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RELATIVE TILTING PAIRS

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Abstract. Let Λ be an artin algebra. We extend Miyashita's theory of tilting pairs to the setting of relative tilting pairs with respect to an additive subfunctor F of $\mathrm{Ext}^1_\Lambda(-,\ -)$. Building on Wei's work on Miyashita's ordinary tilting pairs, we establish an equivalent characterization for relative tilting pairs, which provides a unified framework that generalizes several important results in tilting theory, including classical tilting pairs, F-tilting modules, and Gorenstein tilting pairs.

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1. Introduction

The concept of tilting first emerged in Bernstein et al. [7] in 1973, as a part of their proof of Gabriel's theorem. Later, Auslander and coauthors [2] introduced a specialized tilting module in 1979, now known as the APR-tilting module, which played a pivotal role in subsequent developments. A major advancement came in 1980 with Brenner-Butler [8], who laid the foundation for classical tilting theory, and their conditions were later weakened by Happel-Ringel [12] in 1982. The theory was further extended to infinitely generated modules over arbitrary associative rings by Colpi-Trlifaj [9] and Angeleri Hügel-Coelho [1], utilizing Enochs's approximation theory. A comprehensive overview of earlier work was provided by Happel [11, Section III]. In 2004, Bazzoni [6] generalized classical tilting modules to n-tilting and n-cotilting modules, establishing an equivalent condition for n-tilting modules (resp., n-cotilting modules). In addition, Auslander-Solberg [3,4,5] developed a general theory of relative cotilting modules over artin algebras. Wei [15] demonstrated that a characterization analogous to Bazzoni's holds for relative tilting modules. Finally, Hu *et al.* [13] derived a triangulated version of this characterization in the context of triangulated categories.

In order to construct the tilting modules, Miyashita [14] introduced the notion of a tilting pair. Let A be a ring, and $T^{'}$, T be self-orthogonal left A-modules. If

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$T^{'} \in add\check{d}_A T$ and $T \in \mathrm{add}\hat{_A}T'$, then $(T^{'},\ T)$ is called a tilting pair. In fact, a left A-module T is tilting if and only if $(A,\ T)$ is a tilting pair. Wei-Xi [16] generalized the above Bazzoni's results to the tilting pairs over artin algebras. They proved that if a self-orthogonal Λ-module C satisfies $C \in Copres^n(Pres^n_{{}_C\mathcal{X}}(T))$, where Λ is an artin algebra, then $(C,\ T)$ is an n-tilting pair if and only if $T^{\perp} \cap {}_C\mathcal{X} = Pres^n_{{}_C\mathcal{X}}(T)$. Furthermore, Liu-Wei [10] generalized it to the Gorenstein tilting pair (which is the generalization of Gorenstein tilting modules induced by Yan *et al.* [17]) over an n-Gorenstein ring.

The transition from classical tilting modules to relative tilting modules marks a significant step in accommodating more general homological settings. The present work continues this line by introducing tilting pairs in the relative context, thus offering a broader framework that encompasses previous results as special cases. Inspired by this series of works, we explore relative tilting pairs, which can be viewed as a natural generalization of the results in [15,16]. Relative tilting pairs are important because they allow us to study tilting phenomena in categories equipped with a subfunctor of Ext, a structure that arises naturally in representation theory, homological algebra, and related fields.

In Section 2, we review Auslander's relative tilting theory and recall the basic notions that will be used throughout. Section 3 is devoted to establishing useful properties of relative orthogonal classes, which prepare the ground for the study of relative tilting pairs in Section 4. The main result of this paper is Theorem 4.8, which provides an equivalent characterization of relative tilting pairs. In particular, we show that if $(\mathcal{C},\ \mathcal{T})$ is an r-F-tilting pair, then, for some integer $r \geq 1$, $\mathcal{T}^{F^{\perp}} \bigcap {}_{\mathcal{C}}\mathcal{X}^F = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$. And the converse also holds true if $\mathcal{C} \subseteq FCopres^r(FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}))$. The specific explanations of these notations are provided in the next section. This result extends Wei's characterization of tilting pairs to the relative setting, providing a unifying perspective for several known tilting and cotilting theories.

2. Preliminaries

Throughout this manuscript, let Λ be an artin algebra and denote by mod-Λ the category of finitely generated Λ-modules. For any module $T \in$ mod-Λ, $\mathrm{add}_\Lambda T$ denotes the subcategory consisting of all Λ-modules which are isomorphic to the direct summands of finite direct sums of copies of T. And $\mathcal{P}(\Lambda)$ (resp., $\mathcal{I}(\Lambda)$) denotes the class of all projective Λ-modules (resp., injective Λ-modules).

We first recall some basic notions and results about relative homology. More details can be found in the literature [3,4,5,15].

Let F be an additive subfunctor of $\mathrm{Ext}^1_\Lambda(-,\ -)$. A short exact sequence

$$\eta:\ 0 \longrightarrow A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C \longrightarrow 0$$

is said to be F-exact if $\eta \in F(C,\ A)$. In this case, we say that f is F-monomorphic, and g is F-epimorphic. The following properties hold.

Lemma 2.1. [15, Lemma 2.2]

(1) *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be an exact sequence in mod-*Λ*, then* f *is* F*-monomorphic if and only if* g *is* F*-epimorphic;*

(2) *Let* gf *be the composition of* $f : A \to B$ *and* $g : B \to C$*, then* gf *is* F*-monomorphic (resp.,* F*-epimorphic) if and only if* f *is* F*-monomorphic (resp.,* g *is* F*-epimorphic).*

Remark 2.2. Note that when the functor F happens to be $\mathrm{Ext}^1_\Lambda(-,\ -)$, all these notations and definitions in this manuscript coincide with the ordinary ones, in this case, F is typically omitted.

A long exact sequence

$$\cdots \longrightarrow M_n \xrightarrow{d_n} \cdots \longrightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots \longrightarrow M_{-n} \xrightarrow{d_{-n}} \cdots$$

is F-exact if every

$$0 \to \mathrm{Im} d_{i+1} \to M_i \to \mathrm{Im} d_i \to 0$$

is F-exact for every integer i. The F-monomorphisms are stable under pushouts, dually, the F-epimorphisms are stable under pullbacks. A subfunctor F of $\mathrm{Ext}^1_\Lambda(-,\ -)$ is additive if and only if F is closed under direct sums of F-exact sequences by [3, Lemma 1.1].

A Λ-module P is said to be F-projective if all F-exact sequences that end in P split. Dually, a Λ-module I is said to be F-injective if all F-exact sequences starting in I split. We denote by $\mathcal{P}(F)$ (resp., $\mathcal{I}(F)$) the full subcategory of mod-Λ consisting of F-projective modules (resp., F-injective modules). Clearly, $\mathcal{P}(\Lambda) \subseteq \mathcal{P}(F)$ and $\mathcal{I}(\Lambda) \subseteq \mathcal{I}(F)$. A Λ-module $P \in \mathcal{P}(F)$ (resp., $I \in \mathcal{I}(F)$) if and only if $F(P,\ X) = 0$ (resp., $F(X,\ I) = 0$) for all Λ-modules X by [3, Corollary 1.4]. Moreover, we say F has enough projectives if for every Λ-module X, there exists an F-exact sequence $0 \to X^{'} \to P \to X \to 0$ with $P \in \mathcal{P}(F)$. And that F has enough injectives is defined dually.

The following results on F-exact sequences (assuming F has enough projectives or injectives) are fundamental to this manuscript. Detailed proofs are given in [3, Proposition 1.5].

Lemma 2.3. *Let* $\eta : 0 \to A \to B \to C \to 0$ *be an exact sequence in mod-*Λ*, we have:*

(1) *If* F *has enough projectives, then* η *is* F*-exact if and only if* $0 \to \mathrm{Hom}_\Lambda(P,\ A) \to \mathrm{Hom}_\Lambda(P,\ B) \to \mathrm{Hom}_\Lambda(P,\ C) \to 0$ *is exact for all* F*-projective module* P*;*

(2) *If F has enough injectives, then η is F-exact if and only if* $0 \to \mathrm{Hom}_\Lambda(C,\ I) \to \mathrm{Hom}_\Lambda(B,\ I) \to \mathrm{Hom}_\Lambda(A,\ I) \to 0$ *is exact for all F-injective module I.*

We always assume that F has enough projectives and injectives in this manuscript, which is equivalent to the fact that $\mathcal{P}(F)$ and $\mathcal{I}(F)$ are functorially finite in mod-Λ by [3, Corollary 1.13]. In this case, we say every Λ-module M has an F-projective resolution and an F-injective coresolution, respectively, that is, there exist a long F-exact sequence

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

with every $P_i \in \mathcal{P}(F)$, and a long F-exact sequence

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to \cdots$$

with every $I_i \in \mathcal{I}(F)$.

Let $M, N \in$ mod-Λ, we calculate the right derived functors of $\mathrm{Hom}_\Lambda(M,\ N)$ using an F-projective resolution of M or an F-injective coresolution of N. Denote this derived functor by $\mathrm{Ext}^i_F(M,\ N)$, it is clear that $\mathrm{Ext}^1_F(M,\ N) = F(M,\ N)$.

According to [15, Section 2], a Λ-module X has F-projective dimension at most n, denoted $\mathrm{pd}_F X \leq n$, if there exists an F-projective resolution of X of length no more than n. Equivalently, this means that $\mathrm{Ext}^{n+i}_F(X,\ N) = 0$ for all $i \geq 1$ and every Λ-module N. The F-injective dimension can be defined dually.

More generally, let $\mathcal{T} \subseteq$ mod-Λ, we denote by $\hat{\mathcal{T}}_F$ the subcategory of mod-Λ consisting of all Λ-module M such that there exists a long F-exact sequence $0 \to T_r \to \cdots \to T_1 \to T_0 \to M \to 0$ for some nonnegative integer r with each $T_i \in \mathcal{T}$. For $M \in \hat{\mathcal{T}}_F$, we denote $\dim_F M \leq m$ if m is the minimum integer that makes the above sequence exist, and $(\hat{\mathcal{T}}_F)_m := \{M \in \hat{\mathcal{T}}_F \mid \dim_F M \leq m\}$. On the other hand, the definitions of $\check{\mathcal{T}}_F$, $\mathrm{codim}_F M$ and $(\check{\mathcal{T}}_F)_m$ can be defined dually. Let $\mathcal{X} \subseteq$ mod-Λ, then we denote by $FPres^r_{\mathcal{X}}(\mathcal{T})$ the subcategory of all Λ-modules M such that there exists an F-exact sequence $0 \to X \to T_r \to \cdots \to T_1 \to M \to 0$ with every $T_i \in \mathcal{T}$ for all integers $i = 1, 2, \cdots, r$ and $X \in \mathcal{X}$. Specially, if $\mathcal{X} =$ mod-Λ, then we denote $FPres^r_{\text{mod-}\Lambda}(\mathcal{T}) = FPres^r(\mathcal{T})$. Moreover, $FCopres^r_{\mathcal{X}}(\mathcal{T})$ and $FCopres^r(\mathcal{T})$ can be defined dually.

Let $\mathcal{T} \subseteq$ mod-Λ, we say $\mathcal{T}$ is closed under F-extensions if for any F-exact sequence $0 \to A \to B \to C \to 0$ with $A, C \in \mathcal{T}$, then $B \in \mathcal{T}$. And $\mathcal{T}$ is an F-orthogonal class if it satisfies:

(i) $\mathrm{Ext}^i_F(M,\ N) = 0$ for any $M, N \in \mathcal{T}$ and integer $i \geq 1$;

(ii) $\mathcal{T}$ is closed under F-extensions, finite direct sums and direct summand.

We denote

$$\mathcal{T}^{F^\perp} := \{X \in \text{mod-}\Lambda \mid \mathrm{Ext}^i_F(T,\ X) = 0 \text{ for any object } T \in \mathcal{T} \text{ and integer } i \geq 1\},$$

dually,

$${}^{\perp}\mathcal{T}^{F} := \{X \in \text{mod-}\Lambda \mid \text{Ext}^{i}_{F}(X,\ T) = 0 \text{ for any object } T \in \mathcal{T} \text{ and integer } i \geq 1\}.$$

Let $\mathcal{T}$ be an F-orthogonal class, then we denote by ${}_{\mathcal{T}}\mathcal{X}^{F}$ (resp., $\mathcal{X}^{F}_{\mathcal{T}}$) the full subcategory of $\mathcal{T}^{F\perp}$ (resp., ${}^{\perp}\mathcal{T}^{F}$) consisting of all Λ-modules M such that there exists an F-exact sequence

$$\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0 \ \ (\text{resp.,}\ \ 0 \longrightarrow M \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots)$$

with each $T_i \in \mathcal{T}$ and $\text{Im}f_i \in \mathcal{T}^{F\perp}$ (resp., $\text{Im}f_i \in {}^{\perp}\mathcal{T}^{F}$). Clearly, we have $\mathcal{T} \subseteq {}_{\mathcal{T}}\mathcal{X}^{F} \subseteq \mathcal{T}^{F\perp}$ and $\mathcal{T} \subseteq \mathcal{X}^{F}_{\mathcal{T}} \subseteq {}^{\perp}\mathcal{T}^{F}$.

Remark 2.4. Let $\mathcal{T}$ be an F-orthogonal class. It is easy to check that $(\hat{\mathcal{T}}_F)_r \subseteq {}_{\mathcal{T}}\mathcal{X}^{F}$ and $(\check{\mathcal{T}}_F)_r \subseteq \mathcal{X}^{F}_{\mathcal{T}}$.

Let $\mathcal{T}$ be a subcategory of some Λ-modules. A homomorphism $f : T \to X$ is a $\mathcal{T}$-precover of the Λ-module X if for every homomorphism $g : T^{'} \to X$, there exists a homomorphism $h : T^{'} \to T$, such that $g = fh$. And we say that $\mathcal{T}$ is precovering if every Λ-module has a $\mathcal{T}$-precover.

Lemma 2.5. *Let $\mathcal{T}$ be a precovering class. Suppose $M \in FPres^1(\mathcal{T})$. Then there exists an F-exact sequence $0 \to M^{'} \to T_M \to M \to 0$ with $T_M \in \mathcal{T}$ which stays exact after applying the functor* $\text{Hom}(T,\ -)$ *for any $T \in \mathcal{T}$.*

Proof. Since $M \in FPres^1(\mathcal{T})$, there is an F-exact sequence $0 \to N \to T^{'} \to M \to 0$ with $T^{'} \in \mathcal{T}$. By the fact that $\mathcal{T}$ is a precovering, we have an epimorphism $\mathcal{T}$-precover $T_M \to M \to 0$ and an exact sequence $\eta : 0 \to M^{'} \to T_M \to M \to 0$. Considering the following commutative diagram

$$\begin{array}{cccccccccc} & 0 & \longrightarrow & N & \longrightarrow & T^{'} & \longrightarrow & M & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \| & & \\ \eta : & 0 & \longrightarrow & M^{'} & \longrightarrow & T_M & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

We know that η is a pushout of an F-exact sequence, and therefore η is also F-exact. It is easy to check that $\text{Hom}(T,\ \eta)$ is exact for any $T \in \mathcal{T}$. $\square$

3. F-orthogonal class

In this section, we establish fundamental closure properties and homological characterizations of F-orthogonal classes. These results will be essential for our study of relative tilting pairs. Throughout this section, we fix $\mathcal{T}$ as an F-orthogonal class and begin by establishing its fundamental properties.

Lemma 3.1. *Let $\mathcal{T}$ be an F-orthogonal class, then the following statements hold.*

(1) $\mathcal{X}_{\mathcal{T}}^{F}$ *is closed under F-extensions, kernels of F-epimorphisms and direct summands;*

(1$^{'}$) ${}_{\mathcal{T}}\mathcal{X}^{F}$ *is closed under F-extensions, cokernels of F-monomorphisms and direct summands;*

(2) $\mathrm{Ext}_F^i(U,\ V)=0$ *for any* $U\in(\check{\mathcal{T}}_F)_r$, $V\in\mathcal{T}^{F\perp}$ *and the integer* $i\geq 1$;

(2$^{'}$) $\mathrm{Ext}_F^i(U,\ V)=0$ *for any* $U\in{}^{\perp}\mathcal{T}^F$, $V\in(\hat{\mathcal{T}}_F)_r$ *and the integer* $i\geq 1$;

(3) $(\check{\mathcal{T}}_F)_r=\{X\in\mathcal{X}_{\mathcal{T}}^F\mid \mathrm{Ext}_F^{r+1}(Y,\ X)=0$ *for any* $Y\in{}^{\perp}\mathcal{T}^F\}=\{X\in\mathcal{X}_{\mathcal{T}}^F\mid \mathrm{Ext}_F^{r+1}(Y,\ X)=0$ *for any* $Y\in\mathcal{X}_{\mathcal{T}}^F\}$;

(3$^{'}$) $(\hat{\mathcal{T}}_F)_r=\{X\in{}_{\mathcal{T}}\mathcal{X}^F\mid \mathrm{Ext}_F^{r+1}(X,\ Y)=0$ *for any* $Y\in\mathcal{T}^{F\perp}\}=\{X\in{}_{\mathcal{T}}\mathcal{X}^F\mid \mathrm{Ext}_F^{r+1}(X,\ Y)=0$ *for any* $Y\in{}_{\mathcal{T}}\mathcal{X}^F\}$;

(4) $(\check{\mathcal{T}}_F)_r$ *is closed under F-extensions, finite direct sums and direct summands;*

(4$^{'}$) $(\hat{\mathcal{T}}_F)_r$ *is closed under F-extensions, finite direct sums and direct summands;*

(5) *If there is an F-exact sequence* $0\to U\to V\to W\to 0$ *with* $V,W\in{}_{\mathcal{T}}\mathcal{X}^F$ *(resp.,* $V,W\in(\hat{\mathcal{T}}_F)_r$*), and* $U\in\mathcal{T}^{F\perp}$*, then* $U\in{}_{\mathcal{T}}\mathcal{X}^F$ *(resp.,* $U\in(\hat{\mathcal{T}}_F)_r$*);*

(5$^{'}$) *If there is an F-exact sequence* $0\to U\to V\to W\to 0$ *with* $U,V\in\mathcal{X}_{\mathcal{T}}^F$ *(resp.,* $U,V\in(\check{\mathcal{T}}_F)_r$*), and* $W\in{}^{\perp}\mathcal{T}^F$*, then* $W\in\mathcal{X}_{\mathcal{T}}^F$ *(resp.,* $W\in(\check{\mathcal{T}}_F)_r$*).*

Proof. It is clear that the statement $(i^{'})$ is dual to the statement (i), so we only prove the statement (i), where $i=1,2,3,4,5$.

(1) Let $0\to A\to B\to C\to 0$ be an F-exact sequence. At first, we assume that $A,C\in\mathcal{X}_{\mathcal{T}}^F\subseteq{}^{\perp}\mathcal{T}^F$, we need to show $B\in\mathcal{X}_{\mathcal{T}}^F$. By the definition, there exist two F-exact sequences $0\to A\to T_A\to A^{'}\to 0$ and $0\to C\to T_C\to C^{'}\to 0$ with $T_A,T_C\in\mathcal{T}$ and $A^{'},C^{'}\in\mathcal{X}_{\mathcal{T}}^F\subseteq{}^{\perp}\mathcal{T}^F$. Consider the following pushout diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & A & \to & B & \to & C & \to & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_A & \to & X & \to & C & \to & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & A^{'} & = & A^{'} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then the middle row is F-exact and it satisfies $\mathrm{Ext}_F^i(C,\ T_A)=0$. So $X\cong T_A\bigoplus C$. Since we can view $0\longrightarrow T_A = T_A\longrightarrow 0$ as an F-exact one, by [3, Lemma 1.2], $0\to T_A\bigoplus C\to T_A\bigoplus T_C\to C^{'}\to 0$ is also F-exact. Hence we have the following diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & T_A \bigoplus C & \to & A^{'} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & B & \to & T_A \bigoplus T_C & \to & Y & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C^{'} & = & C^{'} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then $T_A \bigoplus T_C \in \mathcal{T}$ since $\mathcal{T}$ is closed under direct sums. And $Y \in {}^{\perp}\mathcal{T}^F$ since ${}^{\perp}\mathcal{T}^F$ is closed under F-extensions. Hence, by repeating the previous process with the above last column, we have by induction that $B \in \mathcal{X}_{\mathcal{T}}^F$.

Next, assuming $B, C \in \mathcal{X}_{\mathcal{T}}^F \subseteq {}^{\perp}\mathcal{T}^F$, we say $A \in \mathcal{X}_{\mathcal{T}}^F$, too. Let $0 \to B \to T_B \to B^{'} \to 0$ be an F-exact sequence with $T_B \in \mathcal{T}$ and $B^{'} \in \mathcal{X}_{\mathcal{T}}^F \subseteq {}^{\perp}\mathcal{T}^F$, then we consider the following pushout diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & C & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & T_B & \to & X_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & B^{'} & = & B^{'} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Applying the result of previous to the above last column, we know $X_1 \in \mathcal{X}_{\mathcal{T}}^F$. Hence $A \in \mathcal{X}_{\mathcal{T}}^F$ by the above middle row.

At last, let $B = A \bigoplus C$ and $B \in \mathcal{X}_{\mathcal{T}}^F$, we show $A \in \mathcal{X}_{\mathcal{T}}^F$. Note that the above diagram is still valid here, we take the F-exact sequence $0 \to C \to X_1 \to B^{'} \to 0$ from the last column. Hence $0 \to B \to A \bigoplus X_1 \to B^{'} \to 0$ is obviously F-exact. It implies that $A \bigoplus X_1 \in \mathcal{X}_{\mathcal{T}}^F \subseteq {}^{\perp}\mathcal{T}^F$, too. Then $X_1 \in {}^{\perp}\mathcal{T}^F$. By repeating the process with $0 \to X_1 \to A \bigoplus X_1 \to A \to 0$, we have by induction that $A \in \mathcal{X}_{\mathcal{T}}^F$.

(2) Let $U \in (\check{\mathcal{T}_F})_r$, then there exists a long F-exact sequence $0 \to U \to T_0 \to T_1 \to \cdots \to T_r \to 0$ with every $T_i \in \mathcal{T}$. Since $V \in \mathcal{T}^{F\perp}$, by dimension shifting, we have $\mathrm{Ext}_F^i(U,\ V) \cong \mathrm{Ext}_F^{i+r}(T_r,\ V) = 0$ as expected.

(3) Let $X \in (\check{\mathcal{T}_F})_r$, there exists an F-exact sequence $0 \to X \to T_0 \to \cdots \to T_r \to 0$ with every $T_i \in \mathcal{T}$. Then we have $\mathrm{Ext}_F^{r+1}(Y,\ X) \cong \mathrm{Ext}_F^1(Y, T_r) = 0$ for any $Y \in {}^{\perp}\mathcal{T}^F$ by dimension shifting. So the first "$\subseteq$" hold. And the second "$\subseteq$" hold since ${}_{\mathcal{T}}\mathcal{X}^F \subseteq \mathcal{T}^{F\perp}$. Next we show $\{X \in \mathcal{X}_{\mathcal{T}}^F \mid \mathrm{Ext}_F^{r+1}(Y,\ X) = 0 \text{ for any } Y \in \mathcal{X}_{\mathcal{T}}^F\} \subseteq (\check{\mathcal{T}_F})_r$. Let $X \in \mathcal{X}_{\mathcal{T}}^F$, then there exists an F-exact sequence $0 \longrightarrow X \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots$ with $T_i \in \mathcal{T}$ and $\mathrm{Im}f_i \in \mathcal{X}_{\mathcal{T}}^F$. Hence $\mathrm{Ext}_F^{r+1}(\mathrm{Im}f_i,\ X) = 0$. By dimension shifting, we have $\mathrm{Ext}_F^1(\mathrm{Im}f_{r+1},\ \mathrm{Im}f_r) \cong \mathrm{Ext}_F^{r+1}(\mathrm{Im}f_{r+1},\ X) = 0$, which implies $\mathrm{Im}f_{r+1} \bigoplus \mathrm{Im}f_r \cong T_r$. So $\mathrm{Im}f_{r+1} \in \mathcal{T}$, and $X \in (\check{\mathcal{T}_F})_r$.

(4) It follows from the statements (1) and (3).

(5) Since $W \in {}_{\mathcal{T}}\mathcal{X}^F$, there exists an F-exact sequence $0 \to W' \to T_W \to W \to 0$ with $W' \in {}_{\mathcal{T}}\mathcal{X}^F$ and $T_W \in \mathcal{T}$. We consider the following pullback diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & W' & = & W' & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & X & \longrightarrow & T_W & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then the second column is also F-exact, and we have $X \in {}_{\mathcal{T}}\mathcal{X}^F$ by $(1')$. Moreover, $\mathrm{Ext}_F^1(T_W,\ U) = 0$ by the assumption, which implies $X \cong U \bigoplus T_W$. Hence $U \in {}_{\mathcal{T}}\mathcal{X}^F$ as expected. □

Let $\mathcal{Y} \subseteq \text{mod-}\Lambda$ be a subcategory closed under finite direct sums and summands. A class $\mathcal{Z} \subseteq \text{mod-}\Lambda$ is called an F-relative generator of $\mathcal{Y}$ if for any $Y \in \mathcal{Y}$, there exists an F-exact sequence $0 \to Y' \to Z_Y \to Y \to 0$ with $Z_Y \in \mathcal{Z}$ and $Y' \in \mathcal{Y}$. Dually, $\mathcal{Z}$ is called an F-relative cogenerator of $\mathcal{Y}$ if for any $Y \in \mathcal{Y}$, there exists an F-exact sequence $0 \to Y \to Z_Y \to Y' \to 0$ with $Z_Y \in \mathcal{Z}$ and $Y' \in \mathcal{Y}$.

Remark 3.2. The F-orthogonal class $\mathcal{T}$ is an F-relative generator (resp., F-relative cogenerator) of both $(\hat{\mathcal{T}_F})_r$ and ${}_{\mathcal{T}}\mathcal{X}^F$ (resp., both $(\check{\mathcal{T}_F})_r$ and $\mathcal{X}_{\mathcal{T}}^F$).

Theorem 3.3. *Let $\mathcal{Y}$ be a subcategory closed under F-extensions, finite direct sums and summands, and $\mathcal{Z}$ be an F-relative generator of $\mathcal{Y}$. Suppose A is a Λ-module admitting a long F-exact sequence*

$$0 \to A \to Y_m \to \cdots \to Y_1 \to B \to 0$$

for the object B and some integer $m \geq 1$, where each $Y_i \in \mathcal{Y}$ (resp., $Y_i \in (\hat{\mathcal{Z}_F})_r$). Then

(1) *There exists an F-exact sequence $0 \to U_m \to V_m \to A \to 0$ for some $U_m \in \mathcal{Y}$ (resp., $U_m \in (\hat{\mathcal{Z}_F})_{r-1}$), where V_m admits an F-exact sequence $0 \to V_m \to Z_m \to \cdots \to Z_1 \to B \to 0$ with $Z_i \in \mathcal{Z}$;*

($1'$) *Moreover, if $B \in \mathcal{Y}$ (resp., $B \in (\hat{\mathcal{Z}_F})_{r+1}$), then there exists an F-exact sequence $0 \to U \to V \to A \to 0$ for some $U \in \mathcal{Y}$ (resp., $U \in (\hat{\mathcal{Z}_F})_{r-1}$) and $V \in (\check{\mathcal{Z}_F})_m$;*

(2) *There exists an F-exact sequence $0 \to A \to U \to V \to 0$ for some $U \in \mathcal{Y}$ (resp., $U \in (\hat{\mathcal{Z}_F})_r$), where V admits an F-exact sequence $0 \to V \to Z_{m-1} \to \cdots \to Z_1 \to B \to 0$ with $Z_i \in \mathcal{Z}$. Moreover, if $B \in \mathcal{Y}$ (resp., $B \in (\hat{\mathcal{Z}_F})_{r+1}$), then $V \in (\check{\mathcal{Z}_F})_{m-1}$.*

Proof. (1) We verify this statement by induction on m. At first, when $m = 1$, there exists an F-exact sequence $0 \to U_1 \to Z_1 \to Y_1 \to 0$ with $U_1 \in \mathcal{Y}$ and $Z_1 \in \mathcal{Z}$ by the assumption. Then consider the following pullback diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & U_1 & = & U_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & V_1 & \to & Z_1 & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & Y_1 & \to & B & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Obviously, both the first column and the second row are F-exact, and they happen to be the desired sequence, respectively.

Next, we inductively assume the statement always holds for $\leq m-1$, and we will prove it also holds for m. Denote $L = \operatorname{coker}(A \to Y_m)$. Then, by the assumption, there exists an F-exact sequence $0 \to U \to V \to L \to 0$ with $U \in \mathcal{Y}$ and V admits a long F-exact sequence $0 \to V \to Z_{m-1} \to \cdots \to Z_1 \to B \to 0$, where $Z_i \in \mathcal{Z}$.

Consider the following pullback diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U & = & U & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & Y & \to & V & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & Y_m & \to & L & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Then by the second column, we know $Y \in \mathcal{Y}$. Hence there exists an F-exact sequence $0 \to U_m \to Z_m \to Y \to 0$ with $U_m \in \mathcal{Y}$ and $Z_m \in \mathcal{Z}$. Therefore, we have the following diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & U_m & = & U_m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & V_m & \to & Z_m & \to & V & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & A & \to & Y & \to & V & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Obviously, the first column and the second row are the F-exact sequences that we desired.

($1'$) If $B \in \mathcal{Y}$, then there exists an F-exact sequence $0 \to Y_B \to Z_B \to B \to 0$ with $Y_B \in \mathcal{Y}$ and $Z_B \in \mathcal{Z}$. Denote $K = \ker(Y_1 \to B)$. Hence we have the following pullback diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y_B & = & Y_B & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & M & \to & Z_B & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & K & \to & Y_1 & \to & B & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then $M \in \mathcal{Y}$ by the above second column. Applying the previous results in (1) to the long F-exact sequence $0 \to A \to Y_m \to \cdots \to Y_2 \to M \to Z_B \to 0$, we get an F-exact sequence $0 \to U \to V \to A \to 0$ with $U \in \mathcal{Y}$ and V satisfies the F-exact sequence $0 \to V \to Z^m \to \cdots \to Z^1 \to Z_B \to 0$, where $Z^i \in \mathcal{Z}$ for $i = 1, 2, \cdots, m$, that is, $V \in (\check{\mathcal{Z}}_F)_m$.

(2) By (1), there exists an F exact sequence $0 \to U^{'} \to V^{'} \to A \to 0$ where $U^{'} \in \mathcal{Y}$ and $V^{'}$ admits a long F-exact sequence $0 \to V^{'} \to Z_m \to \cdots \to Z_1 \to B \to 0$, where $Z_i \in \mathcal{Z}$. Let $V = \mathrm{coker}(V^{'} \to Z_m)$. Then by the following pushout diagram,

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & U^{'} & \to & V^{'} & \to & A & \to & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & U^{'} & \to & Z_m & \to & U & \to & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & V & = & V & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

the last column is exactly the desired F-exact sequence. In the case of $B \in \mathcal{Y}$, its proof is similar to the previous by using $(1^{'})$, we omit the details. □

Dually, we give the following results without proof.

Theorem 3.4. *Let $\mathcal{Y}$ be a subcategory closed under F-extensions, finite direct sums and summands, and $\mathcal{Z}$ be an F-relative cogenerator of $\mathcal{Y}$. Suppose A is a Λ-module* *admitting a long F-exact sequence*

$$0 \to B \to Y_m \to \cdots \to Y_1 \to A \to 0$$

for the object B and some integer $m \geq 1$, where each $Y_i \in \mathcal{Y}$ (resp., $Y_i \in (\check{\mathcal{Z}}_F)_r$). Then there exists an F-exact sequence $0 \to A \to V \to U \to 0$ for some $U \in \mathcal{Y}$ (resp., $U \in (\check{\mathcal{Z}}_F)_{r-1}$), where V admits an F-exact sequence $0 \to B \to Z_m \to \cdots \to Z_1 \to V \to 0$ with $Z_i \in \mathcal{Z}$.

4. Relative tilting pair

Following Miyashita's definition of general tilting pairs in the literature [14], we want to define the relative tilting pairs. Note that r is always a fixed positive integer in what follows.

Definition 4.1. A pair $(\mathcal{C},\ \mathcal{T})$ of subcategories is said to be an r-F-tilting pair if it satisfies:

(i) Both $\mathcal{C}$ and $\mathcal{T}$ are F-orthogonal classes;
(ii) $\mathcal{C} \subseteq (\check{\mathcal{T}}_F)_r$;
(iii) $\mathcal{T} \subseteq (\hat{\mathcal{C}}_F)_r$.

In this case, $\mathcal{C}$ is called $\mathcal{T}_F^r$-cotilting class, and $\mathcal{T}$ is called $\mathcal{C}_F^r$-tilting class.

Remark 4.2. (1) Let $\mathcal{C} = \mathcal{P}(F)$, i.e., the subcategory of F-projective modules, $\mathcal{T} = \mathrm{add}(T)$, where T is an F-self-orthogonal module, i.e., $\mathrm{Ext}_F^i(T,\ T) = 0$ for all $i \geq 1$. Then $\mathcal{C}_F^r$-tilting is just the r-F-tilting modules [15, Definition 2.3].

(2) If the subfunctor $F = Gext^1(-,\ -)$, then the r-F-tilting pair coincides with Gorenstein tilting pair [10, Definition 3.1].

(3) If $F = \mathrm{Ext}_\Lambda^1(-,\ -)$, $\mathcal{C} = \{C\}$ and $\mathcal{T} = \{T\}$ where both C and T are self-orthogonal modules, then it returns to the ordinary r-tilting pair [16, Definition 3.1].

The following proposition shows how relative tilting pairs can be composed and related to each other.

Proposition 4.3. *Assume that $(\mathcal{C},\ \mathcal{T})$ is an r-F-tilting pair.*

(1) *If $(\mathcal{B},\ \mathcal{C})$ is an s-F-tilting pair, then $(\mathcal{B},\ \mathcal{T})$ is an $(r+s)$-F-tilting pair;*
(2) *If $(\mathcal{B},\ \mathcal{T})$ is an s-F-tilting pair and $\mathcal{C} \subseteq {}_{\mathcal{B}}\mathcal{X}^F$, then $(\mathcal{B},\ \mathcal{C})$ is an s-F-tilting pair;*
(3) *If $(\mathcal{C},\ \mathcal{D})$ is an s-F-tilting pair and $\mathcal{T} \subseteq \mathcal{X}_{\mathcal{D}}^F$, then $(\mathcal{T},\ \mathcal{D})$ is an s-F-tilting pair;*
(4) *If $(\mathcal{C},\ \mathcal{D})$ is an s-F-tilting pair and $\mathcal{C} \subseteq {}_{\mathcal{D}}\mathcal{X}^F$, then $(\mathcal{D},\ \mathcal{T})$ is an r-F-tilting pair.*

Proof. (1) For any $B \in \mathcal{B}$, by the assumptions, we have an F-exact sequence $0 \to B \to C_0 \to \cdots \to C_s \to 0$ with every $C_i \in \mathcal{C} \subseteq (\check{\mathcal{T}}_F)_r$. Denote $K_i = \ker(C_i \to \mathcal{C}_{i+1}), i = 0, 1, \cdots, s-1$, then it is easy to check that $K_{s-1} \in (\check{\mathcal{T}}_F)_{r+1}, K_{s-2} \in (\check{\mathcal{T}}_F)_{r+2}, \cdots, K_0 = B \in (\check{\mathcal{T}}_F)_{r+s}$. So $\mathcal{B} \subseteq (\check{\mathcal{T}}_F)_{r+s}$. Dually, we also have $\mathcal{T} \subseteq (\hat{\mathcal{B}}_F)_{r+s}$. Hence $(\mathcal{B},\ \mathcal{T})$ is an $(r+s)$-F-tilting pair.

(2) On the one hand, for any $B \in \mathcal{B} \subseteq (\check{\mathcal{T}}_F)_s$, by the assumption, we have an F-exact sequence $0 \to B \to T_0 \to \cdots \to T_s \to 0$ with $T_i \in \mathcal{T} \subseteq (\hat{\mathcal{C}}_F)_r$. Hence by Theorem 3.3($1^{'}$), there exists an F-exact sequence $0 \to U \to V \to B \to 0$ with $U \in (\hat{\mathcal{C}}_F)_{r-1}$ and $V \in (\check{\mathcal{C}}_F)_s$. Since $\mathcal{C} \subseteq {}_{\mathcal{C}}\mathcal{X}^F$, it is easy to check $\mathcal{B} \subseteq {}^{\perp}\mathcal{C}^F$. Then by Lemma 3.1($2^{'}$), we get $\mathrm{Ext}_F^i(B,\ U) = 0$ for the integer $i \geq 1$. So $V \cong B \bigoplus U$, and $B \in (\check{\mathcal{C}}_F)_s$ by Lemma 3.1(4). Therefore $\mathcal{B} \subseteq (\check{\mathcal{C}}_F)_s$.

On the other hand, for any $C \in \mathcal{C} \subseteq {}_{\mathcal{B}}\mathcal{X}^F$, if $\mathrm{Ext}_F^{s+1}(C,\ X) = 0$ for any $X \in \mathcal{B}^{F^{\perp}}$, then $\mathcal{C} \subseteq (\hat{\mathcal{B}}_F)_s$ by Lemma 3.1($3^{'}$). Since $C \in (\check{\mathcal{T}}_F)_r$, we have an F-exact sequence $0 \to T_0 \to \cdots \to T_r \to 0$ with $T_i \in \mathcal{T} \subseteq (\hat{\mathcal{B}}_F)_s$. In fact, we have an F-exact sequence $0 \to T_i \to B_0^i \to \cdots \to B_s^i \to 0$ for every T_i with $B_t^i \in \mathcal{B}, t = 0, 1, \cdots, s$. Then we have $\mathrm{Ext}_F^{s+j}(T_i,\ X) \cong \mathrm{Ext}_F^{j}(B_s^i,\ X) = 0$ for all $j \geq 1$ and $0 \leq i \leq r$. Again by dimension shifting, we get $\mathrm{Ext}_F^{s+1}(C,\ X) \cong \mathrm{Ext}_F^{s+j}(T_i,\ X) = 0$.

The proofs of the statements (3) and (4) are similar to those of (2), we omit their details. □

Wei [16] provided an equivalent characterization of the tilting pairs of Miyashita. Next, we extend Wei's theorem to relative tilting pairs. To this end, we first need to make the following study. Our main result is Theorem 4.8.

Lemma 4.4. *Let $\mathcal{T}$ be an F-orthogonal class.*

(1) *If $\mathcal{C} \subseteq \check{\mathcal{T}}_F$, then $\mathcal{T}^{F^{\perp}} \subseteq \mathcal{C}^{F^{\perp}}$;*

(2) *If $\mathcal{T} \subseteq {}_{\mathcal{C}}\mathcal{X}^F$ and $\mathcal{C} \subseteq \check{\mathcal{T}}_F$, then ${}_{\mathcal{T}}\mathcal{X}^F \subseteq {}_{\mathcal{C}}\mathcal{X}^F$.*

Hence, if $(\mathcal{C},\ \mathcal{T})$ is an r-F-tilting pair, then we have $\mathcal{T}^{F^{\perp}} \subseteq \mathcal{C}^{F^{\perp}}$ and ${}_{\mathcal{T}}\mathcal{X}^F \subseteq {}_{\mathcal{C}}\mathcal{X}^F$.

Proof. (1) Let $C \in \mathcal{C}$, there exists an F-exact sequence $0 \to C \to T_0 \to T_1 \to \cdots \to T_r \to 0$ by the assumption. For any $X \in \mathcal{T}^{F^{\perp}}$, by the dimension shifting, $\mathrm{Ext}_F^i(C,\ X) \cong \mathrm{Ext}_F^{i+r}(T_r,\ X) = 0$, we have $X \in \mathcal{C}^{F^{\perp}}$, i.e., $\mathcal{T}^{F^{\perp}} \subseteq \mathcal{C}^{F^{\perp}}$.

(2) For any $M \in {}_{\mathcal{T}}\mathcal{X}^F$, there exists a long F-exact sequence

$$\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$$

with $T_i \in \mathcal{T}$ and $\mathrm{Im} f_i \in {}_{\mathcal{T}}\mathcal{X}^F \subseteq \mathcal{T}^{F^{\perp}}$. By Remark 3.2, $\mathcal{C}$ is an F-relative generator class of ${}_{\mathcal{C}}\mathcal{X}^F$. We next apply the results of Theorem 3.3, i.e., in the case where $\mathcal{Y} = {}_{\mathcal{C}}\mathcal{X}^F$, $\mathcal{Z} = \mathcal{C}$ and $m = 1$. In fact, for the F-exact sequence $0 \to \mathrm{Im} f_1 \to T_0 \to M \to 0$, note that $T_i \in \mathcal{T} \subseteq {}_{\mathcal{C}}\mathcal{X}^F$ by the assumption. Hence there exists an F-exact sequence $0 \to U_0 \to V_0 \to \mathrm{Im} f_1 \to 0$ with $U_0 \in {}_{\mathcal{C}}\mathcal{X}^F$ and V_0 admits an

F-exact sequence $0 \to V_0 \to C_0 \to M \to 0$, where $C_0 \in \mathcal{C}$. We know $V_0 \in \mathcal{C}^{F\perp}$ since $U_0 \in {}_{\mathcal{C}}\mathcal{X}^F \subseteq \mathcal{C}^{F\perp}$ and $\mathrm{Im} f_1 \in \mathcal{T}^{F\perp} \subseteq \mathcal{C}^{F\perp}$ by (1). Consider the following pullback diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Im} f_2 & = & \mathrm{Im} f_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & U_0 & \to & Y & \to & T_1 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & U_0 & \to & V_0 & \to & \mathrm{Im} f_1 & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Then the second row is also an F-exact sequence. So we have $Y \in {}_{\mathcal{C}}\mathcal{X}^F$ by Lemma 3.1, which implies that there exists an F-exact sequence $0 \to U_1 \to C_1 \to Y \to 0$ with $C_1 \in C$ and $U_1 \in {}_{\mathcal{C}}\mathcal{X}^F$. Hence, we immediately obtain the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & U_1 & = & U_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & V_1 & \to & C_1 & \to & V_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathrm{Im} f_2 & \to & Y & \to & V_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then $V_1 \in \mathcal{C}^{F\perp}$ by the first column. By repeating the previous process to the above second row, we will get that the fact $M \in {}_{\mathcal{C}}\mathcal{X}^F$. □

Proposition 4.5. *If $(\mathcal{C}, \mathcal{T})$ is an r-F-tilting pair and $\mathcal{T}$ is precovering, then*

$${}_{\mathcal{T}}\mathcal{X}^F = \mathcal{T}^{F\perp} \bigcap {}_{\mathcal{C}}\mathcal{X}^F = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}).$$

Proof. The containment of ${}_{\mathcal{T}}\mathcal{X}^F \subseteq \mathcal{T}^{F\perp} \bigcap {}_{\mathcal{C}}\mathcal{X}^F$ is obvious by Lemma 4.4.

Next, we want to verify $\mathcal{T}^{F\perp} \bigcap {}_{\mathcal{C}}\mathcal{X}^F \subseteq FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$. Let $M \in \mathcal{T}^{F\perp} \bigcap {}_{\mathcal{C}}\mathcal{X}^F$, then there exists an F-exact sequence $0 \to M_1 \to C \to M \to 0$ with $M_1 \in {}_{\mathcal{C}}\mathcal{X}^F$

and $C \in \mathcal{C}$. Since $(\mathcal{C},\ \mathcal{T})$ is an r-F-tilting pair, $C \in (\check{\mathcal{T}}_F)_r$, that is, there exists an F-exact sequence $0 \to C \to T \to T_C \to 0$ with $T \in \mathcal{T}$ and $T_C \in (\check{\mathcal{T}}_F)_r$. Consider the following pushout diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & C & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & T & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_C & = & T_C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note the above last column, by Lemma 3.1(2), we have $\mathrm{Ext}^i_F(T_C,\ M) = 0$ with $i \geq 1$. Hence $X \cong M \bigoplus T_C$. Moreover, from the above second row, we get $X \in FPres^1(\mathcal{T})$, so $M \in FPres^1(\mathcal{T})$. Now, combining with the facts in Lemma 2.5 and $M \in \mathcal{T}^{F^{\perp}}$, we get an F-exact sequence $0 \to M^{'} \to T_M \to M \to 0$ with $T_M \in \mathcal{T}$ and $M^{'} \in \mathcal{T}^{F^{\perp}}$. In addition, we know $T_M \in \mathcal{T} \subseteq (\check{\mathcal{C}}_F)_r \subseteq {}_{\mathcal{C}}\mathcal{X}^F$ by the assumption and Remark 2.4, and $M^{'} \in \mathcal{T}^{F^{\perp}} \subseteq \mathcal{C}^{F^{\perp}}$ by Lemma 4.4(1). Hence we have $M^{'} \in {}_{\mathcal{C}}\mathcal{X}^F$ by Lemma 3.1(5), it follows that $M^{'} \in \mathcal{T}^{F^{\perp}} \bigcap {}_{\mathcal{C}}\mathcal{X}^F$. Repeating the previous process to $M^{'}$, we can get that $M \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$.

At last, we say $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) \subseteq {}_{\mathcal{T}}\mathcal{X}^F$. For any $M \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$, by the definition, there exists a long F-exact sequence

$$0 \to X \to T_r \to \cdots \to T_1 \to M \to 0,\ (*)$$

where $X \in {}_{\mathcal{C}}\mathcal{X}^F$ and every $T_i \in \mathcal{T} \subseteq (\hat{\mathcal{C}}_F)_r \subseteq {}_{\mathcal{C}}\mathcal{X}^F$ by the assumption and Remark 2.4. However, ${}_{\mathcal{C}}\mathcal{X}^F$ is closed under cokernels of F-monomorphisms by Lemma 3.1($1^{'}$), we also have $M \in {}_{\mathcal{C}}\mathcal{X}^F$. Hence for any $T \in \mathcal{T}$, by dimension shifting, we get $\mathrm{Ext}^i_F(T,\ M) \cong \mathrm{Ext}^{i+r}_F(T,\ X)$ for all $i \geq 1$. Again since $\mathcal{T} \subseteq (\hat{\mathcal{C}}_F)_r$, there exists an F-exact sequence $0 \to C_r \to \cdots \to C_0 \to T \to 0$ with $C_i \in \mathcal{C}$. Then by dimension shifting again, we get $\mathrm{Ext}^{i+r}_F(T,\ X) \cong \mathrm{Ext}^i_F(C_r,\ X) = 0$, which implies that $\mathrm{Ext}^i_F(T,\ M) = 0$, i.e., $M \in \mathcal{T}^{F^{\perp}}$. Now we consider the above sequence $(*)$ again. If we denote $T^i = T_{i-1}$, then we rewrite $(*)$ to $\cdots \xrightarrow{f_2} T^1 \xrightarrow{f_1} T^0 \xrightarrow{f_0} M \longrightarrow 0$. We apply $\mathrm{Hom}_F(T,\ -)$ to the F-exact sequence $0 \to X \to T^{r-1} \to \mathrm{Im} f_{r-1} \to 0$. It is easy to get $\mathrm{Im} f^{r-1} \in \mathcal{T}^{F^{\perp}}$, it can be verified that each $\mathrm{Im} f_i \in \mathcal{T}^{F^{\perp}}$, naturally. That is, $M \in {}_{\mathcal{T}}\mathcal{X}^F$ as expected. $\square$

Proposition 4.6. *Suppose that* ${\mathcal{T}^{F}}^{\perp}\bigcap {}_{\mathcal{C}}\mathcal{X}^F = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ *for some positive integer* $r \geq 1$*. The following statements hold.*

(1) $\mathcal{T} \subseteq {\mathcal{T}^{F}}^{\perp}\bigcap {}_{\mathcal{C}}\mathcal{X}^F$*;*

(2) *If* $\mathcal{T}$ *is precovering, then* $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) = FPres^{r+1}_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$*. In this case,* $\mathcal{T}$ *is an* F*-relative generator of* $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$*;*

(3) *If* $\mathcal{T}$ *is precovering, then* $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}))$*.*

Proof. (1) It is obvious.

(2) For any $N \in FPres^{r+1}_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$, there exists a long F-exact sequence $0 \to X \to T^{r+1} \to T^r \to \cdots \to T^1 \to N \to 0$ with each $T^i \in \mathcal{T}$ and $X \in {}_{\mathcal{C}}\mathcal{X}^F$. Denoting $K = \operatorname{coker}(X \to T^{r+1})$, we have an F-exact sequence $0 \to X \to T^{r+1} \to K \to 0$. By the statement (1), $T^{r+1} \in {}_{\mathcal{C}}\mathcal{X}^F$. Since ${}_{\mathcal{C}}\mathcal{X}^F$ is closed under the cokernels of F-monomorphisms by Lemma 3.1($1'$), we have $K \in {}_{\mathcal{C}}\mathcal{X}^F$. Hence $N \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$.

Conversely, let $M \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$, there exists an F-exact $0 \to X \to T_r \to \cdots \to T_1 \to M \to 0$ with $T_i \in \mathcal{T}$ and $X \in {}_{\mathcal{C}}\mathcal{X}^F$. Denote $L = \ker(T_1 \to M)$, then similar to K the above, we have $L \in {}_{\mathcal{C}}\mathcal{X}^F$. Consider the F-exact sequence $0 \to L \to T_1 \to M \to 0$, and the fact $M \in {\mathcal{T}^{F}}^{\perp}$ by the assumption, we have $M \in {\mathcal{T}^{F}}^{\perp}\bigcap FPres^1_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$. Similar to the proof of Proposition 4.5, there exists an F-exact sequence $0 \to M_1 \to T_M \to M \to 0$ with $T_M \in \mathcal{T}$ and $M_1 \in {\mathcal{T}^{F}}^{\perp}$. So we have the following pullback diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & = & L & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & M_1 & \to & Y & \to & T_1 & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & M_1 & \to & T_M & \to & M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is obvious that $\operatorname{Ext}^i_F(T_1,\ M_1) = 0$, that is, $Y \cong T_1 \bigoplus M_1$. Moreover, by the middle column, we have $Y \in {}_{\mathcal{C}}\mathcal{X}^F$ since $L \in {}_{\mathcal{C}}\mathcal{X}^F$ and $T_M \in \mathcal{T} \subseteq {}_{\mathcal{C}}\mathcal{X}^F$. This implies $M_1 \in {}_{\mathcal{C}}\mathcal{X}^F$. Therefore $M_1 \in {}_{\mathcal{C}}\mathcal{X}^F \bigcap {\mathcal{T}^{F}}^{\perp} = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$, which implies that $M \in FPres^{r+1}_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ by the last row. At last, $\mathcal{T}$ is an F-relative generator of $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ by the definition.

(3) It is obvious that $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) \subseteq FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}))$ since $\mathcal{T} \subseteq FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$. On the other hand, for any $M \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}))$, there

exists a long F-exact sequence $0 \to X \to F_r \to \cdots \to F_1 \to M \to 0$ with $F_i \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ and $X \in {}_{\mathcal{C}}\mathcal{X}^F$. By the assumption and Lemma 3.1($1^{'}$), we know $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) = \mathcal{T}^{F^{\perp}} \bigcap {}_{\mathcal{C}}\mathcal{X}^F$ is closed under F-extensions, finite direct sums and direct summand. Combined with the fact that $\mathcal{T}$ is an F-relative generator of $FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ in the statement (2), we can apply Theorem 3.3(1) to the case of $\mathcal{Y} = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$. Hence there exists an F-exact sequence $0 \to U \to V \to X \to 0$ for some $U \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$, and V admits a long F-exact sequence $0 \to V \to T_r \to \cdots \to T_1 \to M \to 0$ with $T_i \in \mathcal{T}$. Note that $V \in {}_{\mathcal{C}}\mathcal{X}^F$ since $U \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) \subseteq {}_{\mathcal{C}}\mathcal{X}^F$ and $X \in {}_{\mathcal{C}}\mathcal{X}^F$. It follows that $M \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ as expected. □

Proposition 4.7. *Suppose that* $\mathcal{T}^{F^{\perp}} \bigcap {}_{\mathcal{C}}\mathcal{X}^F = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ *for some positive integer* $r \geq 1$ *and* $\mathcal{C} \subseteq FCopres^r(FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}))$*, The following statements hold.*

(1) *If* $\mathcal{T}$ *is precovering, then* $\mathcal{C} \subseteq (\check{\mathcal{T}}_F)_r$*;*

(2) $\mathcal{T} \subseteq (\hat{\mathcal{C}}_F)_r$.

Proof. (1) For any $C \in \mathcal{C}$, by the assumption, there is a long F-exact sequence $0 \to C \to F_1 \to \cdots \to F_r \to X \to 0$ with $F_i \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$. Note that $\mathcal{C} \subseteq {}_{\mathcal{C}}\mathcal{X}^F$ by the definition, $X \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})) = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ by Proposition 4.6(3), naturally. We next apply the results of Theorem 3.3($1^{'}$), i.e., in the case where $\mathcal{Y} = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$, $\mathcal{Z} = \mathcal{T}$. Then there is an F-exact sequence $0 \to U \to V \to C \to 0$ with $U \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$ and $V \in (\check{\mathcal{T}}_F)_r$. However, by the assumption and Lemma 4.4(1), we have $U \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) \subseteq \mathcal{T}^{F^{\perp}} \subseteq \mathcal{C}^{F^{\perp}}$. This implies that $\mathrm{Ext}^i_F(C,\ U) = 0$ for $i \geq 1$, and $V = U \bigoplus C$. Hence $C \in (\check{\mathcal{T}}_F)_r$.

(2) Let $T \in \mathcal{T}$. By Proposition 4.6(1), $T \in {}_{\mathcal{C}}\mathcal{X}^F$, there exists a long F-exact sequence

$$0 \longrightarrow K \longrightarrow C_r \xrightarrow{f_r} \cdots \longrightarrow C_1 \xrightarrow{f_1} C_0 \xrightarrow{f_0} T \longrightarrow 0$$

with each $C_i \in \mathcal{C}$ and $K \in {}_{\mathcal{C}}\mathcal{X}^F \subseteq \mathcal{C}^{F^{\perp}}$. Then we have $\mathrm{Ext}^1_F(\mathrm{Im}f_r,\ K) \cong \mathrm{Ext}^{r+1}_F(T,\ K)$ by dimension shifting.

On the other hand, we consider the long F-exact sequence $0 \to L \to C^r \to \cdots \to C^1 \to K \to 0$ with $C^i \in \mathcal{C}$ and $L \in {}_{\mathcal{C}}\mathcal{X}^F$. By the statement (1), every $C^i \in (\check{\mathcal{T}}_F)_r$, too. We apply the fact in Theorem 3.4, then there exists an F-exact sequence $0 \to K \to V \to U \to 0$ with $U \in (\check{\mathcal{T}}_F)_{r-1}$ and V admits $0 \to L \to T_1 \to \cdots \to T_r \to V \to 0$ with $T_i \in \mathcal{T}$, which implies that $V \in FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}) = \mathcal{T}^{F^{\perp}} \bigcap {}_{\mathcal{C}}\mathcal{X}^F$. Hence we have $\mathrm{Ext}^{r+1}_F(T,\ K) \cong \mathrm{Ext}^r_F(T,\ U)$. However, $\mathrm{Ext}^r_F(T,\ U)$ vanished by Lemma 3.1(3). This means the above $\mathrm{Ext}^1_F(\mathrm{Im}f_r,\ K) = 0$. Therefore $C_r \cong K \bigoplus \mathrm{Im}f_r$ and $\mathrm{Im}f_r \in \mathcal{C}$. Hence $T \in (\check{\mathcal{C}}_F)_r$ as expected. □

Now, based on the results of Propositions 4.5 and 4.7, we have the following theorem, which is also the main result of our manuscript.

Theorem 4.8. *Let $\mathcal{C}$ and $\mathcal{T}$ be two F-orthogonal classes with $\mathcal{T}$ precovering. If $(\mathcal{C},\ \mathcal{T})$ is an r-F-tilting pair, then, for some positive integer $r \geq 1$, $\mathcal{T}^{F^{\perp}} \bigcap {}_{\mathcal{C}}\mathcal{X}^F = FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T})$. And the converse also holds true if $\mathcal{C} \subseteq FCopres^r(FPres^r_{{}_{\mathcal{C}}\mathcal{X}^F}(\mathcal{T}))$.*

Corollary 4.9. [15, Theorem 3.10] *Let Λ be an artin algebra. A Λ-module T is r-F-tilting if and only if $T^{F^{\perp}} = FPres^r(T)$.*

Proof. Let $\mathcal{C} = \mathcal{P}(F)$ and $\mathcal{T} = \mathrm{add}(T)$ in the above theorem. Note that in this case $\mathcal{T}$ is precovering and ${}_{\mathcal{C}}\mathcal{X}^F$ is just the ordinary Λ-modules. Since $\mathcal{I}(F) \subseteq T^{F^{\perp}} = FPres^r(T)$, we have $\mathcal{P}(F) = \mathcal{C} \subseteq FCopres^r(FPres^r(T))$, naturally. Now, combining the results from Remark 4.2(1) and Theorem 4.8, we immediately obtain this conclusion. □

In this manuscript, we have introduced the notion of relative tilting pairs with respect to an additive subfunctor F of Ext^1, and established a characterization theorem that generalizes earlier work of Wei and others. This framework unifies several existing theories of tilting and cotilting. Future work may include studying the associated torsion pairs, developing relative tilting mutations, or investigating relative tilting in triangulated and derived categories.

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