simple group, and a nonsolvable $\mathrm{T^c}$-group.

Lemma 3.1. *Let G be a nonabelian simple $\mathrm{T^c}$-group. Then G is isomorphic to $\mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$.*

Proof. Let $H < G$ be any maximal subgroup of G. Then from Lemma 2.3, H is solvable, and so, G itself is a non-abelian simple group but its subgroups are solvable. Now by [13], the possibilities for G are $\mathrm{PSL}_2(2^p)$ for p a prime; $\mathrm{PSL}_2(3^p)$ for p an odd prime; $\mathrm{PSL}_2(p)$, for p any prime exceeding 3 such that $p^2+1 \equiv 0 \pmod 5$; $\mathrm{Sz}(2^p)$ for p an odd prime; $\mathrm{PSL}_3(3)$. Thus the following cases will be done with.

Case 1: $\mathrm{PSL}_2(q)$.

Table 3. $\mathrm{PSL}_2(q)$, $q \geq 5$ [7, p. 191]

	$\max(\mathrm{PSL}_2(q))$	Condition
$\mathcal{C}_1$	$E_q : C_{(q-1)/k}$	$k = \gcd(q-1, 2)$
$\mathcal{C}_2$	$D_{2(q-1)/k}$	$q \notin \{5, 7, 9, 11\}$
$\mathcal{C}_3$	$D_{2(q+1)/k}$	$q \notin \{7, 9\}$
$\mathcal{C}_5$	$\mathrm{PSL}_2(q_0).(k, b)$	$q = q_0^b$, b a prime, $q_0 \neq 2$
$\mathcal{C}_6$	S_4	$q = p \equiv \pm 1 \pmod 8$
	A_4	$q = p \equiv 3, 5, 13, 27, 37 \pmod{40}$
$\mathcal{S}$	A_5	$q \equiv \pm 1 \pmod{10}, F_q = F_p[\sqrt{5}]$

Let $q = p^m$ with p a prime. Then Table 3 shows $D_{2n} \in \max \mathrm{PSL}_2(q)$ for some integer $n \geq 3$. By Proposition 2.4 and hypothesis,

$$24 > \mathrm{T^c}(D_{2n}) \geq \tfrac{5}{2} + \tfrac{1}{2}[p(p-1)+1],$$

and so $p \leq 5$. Thus we have that either $2^m - 1 \leq 5$ when q is even or $\frac{p^m-1}{2} \leq 5$ when q is odd, so

$$q = 4 \text{ if } q \text{ is even or } q = 5, 7, 9, 11 \text{ if } q \text{ is odd.}$$

We know from [2], that A_5 is a maximal subgroup of $\mathrm{PSL}_2(q)$ for $q = 9, 11$ and $\mathrm{T^c}(A_5) = 68 \nless 24$, and that $E_{13} : C_6 \in \max \mathrm{PSL}_2(13)$, and $\mathrm{T^c}(E_{13} : C_6) = 57 > 24$, a contradiction. Now we will do $q = 5, 7$. Since $\max \mathrm{PSL}_2(7) = \{S_4, E_7 : C_3\}$ (see [3, p. 6]), we have $\mathrm{T}^c(S_4) = 22 < 24$ and $\mathrm{T}^c(E_7 : C_3) = 21 < 24$. So G is isomorphic to $\mathrm{PSL}_2(7)$. It follows from $\mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$, that $\max \mathrm{PSL}_2(5) = \{A_4, D_{10}, S_3\}$, and that

$$\mathrm{T^c}(A_4) = 1 + (3-1)3 + \frac{(4-1)4}{3} = 11,$$
$$\mathrm{T^c}(D_{10}) = 1 + (2-1)2 + \frac{(5-1)5}{2} = 13,$$
$$\mathrm{T^c}(S_3) = 6,$$

and so for each maximal subgroup H of A_5, we get $\mathrm{T^c}(H) < 24$. Thus G is isomorphic to A_5, the desired result.

Case 2: $\mathrm{Sz}(2^p)$.

In this case, $D_{2(2^p-1)} \in \max \mathrm{Sz}(2^p)$, so by Lemma 2.4, $\frac{5}{2} + \frac{1}{2}(m(m-1)+1) < 24$ and $m \leq 5$. It follows that $2^p - 1 \leq 5$, so $p = 2$, but p is odd, a contradiction.

Case 3: $\mathrm{PSL}_3(3)$.

Then by [3, p. 13], $E_{13} : C_3 \in \max \mathrm{PSL}_3(3)$ and so, Lemma 2.1 forces

$$\mathrm{T^c}(E_{13} : C_3) = 1 + (3-1)\cdot 3 + \tfrac{(13-1)\cdot 13}{3} = 59 \not< 24,$$

a contradiction. □

Lemma 3.2. *Let G be a nonsolvable $\mathrm{T^c}$-group and let N be the largest solvable normal subgroup of G. Then $G/N \cong \mathrm{PSL}_2(q)$, $q = 5, 7$.*

Proof. As G is a nonsolvable $\mathrm{T^c}$-group, by definition, for each maximal subgroup M of G, $\mathrm{T^c}(M) < 24$ and so M is solvable by Lemma 2.3. Thus G is a minimal nonsolvable group. Let N be the largest solvable normal subgroup of G. Then G/N is also a $\mathrm{T^c}$-group by Lemma 2.5, and so, by Lemma 3.1, $G/N \cong \mathrm{PSL}_2(q)$ for $q = 5$ or 7. □

Theorem 3.3. *Let G be a nonsolvable $\mathrm{T^c}$-group. Then G is isomorphic to $\mathrm{PSL}_2(q)$ with $q = 5, 7$.*

Proof. If G is simple, then by Lemma 3.1, G is isomorphic to $\mathrm{PSL}_2(q)$, $q = 5, 7$. In what follows, we assume that G is not a simple group. We see that $|\mathrm{Out}(\mathrm{PSL}_2(q))| = 2$ for $q \in \{5, 7\}$ where $\mathrm{Out}(A)$ is the outer-automorphism group of a group A. Let M be a maximal subgroup of G such that $|G : M| = 2$ and M has a normal subgroup N with $M/N \cong \mathrm{PSL}_2(q)$, $q = 5$ or 7. By hypothesis and Lemma 2.5,

$$\mathrm{T^c}(\mathrm{PSL}_2(q)) < \mathrm{T^c}(M/N) \leq \mathrm{T^c}(M) < 24.$$

But $\mathrm{T^c}(A_5) = 68$, $\mathrm{T^c}(\mathrm{PSL}_2(7)) = 186$, a contradiction. It follows that $G = M$ and G is not an almost simple group as $\mathrm{PSL}_2(q)$ is a maximal subgroup of $\mathrm{PSL}_2(q).2$. In particular $G/N \cong \mathrm{PSL}_2(q)$ with N solvable; see Lemma 3.2.

As G is not an almost simple group, G is isomorphic to $\mathrm{SL}_2(q)N$. If $|N|$ is odd, then $G \cong \mathrm{SL}_2(q) \times N$ since $\gcd(|\mathrm{Z}(\mathrm{SL}_2(q))|, |N|) = 1$. Note that $|\mathrm{Z}(\mathrm{SL}_2(q))| = 2$, so $\mathrm{SL}_2(q)$ is a subgroup of G. Obviously, there is a maximal subgroup M with $\mathrm{SL}_2(q) \times K = M$ with $|N : K|$ a prime. We see by [2], that

$$\mathrm{T^c}(\mathrm{SL}_2(q)) = \begin{cases} 228, & \text{if } q = 5, \\ 508, & \text{if } q = 7. \end{cases}$$

Then hypothesis shows $\mathrm{T^c}(\mathrm{SL}_2(q)) \leq \mathrm{T^c}(M) < 24$, a contradiction. Thus $|N| = 2^a m$ with $a > 1$ and $(m, 2) = 1$, and $G \cong \mathrm{SL}(2, q) \circ N$. Let M be a maximal subgroup of G with $\mathrm{SL}(2, q) \leq M$ and let $Z = \mathrm{Z}(\mathrm{SL}(2, q))$. Then $M/Z \cong \mathrm{SL}(2, q)/Z \times LZ/Z$ with L a maximal subgroup of N, so by Lemma 2.5, $\mathrm{T^c}(\mathrm{SL}(2, q)/Z) < \mathrm{T^c}(M/Z) \leq \mathrm{T^c}(M) < 24$, a contradiction as $\mathrm{SL}(2, q)/Z \cong \mathrm{PSL}_2(q)$. Now we have $|N| = 2$ and so $G \cong \mathrm{SL}(2, q)$. Note that $\mathrm{SL}_2(3)$ and $C_2 \times E_7 : C_3$ are maximal subgroups of $\mathrm{SL}_2(5)$ and $\mathrm{SL}_2(7)$, respectively. By [2], $\mathrm{T^c}(\mathrm{SL}_2(3)) = 47$, and $\mathrm{T^c}(C_2 \times E_7 : C_3) = 63$, a contradiction to the hypothesis.

This completes the proof. □

Corollary 3.4. *Let G be a $\mathrm{T^c}$-group such that G has no subgroup isomorphic to S_3. Then G is solvable.*

Proof. Assume the result is not true, then G is a nonsolvable $\mathrm{T^c}$-group. Now by Theorem 3.3, G is isomorphic to $\mathrm{PSL}_2(5)$ or $\mathrm{PSL}_2(7)$. But by [3], $\mathrm{PSL}_2(5)$ has a subgroup S_3 and $\mathrm{PSL}_2(7)$ has a subgroup S_4 in which S_3 is contained, a contradiction. □

Note that, if the common divisors of any two different codegrees of a group G is not divisible by 4, then G is solvable [10].

3.2. Solvable $\mathrm{T^c}$-groups. In this section, we will first show the structure of nilpotent $\mathrm{T^c}$-groups by using Lemma 2.5 and then give that of non-nilpotent $\mathrm{T^c}$-groups by using the properties of $\mathrm{C}_G(N)$ and finally the proof of the main theorem is shown.

Theorem 3.5. *Let G be a nilpotent $\mathrm{T^c}$-group. Then one of the following holds:*

(1) *G is isomorphic to E_2^k with $k \leq 4$, C_4, D_8, Q_8, $M_2(2, 1, 1)$, D_{16}, SD_{16}, Q_{16}, $C_2 \times D_8$, $C_2 \times Q_8$, $D_8 \circ C_4 \cong Q_8 \circ C_4$;*
(2) *G is a group of prime order;*
(3) *G is a p-group of order p^2, $p = 3, 5$;*
(4) *G is isomorphic to $C_p \times C_q$ with $p, q \in \{2, 3, 5\}$ and $p \neq q$.*

Proof. Let p_i, $i = 1, 2, \cdots, s$ be different prime divisors of $|G|$. Then say $G = P_1 \times P_2 \times \cdots \times P_s$ where the P_i's are Sylow p_i-subgroups of G. Let M be a maximal subgroup of G.

Case 1: $s = 1$. If $M = 1$, G is a group of prime order. Thus we assume that M is non-trivial. Let $\Phi(A)$ be the Frattini subgroup of a group A. Note that $M/\Phi(M)$ is an elementary abelian p-group of order p^k, so $\mathrm{T^c}(M) \geq \mathrm{T^c}(M/\Phi(M)) = p(p^k - 1) + 1 \geq p(p-1) + 1$ as a character of $M/\Phi(M)$ can be viewed as a character of M. Hypothesis implies $p(p-1) + 1 < 24$, so $p \leq 5$. Let $|M| = p^n$.

Let G **be a 5-group**. Then for $a \geq 2$, $5^a \notin \mathrm{cd}_c(M)$ as $5^a < \mathrm{T^c}(M) \not< 24$, and so, $\mathrm{cd}_c(M) = \{1, 5\}$, and M is an elementary abelian 5-group. If $n \geq 2$, then $121 = 1 + (5^2 - 1) \cdot 5 \leq \mathrm{T^c}(M) \leq 24$, a contradiction. Therefore $n = 1$ and G is a 5-group of order at most 5^2.

Assume that G **is a 3-group**. If 3^a for some $a \geq 3$, belongs to $\mathrm{cd}_c(M)$, then by hypothesis, $27 \leq 3^a < \mathrm{T^c}(M) < 24$, a contradiction. Thus $\mathrm{cd}_c(M) = \{1, 3\}$ or $\mathrm{cd}_c(M) = \{1, 3, 3^2\}$. If $\mathrm{cd}_c(M) = \{1, 3\}$, then M is an elementary abelian 3-group. If $n \geq 2$, then $25 = 1 + (3^2 - 1) \cdot 3 \leq 1 + (3^n - 1)3 \leq \mathrm{T^c}(M) < 24$, a contradiction. Thus $n = 1$ and G is of order 9. If $\mathrm{cd}_c(M) = \{1, 3, 9\}$, then $\mathrm{T^c}(M) = 1 + 3m + 9l < 24$ for some $m, l \geq 1$, i.e., $m + 3l \leq \left[\frac{23}{3}\right] = 7$. Note that $m, l \geq 1$, so $(m, l) \in \{(1, 1), (1, 2), (2, 1), (3, 1), (4, 1)\}$. If $l = 1$, then M has only one codegree 9, and so M has only one character corresponding to such codegree. Thus the character with codegree 9 is faithful and $|M| \leq 3^2(3^2 - 1) = 72$. Recall that M is a 3-group, so $|M|$ is equal to 27 and if M is nonabelian, then by [2], M has the form 3^{1+2}, an extraspecial 3-group and $\mathrm{T^c}(M) = 1 + 8 \cdot 3 + 9 \cdot 2 = 53 < 24$, a contradiction. If M is abelian, then $M \cong C_9 \times C_3$ or C_{27}. If $M \cong C_{27}$, then $27 \in \mathrm{cd}_c(M)$, so $27 < \mathrm{T^c}(M) < 24$, a contradiction. If $M \cong C_9 \times C_3$, then by [2], $\mathrm{T^c}(M) = 1 + 3m + 9l < 24$, i.e., $m + 3l \leq 7$. Observe that M is abelian, so m and n are all > 1. It follows that the inequality $m + 3n \leq 7$ has no solution in $\mathbf{N}$, the natural number set. Similarly, if $m = 1$, then we also have $|G| \leq 3(3-1)$, but $3^2 \mid |G|$, a contradiction.

Now we will do with the case: G **is a 2-group**. We **claim that** $2^a \notin \mathrm{cd}_c(M)$ with $a \geq 3$. In fact, if not, then $1 + 2m + 4n + 8p < \mathrm{T^c}(M) < 24$ with m, n, p positive integers, i.e., $m + 2n + 4p < \left[\frac{23}{2}\right] = 11$. It follows that $p = 1, 2$. **Let** $p = 1$, then in M, there is a faithful character χ with codegree 8, so $|M| < 8 \cdot (8 - 1) = 56$. Now $|M| \in \{32, 16, 8\}$. Let $|M| = 8$. Since $8 \in \mathrm{cd}_c(M)$, we obtain $M \cong C_8$, and so by Lemma 2.7, $\mathrm{T^c}(C_8) = \frac{2^{2 \cdot 3 + 1} + 1}{2 + 1} = 43 \not< 24$, a contradiction. If $|M| = 16$, then since M has a faithful character with codegree 8, $2 \in \mathrm{cd}(M)$ and the nilpotence of M

shows that $2^2 \mid |M/M'|$, and so M has 4 linear characters and now, $1+m+n>4$ and $m+2n \leq 6$. It follows that $(m,n)=(1,2)$ or $(m,n)=(2,1)$, so in both cases, there is a faithful character χ with codegree 2 or 4, so $|G|<2$ or $|G|<12$. Order consideration rules out. Now consider $|M|=32$. Similarly, we obtain $4 \in \mathrm{cd}(M)$ as the existence of faithful character with codegree 8, and because of the nilpotence of M, $4^2 \mid |M/M'|$, so M has 16 linear characters, but $16 < m+n < m+2n \leq 6$, a contradiction. **Let** $p=2$. Then $3 > m+2n > 2$, we have no solution in natural number set $\mathbf{N}$, a contradiction. **This proves the claim**. Thus $\mathrm{cd}_c(M)=\{1,2\}$, or $\{1,2,4\}$.

Let $\mathrm{cd}_c(M)=\{1,2,4\}$. Then by [4, Theorem A], M is isomorphic to $C_4 \times C_2$, Q_8 or D_8. If $M \cong C_4 \times C_2$, then G is isomorphic to $C_4 \times C_4$, $C_4 \times E_2^2$, $D_8 \times C_2$ or $Q_8 \times C_2$. If G has a maximal subgroup $M \cong D_8$ or Q_8, then by [15, Theorem 2.4.1(C)], G is isomorphic to

$$D_{16},\ SD_{16},\ Q_{16},\ C_2 \times D_8,\ C_2 \times Q_8,\ D_8 \circ C_4 \cong Q_8 \circ C_4.$$

Let $\mathrm{cd}_c(M)=\{1,2\}$. Then $M \cong E_2^k$ for certain integer $k \geq 1$. If $k \geq 4$, then

$$31 = 1+(2^4-1)\cdot 2 \leq 1+(2^k-1)\cdot 2 = \mathrm{T^c}(M) < 24,$$

a contradiction. If $k \leq 3$, then $24 > \mathrm{T^c}(M) = 1+(2^k-1)\cdot 2 \geq 1+(2^3-1)\cdot 2 = 15$, and $M \cong E_2^k$ with $k=1,2,3$. It follows from [15, Theorem 2.4.1(B)] and [2], that G is isomorphic to E_2^l with $l=1,2,3,4$, C_4 or $M_2(2,1,1)$.

Case 2: $s=2$. Then $G \cong P_1 \times P_2$. From the proof of Case 1, we have that

$$(p_1,p_2)=(2,3),(2,5) \text{ or } (3,5)$$

and that, if $(p_1,p_2)=(2,3)$ or $(2,5)$, then $|P_1| \leq 8$ and $|P_2|=p$ with $p=3,5$ and that, if $(p_1,p_2)=(3,5)$, then $|P_1|=p_1$, and $|P_2|=p_2$.

Let $(p_1,p_2)=(2,3)$ or $(2,5)$. Then $M \cong N \times P_2$ or P_1 where the N is a maximal subgroup of P_1 and $|P_1| \leq 8$ (See **Case 1**). Note that $|P_2|=3$ or 5, so we need to consider only $p_1=2$. If $|P_1|=8$, then $P_1 \cong C_8$, $C_4 \times C_2$, E_2^3, Q_8 or D_8; see [15, Example 2.1.7] for instance. We know from Lemmas 2.7 and 2.5(1), that $\mathrm{T^c}(C_8)=43$, $\mathrm{T^c}(C_4 \times C_3)=77$ and $\mathrm{T^c}(C_4 \times C_5)=231$, $\mathrm{T^c}(E_2^2 \times C_3)=49$ and $\mathrm{T^c}(E_2^2 \times C_5)=107$, so we from Lemma 2.5, have that for any $M \in \max P_1$, $\mathrm{T^c}(M) \not< 24$ when $|P_1|=8$, a contradiction. If $|P_1|=4$, then $P_1 \cong C_4$ or E_2^2 and so, $M \cong C_2 \times C_3$ or $M \cong C_2 \times C_5$. Note from Lemma 2.5, that, $\mathrm{T^c}(C_2 \times C_3)=21<24$, $\mathrm{T^c}(C_2 \times C_5)=63 \not< 24$ and so the nilpotence of G forces that $G \cong C_4 \times C_3$ or $E_2^2 \times C_3$. If $|P_1|=2$, then $G \cong C_2 \times C_3$ or $G \cong C_2 \times C_5$.

Let $(p_1, p_2) = (3, 5)$. Then using Lemma 2.5, $\mathrm{T^c}(C_3 \times C_5) = 147$, but $\mathrm{T^c}(C_3) = 1 + (3-1) \cdot 3 = 7 < 24$, $\mathrm{T^c}(C_5) = 21 < 24$, so $G \cong C_3 \times C_5$.

Case 3: $s \geq 3$. Then for each $i > 1$, $|P_i| = p_i$ by Case 1. Let L be a maximal subgroup of P_1. Note that $2 \leq p_1 < p_2 < \cdots < p_t$, and

$$\mathrm{T^c}(C_{p_2} \times C_{p_3}) \leq \mathrm{T^c}(L \times C_{p_2} \times C_{p_3} \times \cdots \times C_{p_s}) = \mathrm{T^c}(M) < 24$$

by hypothesis and Lemma 2.5(1). But we see from Lemmas 2.5(1) and 2.7, that

$$\begin{aligned}\mathrm{T^c}(C_{p_2} \times C_{p_3}) &= (1 + p_2(p_2 - 1))(1 + p_3(p_3 - 1)) \\ &= 1 + p_2p_3(p_2 - 1)(p_3 - 1) + p_2(p_2 - 1) + p_3(p_3 - 1) \\ &> 1 + p_2^2(p_2 - 1)^2 + 2p_2(p_2 - 1) \\ &= (1 + p_2(p_2 - 1))^2 \geq (1 + 3 \cdot (3 - 1))^2 = 49,\end{aligned}$$

so, $49 \leq \mathrm{T^c}(M) < 24$, a contradiction. □

Lemma 3.6. *Let N be a minimal normal subgroup of a non-nilpotent solvable* $\mathrm{T^c}$-*group G. Then the order of N is bounded, in particular, $|N| \leq 8$ when $p = 2$, and otherwise $|N| = p$ when $p = 3, 5, 7$.*

Proof. The two facts that the minimality of N and the solvability of G imply that N is an elementary abelian p-group for certain prime p. As G is non-nilpotent, we have $N < G$ and in particular, N is contained in some maximal subgroup M of G. Assume that $|N| = p^n$ for certain integer $n \geq 1$.

Case 1: N is a maximal subgroup. Then $N = M$ and $\mathrm{T^c}(N) < 24$. If $p \geq 7$, then by Lemmas 2.5 and 2.7,

$$\begin{aligned}43 = 1 + 7 \cdot (7 - 1) \leq \mathrm{T^c}(C_p) &\leq \mathrm{T^c}(C_p \times C_p \cdots \times C_p) \\ &= \mathrm{T^c}(N) < 24,\end{aligned}$$

a contradiction. It follows that the possibilities for p are 2, 3, and 5.

If $p = 2$, then $1 + 2 \cdot (2^n - 1) = \mathrm{T^c}(E_{2^n}) = \mathrm{T^c}(N) < 24$, so $n \leq 3$; if $p = 3$, then $1 + 3 \cdot (3^n - 1) = \mathrm{T^c}(E_{3^n}) = \mathrm{T^c}(N) < 24$, so $n = 1$; if $p = 5$, then $1 + 5 \cdot (5^n - 1) = \mathrm{T^c}(E_{5^n}) = \mathrm{T^c}(N) < 24$, so $n = 1$. It follows that $|N|$ is bounded, the desired result.

Case 2: N is not a maximal subgroup of G. Then $N < M$. If M is nilpotent, then by Case 1, we have the result. Thus now we think that M is non-nilpotent. In order to show the bound of N, we think that it is maximal in M and so, M/N is of prime order and by hypothesis, $\mathrm{T^c}(M/N) \leq 24$. If $M/N \cong C_q$ with $q \geq 7$, a prime, then $\mathrm{T^c}(M/N) = 1 + q(q - 1) < 24$ shows $q \leq 5$, a contradiction. Hence $q = |M/N|$ equals to 2, 3, or 5. By NC theorem, $M/\mathrm{C}_M(N)$ is isomorphic to a subgroup of $\mathrm{GL}_n(p)$. Observe that N is maximal in M and M is non-nilpotent, so $\mathrm{C}_M(N) = N$. Thus M is a Frobenius group.

Subcase 1: $p = 2$. Then $q = 3$ or 5 and so, $M/\mathrm{C}_M(N) = M/N \cong C_3$ or $M/\mathrm{C}_M(N) = M/N \cong C_5$. Then $24 > \mathrm{T^c}(M) \geq 1 + q|M| + 2q$, i.e.,

$$(2^{n-1} + 1)q \leq 11.$$

It follows from nonnilpotence of M that $|N| \geq 4$, and so, $(q, n) = (3, 2)$.

Subcase 2: $p = 3$. Then $q = 2, 5$ and so, $M/N \cong C_2$ or $M/N \cong C_5$. Hence $24 > \mathrm{T^c}(M) = 1 + q|N| + 3q$ and so, $(q, n) = (2, 2)$ or $(2, 1)$. If $n = 2$, then $M \cong E_3^2 : C_2$ and $\mathrm{T^c}(M) = 1 + 3 \cdot 2 + 2 \cdot 3^2 > 24$, a contradiction. Thus $n = 1$ is possible.

Subcase 3: $p = 5$. Then similarly we obtain $q = 2, 3$ and, $24 > \mathrm{T^c}(M) = 1 + q|N| + 5q$. It means $(q, n) = (2, 1)$.

Subcase 4: $p \geq 7$. Then $q = 2, 3, 5$ and so, by Lemma 2.1,

$$\mathrm{T^c}(M) = 1 + q(q-1) + \tfrac{p^n-1}{q}p^n < 24.$$

It follows that $p = 7$, $q = 3$, and $n = 1$.

It follows from above four subcases, that the result is true. □

Theorem 3.7. *Let G be a non-nilpotent solvable $\mathrm{T^c}$-group. Then G is isomorphic to D_{2p} with $p \in \{3, 5\}$, $C_2 \times S_3$, A_4, $E_{2^3} : C_3$, $A_4 \times C_2$, $\mathrm{SL}_2(3)$, S_4 or $C_5 : C_4$.*

Let p be a prime divisor of a positive integer n, then denote by n_p, the p-part of n, i.e., $n_p \mid n$ but $pn_p \nmid n$.

Proof. Let N be a minimal normal subgroup of G and M be a maximal subgroup of G. Then the solvability of G shows that N is an elementary abelian p-group and $N < G$ (otherwise G is nilpotent). If $N \leq M$, then by hypothesis, $\mathrm{T^c}(M/N) \leq \mathrm{T^c}(M) < 24$. We know that N is nilpotent, so by Lemma 3.6,

$$|N| \begin{cases} \leq 8, & \text{if } p = 2, \\ = p, & \text{if } p \in \{3, 5, 7\}. \end{cases} \tag{1}$$

If $p \in \{3, 5, 7\}$, then by NC theorem, $\frac{G}{\mathrm{C}_G(N)}$ is isomorphic to a subgroup of $\mathrm{Aut}(N) \cong C_{p-1}$ where $\mathrm{Aut}(A)$ denotes the automorphism group of a group A.

Case 1: p is odd.

Then $p = 3$, 5, or 7 and N is a Sylow p-subgroup of G with order p. If $G = \mathrm{N}_G(N) = \mathrm{C}_G(N)$, then $N \leq \mathrm{Z}(\mathrm{C}_G(N))$, so G has a normal p-complement K, in particular, $G = N \times K$ since $N \trianglelefteq G$. As G is non-nilpotent, K is non-nilpotent, in particular, K is non-abelian and so, by Lemma 3.6,

$$|K| \mid 8p_1p_2 \text{ with } p_1, p_2 \in \{3,5,7\}\backslash\{p\}.$$

It follows that G has a maximal subgroup $L := C_p \times M$ where M is a maximal subgroup of K. If $|M| = 2$, then $\mathrm{T^c}(L) = (1 + p(p-1))(1 + 2) < 24$ and so, $p = 3$; if $|M| = 4$, then

$$\begin{cases} \mathrm{T^c}(L) = (1 + p(p-1))(1 + 2(2^2-1)) < 24, & \text{if } M \cong E_{2^2}, \\ \mathrm{T^c}(L) = (1 + p(p-1)) \cdot \frac{2^5+1}{2+1} < 24, & \text{if } M \cong C_{2^2}, \end{cases} \tag{2}$$

but the equation (2) has no solution since $p \geq 3$; if $|M| = 3$, 5, or 7, then by Lemma 2.5,

$$\mathrm{T^c}(L) = \mathrm{T^c}(C_p \times C_{p_i}) = (1+p(p-1))(1+p_i(p_i-1)) \geq (1+3(3-1))(1+5(5-1)) > 24,$$

a contradiction. It also means that $p_i \nmid |M|$, and so, K must be a 2-group. Thus G is nilpotent, a contradiction.

Thus in what follows, we assume that $\mathrm{C}_G(N) < G$ and three subcases will be considered.

Subcase 1a: $p = 7$. Then because of the non-nilpotence of G, we obtain that $G/\mathrm{C}_G(N) \cong C_2$ or C_3 or C_6.

Let $G/\mathrm{C}_G(N) \cong C_2$, then $M = \mathrm{C}_G(N)$. If $\mathrm{C}_G(N) = N = M$, we rule out this case as $\mathrm{T^c}(M) = \mathrm{T^c}(N) = \mathrm{T^c}(C_7) = 43$. Hence $M = \mathrm{C}_G(N) > N$. Let L be a maximal subgroup of G such that $H > N$ be a maximal subgroup of M and $L > C_2$. Observe that $H > N$, so we suppose that H/N is a prime, in particular $H/N = 2$ or 3. It follows that L is of order 28 or 42 and so by [2], L isomorphic to $D_{14} \times C_2$, $C_{14} \times C_2$, $D_{14} \times C_3$, $C_2 \times (C_7 : C_3)$, $C_7 : C_6$ or C_{42}, a contradiction since in all these cases, $\mathrm{T^c}(L) > 24$. Similarly we can exclude the case: $G/\mathrm{C}_G(N) \cong C_3$.

Let $G/\mathrm{C}_G(N) \cong C_6$. Let $\mathrm{C}_G(N) = M > N$. Let $L \in \max G$ such that $N \leq L$ and $C_6 < L$, hence $L = HC_6$ where $H \geq N$ is some maximal subgroup of M. By the similar arguments as Subcases 1a, we have $\mathrm{T^c}(L) = \mathrm{T^c}(HC_6) > 24$, a contradiction. If $\mathrm{C}_G(N) = M = N$, then as G is non-nilpotent, G is isomorphic to $C_7 : C_6$. Then $D_{14} \in \max G$ and by Lemma 2.4, $\mathrm{T^c}(D_{14}) = 24$, a contradiction. Let $\mathrm{C}_G(N) < M < G$. Then we also can show that $N = \mathrm{C}_G(N)$. Thus $|M/N| = 2$ or 3. Non-nilpotence of G implies that $G \cong C_7 : C_6$ and we rule out similarly.

Subcase 1b: $p = 5$. Then $G/\mathrm{C}_G(N) \cong C_2$ or C_4.

- If $G/\mathrm{C}_G(N) \cong C_2$, then $M = \mathrm{C}_G(N)$. If $N = \mathrm{C}_G(N)$, then $G \cong C_5 : C_2$, the desired result. Now we let $\mathrm{C}_G(N) > N$, then by (1), we think that $|M/N| = |\mathrm{C}_G(N)/N| = 2$ or 3. It follows that $M \cong C_5 : C_2$, $C_5 \times C_2$ or $C_5 \times C_3$. If $M \cong C_5 \times C_3$ or $C_5 \times C_2$, then by Lemma 2.5, $\mathrm{T^c}(M) > 24$, a contradiction. If $M \cong C_5 : C_2$, then $G \cong C_5 : C_4$ by hypothesis.

• Let $G/\mathrm{C}_G(N) \cong C_4$. If $\mathrm{C}_G(N) < M$, then we have similarly that $\mathrm{C}_G(N) = N$ and $|M/N| = 2$. Thus $|G| = 20$ and the non-nilpotence of G forces that G is isomorphic to D_{20}, $C_2 \times D_{10}$, or $C_5 : C_4$. If $G \cong D_{20}$, $C_2 \times D_{10}$, then G has a maximal subgroup C_{10} but $\mathrm{T}^{\mathrm{c}}(C_{10}) = \mathrm{T}^{\mathrm{c}}(C_2)\mathrm{T}^{\mathrm{c}}(C_5) > 24$, a contradiction. If $G \cong C_5 : C_4$, then $\max G = \{D_{10}, C_5\}$, and $\mathrm{T}^{\mathrm{c}}(M) < 24$ for all $M \in \max G$, so G is a T^{c}-group.

Subcase 1c: $p = 3$. Then $G/\mathrm{C}_G(N) \cong C_2$ and $M \cong \mathrm{C}_G(N)$. If $\mathrm{C}_G(N) > N$, then by Subcase 1a above, we can suppose $|\mathrm{C}_G(N)/N| = 2$. It follows that G is isomorphic to $C_2 \times S_3$. Now we remain to consider the case $\mathrm{C}_G(N) = N$. Observe that N is a 3-group and $|N| = 3$, then $|G| = 6$ and $G \cong C_3 : C_2 \cong S_3$.

Case 2: $p = 2$. Then by (1), $|N| \leq 8$. And from Theorem 3.5, we have $|G|_2 \leq 8$. If $|G|_2 = |N|$, $G = \mathrm{N}_G(N) = \mathrm{C}_G(N)$, then G has normal 2-complement K, and so $G = N \times K$. Note that K is of odd order and K is non-abelian, in particular, by Lemma 3.6, $K \cong C_7 : C_3$. It follows that $L \cong N \times C_7 \in \max G$, but $\mathrm{T}^{\mathrm{c}}(L) > 24$ by Lemmas 2.5 and 2.7. We will consider three subcases under the condition: $G > \mathrm{C}_G(N)$.

Subcase 2a: $|N| = 8$. Then N is a Sylow 2-subgroup of G by (1), and $N \cong E_{2^3}$ as N is elementary abelian. We see that $G/\mathrm{C}_G(N)$ is isomorphic to a subgroup of $\mathrm{GL}(3,2)$ and that $K := G/\mathrm{C}_G(N)$ is of odd order, so $|K|$ is of order 3, or 7 and $M = \mathrm{C}_G(N)$. If $M > N$, then M/N is of odd order and so, $|M/N| = 3$ (if $|M/N| = 7$, we have $43 = \mathrm{T}^{\mathrm{c}}(M/N) \leq \mathrm{T}^{\mathrm{c}}(M) < 24$; if $|M/N| = 5$, then by Sylow's theorem, $M \cong N \times C_5$ and so by Lemma 2.5, $\mathrm{T}^{\mathrm{c}}(M) > 24$). Thus $|M| = 24$ and $M \cong E_{2^3} \times C_3$ or $E_{2^3} : C_3$. Observe that G/M is of odd order, then G/M has order 3 or 5, and so G has a maximal subgroup L isomorphic to E_{3^2} or $C_5 \times C_3$, but $\mathrm{T}^{\mathrm{c}}(L) > 24$, a contradiction. Hence $M = N$. Note that G is non-nilpotent and solvable, so G is probably isomorphic to $E_2^3 : C_3$. As A_4 is a maximal subgroup of $E_2^3 : C_3$ and $\mathrm{T}^{\mathrm{c}}(A_4) = 11$, we get $G \cong E_2^3 : C_3$, the wanted result.

Subcase 2b: $|N| = 4$, then $N \cong E_{2^2}$ and so, by NC theorem, $G/\mathrm{C}_G(N)$ is isomorphic to a subgroup of $\mathrm{GL}_2(2)$ and $N \leq \mathrm{C}_G(N)$.

If $|G|_2 = |N| = 4$, then $G/\mathrm{C}_G(N) \cong C_3$, and so, $M = \mathrm{C}_G(N)$. Let $H \geq N$ be a maximal subgroup of M. Then H/N has odd order, so we assume that H/N is of order 3. Then the non-nilpotence of G implies that $G \cong A_4 \times C_3$, but $E_{3^2} \in \max G$ and $\mathrm{T}^{\mathrm{c}}(E_{3^2}) > 24$, a contradiction. Hence $\mathrm{C}_G(N) = N$ and $G \cong A_4$, the desired result.

Now let $|G|_2 = 8$. Then $G/\mathrm{C}_G(N) \cong C_3$ or C_2.

Let $G/\mathrm{C}_G(N) \cong C_3$. Then $M = \mathrm{C}_G(N)$ has a Sylow 2-subgroup P of the form E_{2^3}, $C_4 \times C_2$, Q_8, or D_8 and $P \leq M$.

• If $P = M$, then G is isomorphic to $E_{2^3} : C_3$, $Q_8 : C_3$, or $C_2 \times A_4$. If $G \cong Q_8 : C_3$, then $M := C_4 \times C_3 \in \max G$ and $\mathrm{T^c}(M) > 24$, a contradiction. If G is isomorphic to $E_{2^3} : C_3$, then $\max G = \{E_{2^3}, A_4\}$. We know that $\mathrm{T^c}(E_{2^3}) = 15$, and $\mathrm{T^c}(A_4) = 11$, so $G \cong E_{2^3} : C_3$, the desired result. If $G \cong C_2 \times A_4$, then $\max G = \{A_4, E_{2^3}, C_6\}$ implies that $G \cong C_2 \times A_4$ is a $\mathrm{T^c}$-group.

• If $P < M$, then $|M : P| = 3$, or 5 and so $M \cong P \times C_3$, $P \times C_5$ or $P : C_3$. If $M \cong P \times C_3$, $P \times C_5$, then Lemma 2.5 shows that $\mathrm{T^c}(M) > 24$, a contradiction. If $M \cong P : C_3$ where $P = E_{2^3}$, Q_8, or $C_4 \times C_2$, then by [2], we get that $\mathrm{T^c}(C_2 \times A_4) = 33$ and $\mathrm{T^c}(S_4) = 22$. It follows that $M \cong Q : C_3 \cong S_4$ and $G \cong C_3 \times S_4$ as $\mathrm{Aut}(S_4) = S_4$ and the Schur multiplier of S_4 is trivial. Thus $E_{3^2} \in \max G$, but $\mathrm{T^c}(E_{3^2}) > 24$, a contradiction.

Now let $G/\mathrm{C}_G(N) \cong C_2$, then $M = \mathrm{C}_G(N)$, and $N \trianglelefteq \mathrm{C}_G(N)$. By hypothesis, we can assume that M/N is of odd order, so $M/N \cong C_3$ or C_5. It means that $M \cong A_4$ or $E_{2^2} \times C_5$. If $M \cong A_4$, then $G \cong C_2 \times A_4$ or S_4. As $G \cong C_2 \times A_4$ is considered above, we need to deal with $G \cong S_4$. Since $\max S_4 = \{A_4, S_3, D_8\}$ and $\mathrm{T^c}(M) < 24$ for $M \in \max S_4$, we get $G \cong S_4$, the desired result. If $M \cong E_{2^2} \times C_5$, then $\mathrm{T^c}(M) > 24$, a contradiction.

Subcase 2c: $|N| = 2$. Then $G = \mathrm{N}_G(N) = \mathrm{C}_G(N)$, and so, $N \leq \mathrm{Z}(G)$. Obviously, the Sylow 2-subgroup P of G is of order ≤ 8 and $\mathrm{C}_G(N) \geq N$.

If N is a Sylow subgroup, then we rule out as above arguments.

If $|P| = 4$, then $P \cong E_{2^2}$ or C_4. Let $M \in \max G$ with $N \leq M$ and $P \leq M$. If $P < M$, then by Sylow's theorem, $[M : P] = 3$, and so, $M \cong A_4$, $C_2 \times S_3$, $E_{2^2} \times C_3$, or $C_4 \times C_3$. If $M \cong C_4 \times C_3$ or $E_{2^2} \times C_3$, we have $\mathrm{T^c}(M) > 24$. If $M \cong A_4$, or $C_2 \times S_3$, then $[G : M] = 3$. Now G has a maximal subgroup L isomorphic to E_{3^2}, C_9 or $C_3 \times S_3$, but $\mathrm{T^c}(L) > 24$, a contradiction.

Let $|P| = 8$. Let M be a maximal subgroup with $P \leq M$. If $P < M$, then Sylow's theorem shows that the cardinality of $M : P$ is 3 and so, M is isomorphic to $\mathrm{SL}_2(3)$, $C_2 \times A_4$, $D_8 \times C_3$, $Q_8 \times C_3$, or S_4. If M is isomorphic to $\mathrm{SL}_2(3)$, $D_8 \times C_3$, $Q_8 \times C_3$, then by Lemma 2.5 and [2], $\mathrm{T^c}(M) > 24$, a contradiction. If $M \cong C_2 \times A_4$ or S_4, then $[G : M] = 3$ as $|M|_2 = |G|_2$ and Sylow's theorem. It means that G is isomorphic to $C_6 \times A_4$, $S_3 \times A_4$, $C_3 \times S_4$ as $\mathrm{Aut}(S_4) = S_4$ and the Schur multiplier of S_4 is trivial. Thus G has a maximal subgroup K of the form: $C_6 \times C_3$, $S_3 \times C_3$, $C_3 \times A_4$, but by [2], $\mathrm{T^c}(K) > 24$, a contradiction. Therefore $M = P$, and $[G : M] = 3$. Since G is non-nilpotent, we get that G is isomorphic to $\mathrm{SL}_2(3)$, or $C_2 \times A_4$. Observe that $\max \mathrm{SL}_2(3) = \{Q_8, S_3\}$ and $\mathrm{T^c}(M) < 24$ for $M \in \max \mathrm{SL}_2(3)$. Thus $G \cong \mathrm{SL}_2(3)$ or $C_2 \times A_4$, the desired result. □

Now we will prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Theorems 3.3, 3.5 and 3.7. □

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