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FINITE GROUPS ADMITTING MAXIMAL SUBGROUP SERIES WITH CERTAIN NORMALITY

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Abstract. Let G be a finite group and H be a subgroup of G. Then H is said to be S-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G. H is said to be c-c-permutable in G if for each subgroup A of G, there exists an element $g \in \langle A, H\rangle$ such that $AH^g = H^gA$. H is said to be an SS-quasinormal subgroup of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. A subgroup series $\Omega : G = G_0 > G_1 > \cdots > G_i > \cdots > G_{n-1} > G_n = 1$ is said to be a maximal subgroup series of G if G_i is a maximal subgroup of G_{i-1} for each $i \in \{1, 2, \ldots, n\}$. In this paper, we first prove that G is supersolvable if and only if G possesses subnormal maximal series Ω such that either G_i is S-quasinormally embedded in G, or G_i is SS-quasinormal in G for each $i \in \{1, 2, \ldots, n\}$. Second, we prove that if G possesses a maximal subgroup series Ω such that either G_i is c-c-permutable in G, or G_i is SS-quasinormal in G, then G is solvable.

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. As a fundamental embedding property in finite group theory, the normality of subgroups has long been recognized as pivotal. This significance has spurred research aimed at weakening the restrictive condition of normality while preserving key structural features. A notable characteristic of normal subgroups lies in their permutability with all other subgroups of the group: specifically, if N is a normal subgroup of G, then $NH = HN$ for every subgroup H of G. Motivated by this

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permutability property, a natural generalization emerges: a subgroup H of a group G is said to be quasinormal (or permutable) in G if it satisfies $HK = KH$ for all $K \leq G$ (see [3]). In 1962, Kegel [8] generalized the concept of quasinormal subgroup to the S-quasinormal subgroup (or S-permutable subgroup): a subgroup H of G is said to be S-quasinormal in G if H is permutable with all Sylow subgroups of G. In 1998, Ballester-Bolinches and Pedraza-Aguilera [1] extended those concepts to S-quasinormally embedded subgroups. A subgroup H of G is said to be S-quasinormally embedded in G if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some S-quasinormal subgroup of G. In 2005, Guo, Shum and Skiba [5] introduced the concept of completely conditional permutable (abbreviated as c-c-permutable) subgroup. A subgroup H of G is called completely conditional permutable (abbreviated as c-c-permutable) in G if for each subgroup A of G, there exists an element $g \in \langle A, H\rangle$ such that $AH^g = H^gA$.

In 2008, Li [10] studied another generalization of S-quasinormal subgroup in a new way. Recall that a supplement of H to G is a subgroup B such that $G = HB$. There is at least one such supplement for every subgroup, for instance, let $B = G$. Based on the above concepts, Li [10] gave a new generalization of S-quasinormal subgroup to SS-quasinormal subgroup: A subgroup H of G is said to be an SS-quasinormal subgroup (supplement-Sylow-quasinormal subgroup) of G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B.

On the other hand, the relationship between the properties of maximal subgroups of a finite group G and the structure of G has been studied extensively. It is well known that a finite group G is nilpotent if and only if every maximal subgroup of G is normal in G. Huppert's well known theorem shows that a finite group G is supersolvable if and only if every maximal subgroup of G has prime index in G. Furthermore, let

$$\Omega : G = G_0 > G_1 > \cdots > G_i > \cdots > G_{n-1} > G_n = 1$$

be a maximal subgroup series of G, meaning that G_i is a maximal subgroup of G_{i-1} for every $i = 1, \ldots, n$. The structure of G can be investigated by under the assumption that all G_i have well-behaved properties. For example, the series Ω is said to be central in G if $[G, G_{i-1}] \leq G_i$ for every $i = 1, \ldots, n$; and it is said to be normal (or subnormal) in G if all G_i are normal (or subnormal) in G. The following results are well known:

(1) G is nilpotent if and only if G possesses a maximal subgroup series that is central in G.

(2) G is supersolvable if and only if G possesses a maximal subgroup series that is normal in G.

(3) G is solvable if and only if G possesses a maximal subgroup series that is subnormal in G.

Recently, Qian and Tang [12] studied the finite groups G that admit an S-quasinormal (c-c-permutable, resp.) maximal subgroup series, i.e., a series $G = G_0 > G_1 > \cdots > G_i > \cdots > G_{n-1} > G_n = 1$ where all G_i are S-permutable (c-c-permutable, resp.) in G. They proved that G is supersolvable if and only if G possesses an S-quasinormal (c-c-permutable, resp.) maximal subgroup series.

Meng and Lu [11] investigated finite groups G that admit an SS-quasinormal maximal subgroup series, i.e., a series $G = G_0 > G_1 > \cdots > G_i > \cdots > G_{n-1} > G_n = 1$ where all G_i are SS-quasinormal in G. They showed that if G possesses an SS-quasinormal maximal subgroup series, then G is solvable. Furthermore, G is supersolvable if and only if G possesses an SS-quasinormal maximal subgroup series which is subnormal in G.

In the light of the above investigations, it seems meaningful to investigate finite groups which possess an S-quasinormally embedded maximal subgroup series. However, such groups need not be solvable, we have the following example:

Example 1.1. Let $G = PSL(2, 7)$. Then G has a maximal subgroup of order 21, say G_1. Let G_2 be a subgroup of G_1 of order 3. Obviously, both G_1 and G_2 are Hall subgroups of G. In particular, they are S-quasinormally embedded in G. So G possesses an S-quasinormally embedded maximal subgroup series: $G = G_0 > G_1 > G_2 > G_3 = 1$. But, G is not solvable.

Inspired by Example 1.1, after checking many examples, it seems reasonable to conjecture that if G possesses an S-quasinormally embedded maximal subgroup series $G = G_0 > G_1 > \cdots > G_i > \cdots > G_{n-1} > G_n = 1$ such that $|G_{i-1} : G_i|$ is a prime for each $i \in \{1, 2, \ldots, n\}$, then G is solvable. However, we cannot yet prove it in this paper.

Furthermore, the following example shows that a solvable group which possesses an S-quasinormally embedded maximal subgroup series need not be supersolvable.

Example 1.2. Let $G = A_4$ and H be a subgroup of G of order 3. Then

$$\Omega_1 : G > H > 1$$

is an S-quasinormally embedded maximal subgroup series of G. However, G is not supersolvable. On the other hand, let K be a subgroup of G of order 4 and L be any subgroup of K of order 2. Then

$$\Omega_2 : G > K > L > 1$$

is a subnormal maximal subgroup series of G. But Ω_2 is not S-quasinormally embedded in G.

Observe that every S-quasinormal subgroup of G is SS-quasinormal and S-quasinormally embedded in G. In general, an SS-quasinormal subgroup need not be S-quasinormally embedded. Conversely, an S-quasinormally embedded subgroup need not be SS-quasinormal too. In fact, there is no inclusion-relationship between the two concepts (see [9,10]). So the first aim of this paper is to investigate the finite groups G that admit a subnormal maximal subgroup series:

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_i \rhd \cdots \rhd G_{n-1} \rhd G_n = 1$$

such that either G_i is S-quasinormal embedded in G, or SS-quasinormal in G for each $i \in \{1, \ldots, n\}$.

Theorem 1.3. *Let G be a finite group. Then G is supersolvable if and only if G possesses a subnormal maximal subgroup series:*

$$G = G_0 \rhd G_1 \rhd \cdots \rhd G_i \rhd \cdots \rhd G_{n-1} \rhd G_n = 1$$

such that either G_i is S-quasinormally embedded in G, or G_i is SS-quasinormal in G for each $i \in \{1, \ldots, n\}$.

By Theorem 1.3, the following two corollaries are immediate.

Corollary 1.4. [11, Theorem 1.4] *Let G be a finite group. Then G is supersolvable if and only if G possesses a subnormal maximal subgroup series which is SS-quasinormal in G.*

Corollary 1.5. *Let G be a finite group. Then G is supersolvable if and only if G possesses a subnormal maximal subgroup series which is S-quasinormal embedded in G.*

Moreover, note that if G possesses a c-c-permutable (SS-quasinormal, resp.) maximal subgroup series, then G is solvable (see [11,12]). So the second aim of this paper is to prove the following results.

Theorem 1.6. *Let G be a finite group. If G possesses a maximal subgroup series:*

$$G = G_0 > G_1 > \cdots > G_i \cdots > G_{n-1} > G_n = 1$$

such that either G_i is c-c-permutable in G, or G_i is SS-quasinormal in G, then G is solvable.

Applying Theorem 1.6, we can obtain the following two corollaries.

Corollary 1.7. [12, Theorem 1.2] *If G possesses a c-c-permutable maximal subgroups series, then G is solvable.*

Corollary 1.8. [11, Theorem 1.2] *If G possesses an SS-quasinormal maximal subgroups series, then G is solvable.*

All unexplained notations and terminologies are standard and can be found in [4,7].

2. Preliminaries

In this section, we collect some results which will be used in the proof of the main results.

Lemma 2.1. [1, Lemma 1] *Suppose that H is an S-quasinormally embedded subgroup of G, $K \leq G$ and N is a normal subgroup of G. Then, we have the following:*

(1) *If $H \leq K$, then H is an S-quasinormally embedded subgroup of K.*
(2) *HN/N is an S-quasinormally embedded subgroup of G/N.*

Lemma 2.2. [10, Lemma 2.1] *Suppose that H is an SS-quasinormal subgroup of G, $K \leq G$ and N is a normal subgroup of G. Then, we have the following:*

(1) *If $H \leq K$, then H is an SS-quasinormal subgroup of K.*
(2) *HN/N is an SS-quasinormal subgroup of G/N.*

Lemma 2.3. [10, Lemma 2.2] *Let H be a nilpotent subgroup of G. Then, the following statements are equivalent:*

(1) *H is an S-quasinormal subgroup of G.*
(2) *$H \leq F(G)$ and H is an SS-quasinormal subgroup of G.*

Lemma 2.4. [11, Lemma 2.3] *Let M be a maximal subgroup of G. If M is SS-quasinormal in G, then $|G : M|$ is a prime power.*

Lemma 2.5. [5, Lemma 2.1] *Suppose that H is a c-c-permutable subgroup of G, $K \leq G$ and N is a normal subgroup of G. Then, we have the following:*

(1) *If $H \leq K$, then H is a c-c-permutable subgroup of K.*
(2) *HN/N is a c-c-permutable subgroup of G/N.*

Lemma 2.6. [5, Lemma 2.4] *Let A be a maximal subgroup of H where $H \leq G$. If A is c-c-permutable in G, then $|H : A|$ is a prime.*

Lemma 2.7. *Let G be a finite group and N be a normal subgroup of G. Suppose that G possesses a maximal subgroup series Ω such that each member of Ω is either*

S-quasinormally embedded in G, or SS-quasinormal in G. Then G/N also possesses a maximal subgroup series $\overline{\Omega}$ such that each member of $\overline{\Omega}$ is either S-quasinormally embedded in G, or SS-quasinormal in G.

Proof. Let G be a finite group and N be a normal subgroup of G. Suppose that

$$\Omega : G = G_0 > G_1 > \cdots > G_i > \cdots > G_{n-1} > G_n = 1$$

is a maximal subgroup series of G such that G_i is either S-quasinormally embedded in G, or SS-quasinormal in G, for each $i \in \{1, 2, \ldots, n\}$. It follows that every G_iN/N is either S-quasinormally embedded in G, or SS-quasinormal in G/N by Lemmas 2.1(2) and 2.2(2). Write $\overline{G} = G/N$ and $\overline{G_i} = G_iN/N$. Let us investigate the following subgroup series of $\overline{G}$:

$$\overline{\Omega} : \overline{G} = \overline{G_0} \geq \cdots \geq \overline{G_i} \geq \cdots \geq \overline{G_n} = 1.$$

For each $i = 1, \ldots, n$, we see that either $\overline{G_i} = \overline{G_{i-1}}$ or $\overline{G_i}$ is maximal in $\overline{G_{i-1}}$. Therefore, after removing the equal terms in $\overline{\Omega}$, we obtain that $\overline{G}$ possesses a maximal subgroup series $\overline{\Omega}$ which satisfies the conclusion of the Lemma. □

Lemma 2.8. *Let G be a finite group and N be a normal subgroup of G. Suppose that G possesses a maximal subgroup series Ω such that each member of Ω is either c-c-permutable in G, or SS-quasinormal in G. Then G/N also possesses a maximal subgroup series $\overline{\Omega}$ such that each member of $\overline{\Omega}$ is either c-c-permutable in G, or SS-quasinormal in G.*

Proof. By Lemmas 2.2(2) and 2.5(2), the proof is similar to Lemma 2.7, and thus we omit it. □

Lemma 2.9. [2, Lemma 2] *Let $N = S_1 \times \cdots \times S_t$ be a direct product of isomorphic non-abelian simple groups, and let M be a maximal subgroup of N with $S_1 \nleq M$. Then one of the following assertions holds:*

(1) *$M = D \times S_2 \times \cdots \times S_t$, where D is maximal in S_1.*
(2) *One of the subgroups $S_2, \ldots, S_t$, say S_2, is not contained in M, then $M = D \times S_3 \times \cdots \times S_t$, where $D \cap S_1 = D \cap S_2 = 1$ and $S_1 \cong S_2 \cong D < S_1 \times S_2$.*

Lemma 2.10. [6, Theorem 1] *Let G be a non-abelian simple group. If H is a proper subgroup of G with index p^a, where p is a prime, then one of the following holds:*

(1) *$G =\mathrm{A}_n$ and $H =\mathrm{A}_{n-1}$, $n = p^a$.*
(2) *$G =\mathrm{PSL}(n, q)$ and H is the stabilizer of a line or hyperplane, $|G : H| = (q^n - 1)/(q - 1) = p^a$ and n is a prime.*

(3) $G=$PSL$(2,11)$ *and* $H=$A$_5$.
(4) $G=$M$_{23}$ *and* $H=$M$_{22}$.
(5) $G=$M$_{11}$ *and* $H=$M$_{10}$.
(6) $G=$PSU$(4,2)$ *and* $|G:H|=27$.

3. Proofs of the theorems

Proof of Theorem 1.3. The necessity is trivial as every supersolvable group has a normal maximal subgroup series. So we only need to prove the sufficiency. Suppose that the theorem is not true and let G be a counterexample of the smallest order. Let

$$\Omega : G = G_0 \rhd G_1 \rhd \cdots \rhd G_i \rhd \cdots \rhd G_{n-1} \rhd G_n = 1$$

be a subnormal maximal subgroup series of G such that G_i is either S-quasinormally embedded in G, or SS-quasinormal in G, for each $i \in \{1, 2, \dots, n\}$. It is obvious that G is solvable and G_i is normal in G_{i-1} with prime index. By Lemmas 2.1(1) and 2.2(1), every member G_i of Ω satisfies the hypothesis of the theorem, so G_i is supersolvable by choice of G for $i \geq 1$.

Let N be a minimal normal subgroup of G. Then G/N satisfies the hypothesis of the theorem by Lemma 2.7 and hence G/N is supersolvable. Consequently, if G has two distinct minimal normal subgroups, say N_1 and N_2, then both G/N_1 and G/N_2 are supersolvable, and so is G. It contradicts the choice of G. Therefore, G possesses a unique minimal normal subgroup, says N. Since G is solvable, we may assume that N is an elementary abelian p-group for some prime p. Furthermore, if $\Phi(G) \neq 1$, then $N \leq \Phi(G)$ and hence $G/\Phi(G)$ is supersolvable and so is G. This is another contradiction. Therefore, we may assume that $\Phi(G) = 1$. Moreover, applying the solvability of G again, there is a maximal subgroup H of G such that $G = HN = H \ltimes N$, where $H \cong G/N$ is supersolvable. Now, it is easy to see that

$$N = O_p(G) = F(G) = C_G(N), \text{ and } C_H(N) = 1.$$

Furthermore, observe that G_1 is normal in G and G_{n-1} is a subnormal subgroup of G of prime order. We have

$$G_{n-1} \leq N = O_p(G) = F(G) < G_1.$$

On the other hand, we have $G_1 = G_1 \cap G = G_1 \cap HN = (G_1 \cap H) \ltimes N$. Set $H_1 = G_1 \cap H$, then H_1 is normal in H as G_1 is normal in G. By the supersolvability of G_1, we have $O_{p',p,p'}(G_1) = G_1$. Furthermore, since $O_{p'}(G_1) \leq O_{p'}(G) = 1$, G_1

has a normal Sylow p-subgroup which is also normal in G. Consequently, N is exactly the normal Sylow p-subgroup of G_1. Hence, H_1 is a p'-group.

We claim that H is a Hall p'-subgroup of G. If not, then $H = P_0H_1$, where $P_0 \in Syl_p(H)$ has order p. By the supersolvability of G_1, we conclude that $H_1 \cong G_1/N = G_1/C_{G_1}(N)$ is an abelian group with exponent dividing $p-1$. In particular, p is the largest prime divisor of $|H|$. Since H is also supersolvable, H has a normal Sylow p-subgroup. This leads to $O_p(G) = P_0N > N$, a contradiction. Hence, H is a Hall p'-subgroup of G, the claim as desired.

Finally, by the hypothesis of the theorem, we know that G_{n-1} is either S-quasinormally embedded in G, or SS-quasinormal in G. If G_{n-1} is S-quasinormally embedded in G, then there exists an S-quasinormal subgroup M of G such that G_{n-1} is a Sylow p-subgroup of M. Since H is a Hall p'-subgroup of G, we have $MH = HM$ and G_{n-1} is also a Sylow p-subgroup MH. However, H is a maximal subgroup of G implies that $G = MH$. It follows that $N = G_{n-1}$ is a subgroup of order p. This implies that G is supersolvable, a contradiction. So we assume that G_{n-1} is SS-quasinormal in G. Applying Lemma 2.3, G_{n-1} is S-quasinormal in G. It follows that $G_{n-1}H = HG_{n-1} \leq G$ as H is a Hall p'-subgroup of G. Since H is a maximal subgroup of G, we have $G = HG_{n-1}$ which implies $|G_{n-1}| = |G : H| = |N|$. Therefore, G is supersolvable. This is a final contradiction. The proof of the theorem is complete. □

Proof of Theorem 1.6. Suppose that the theorem is not true and let G be a counterexample of the smallest order. Let

$$\Omega : G = G_0 > G_1 > \cdots > G_i > \cdots > G_{n-1} > G_n = 1$$

be a maximal subgroup series of G such that G_i is either c-c-permutable in G, or SS-quasinormal in G for each $i \in \{1, 2, \ldots, n\}$.

Let N be a minimal normal subgroup of G. Then G/N satisfies the hypothesis of the theorem by Lemma 2.8. By induction, G/N is solvable. Suppose that G has two distinct minimal normal subgroups, say N_1 and N_2, then both G/N_1 and G/N_2 are solvable, and so is G, a contradiction. Therefore, we may assume that G possesses a unique minimal normal subgroup, say N. Since G is non-solvable, N is a direct product of some isomorphic non-abelian simple groups and $C_G(N) = 1$.

Applying Lemma 2.2(1) and Lemma 2.5(1), we know that G_1 satisfies the hypothesis of the theorem. It follows that G_1 is solvable by induction. So $N \nleq G_1$ and hence $G_1 \cap N$ is a proper subgroup of N. Since G_i is either c-c-permutable in G, or SS-quasinormal in G, we get G_i is either c-c-permutable in G, or SS-quasinormal in G by Lemma 2.2(1) and Lemma 2.5(1). Furthermore, applying Lemma 2.4 and Lemma 2.6, we have $|G_{i-1} : G_i|$ is a prime power. Observe that $|N \cap G_{i-1} : N \cap G_i|$ divides $|G_{i-1} : G_i|$, we have $|N \cap G_{i-1} : N \cap G_i|$ is also a prime power for each $i \in \{1, 2, \ldots, n\}$. In particular, assume that $|N : N \cap G_1| = |N \cap G_0 : N \cap G_1| = p^\alpha$ for some prime p. This shows that N possesses a solvable subgroup with index prime power, namely $N \cap G_1$. By Lemma 2.9, we get that N is a non-abelian simple group.

Let j be the largest index such that $|N : N \cap G_j|$ is a p-power. Let $B = N \cap G_j$ and $C = N \cap G_{j+1}$. Then $|B : C| = q^\beta$, where q is a prime different from p.

By hypothesis of the theorem, we know that G_{j+1} is either c-c-permutable in G, or SS-quasinormal in G. If G_{j+1} is c-c-permutable in G, then there exists a Sylow p-subgroup N_p of N such that $G_{j+1}N_p \leq G$. Consequently, we have

$$CN_p = (N \cap G_{j+1})N_p = N \cap G_{j+1}N_p \leq N.$$

On the other hand, if G_{j+1} is SS-quasinormal in G, there exists a subgroup B of G such that $G = G_{j+1}B$ and $G_{j+1}B_p = B_pG_{j+1}$, where $B_p \in Syl_p(B)$. This implies that $G_{j+1}B_p$ is a proper subgroup of G and $|G|_p = |G_{j+1}B_p|_p$. Consequently, there exists a subgroup $G_p \in Syl_p(G)$ such that $G_{j+1}G_p = G_pG_{j+1}$. It follows that

$$\begin{aligned} C(N \cap G_p) &= (N \cap G_{j+1})(N \cap G_p) \\ &= N \cap (G_{j+1}(N \cap G_p)) \\ &= (G_{j+1}(N \cap G_p)) \cap N \\ &= ((N \cap G_p)G_{j+1}) \cap N \\ &= (N \cap G_p)C. \end{aligned}$$

Set $N_p = N \cap G_p$, then $N_p \in Syl_p(N)$ as N is normal in G. We also get that $CN_p = N_pC \leq N$. Moreover, by calculating the p'-part of $|N : CN_p|$, we have

$$|N : CN_p| = |N : CN_p|_{p'} = |N : C_{p'}| = |B : C| = q^\beta.$$

This implies that the non-abelian simple group N admits subgroups $G_1 \cap N$ and CN_p such that $|N : G_1 \cap N|$ and $|N : CN_p|$ are distinct prime powers. So N will be isomorphic to one of the group in Lemma 2.10. Moreover, since $\mathrm{PSL}(2, 7)$ is the only simple group with subgroups of two different prime powers indices (see [6]), we have $N \cong \mathrm{PSL}(2, 7)$. Finally, observe that N is the unique minimal normal subgroup G and $C_G(N) = 1$. Applying N/C-theorem, we get

$$N \leq G = G/C_G(N) \lesssim \mathrm{Aut}(N).$$

Consequently, since Aut(PSL(2, 7)) $\cong$PSL(2, 7).Z_2, we have

$$\mathrm{PSL}(2,7) \cong N \leq G \lesssim \mathrm{Aut}(\mathrm{PSL}(2,7)) \cong \mathrm{PSL}(2,7).Z_2.$$

So $G = N \cong$PSL(2, 7) or $G \cong$PSL(2, 7).Z_2. By checking the maximal subgroup series of G, we can conclude that G does not satisfy the hypothesis of the theorem. This is a final contradiction. The proof of the theorem is complete. □

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